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Rough sets with involution

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Abstract The aim of this study is to introduce the notion of rough sets with involution. It is given some conditions for characterization of lower and upper aproximation of a set in approximation spaces with involution. Further, the relationship between topological spaces with involution and approximation spaces are presented in view of categorical aspect.

Key Words Rough sets, involution

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1 Introduction

Structures such as fuzzy sets and rough sets make them suitable for data analysis by editing incomplete, inadequate and ambiguous information. The rough set theory first proposed by Pawlak in 1982 is a mathematical method used to obtain information from indefinite and incomplete data [6]. The main concepts of classical rough sets are lower and upper approximation operators based on equivalence relations. Generalized rough set is defined based on binary relation. Further, a pair (X, r) is called a generalized approximation space where X is a non-empty set and r is a relation on X [7, 8]. Note that textural rough set algebra was introduced to approach for generalized rough set, and it is obtained effective results for classical rough sets in [2, 3, 4].

On the other hand, an involution is a function $'$ that is its own inverse, that is, $(x')' = x$ for all x in the domain of $'$ [5]. It is well known that the concept of involution is an important tool for fuzzy lattices. Further, a pair $(X,')$ will be called a set with involution in this work. The main aim of this study is to present a discussion on rough sets with involution which may give more suitable environments for rough set theory.

This paper is organized into three sections: the concept of set with involution is introduced, and some properties are given in the next section. In the section 2, the categories of topological spaces with involution and some results are given. The section 3 is devoted to the rough sets with involution, and some properties of approximation spaces with involution are presented. The reader is referred to [5] for terms from lattice theory not defined here. Generally we follow the terminology of [1] for general concepts relating to category theory.

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2 Sets and Involution

In this section, the notions of set with involution and complemented function are introduced, and some categorical properties are given.

Definition 2.1. Let X be a set. A function $' : X \to X$, $x \to x'$ is called an involution on X if it is satisfies

$$(x')' = x, \quad \text{for all} \quad x \in X.$$

$(X,')$ is called a set with involution.

Let $(X,')$ be a set with involution and $A \subseteq X$. Then we will set

$$A' = \{a' \mid a \in A\}.$$

We have the following basic properties.

Lemma 2.2. *For $A, B \subseteq X$,*

(i) $(A')' = A$.

(ii) $A \subseteq B \Longrightarrow A' \subseteq B'$.

(iii) $(X \setminus A)' = X \setminus A'$.

Proof. Since $x \in A \Longleftrightarrow x' \in A'$ for all $x \in X$, the results are obvious. □

Examples 2.3.

(1) The identity involution is defined by $x \to x' = x$. Then we have $A = A'$, for all $A \subseteq X$.

(2) Let $X = \{a, b, c, d\}$. We consider the following two involutions,

$$'_1 : X \to X, a \to b, \quad b \to a, \quad c \to d, \quad d \to c$$

and

$$'_2 : X \to X, a \to b, \quad b \to a, \quad c \to c, \quad d \to d$$

Now let $A = \{b, c\}$. Then $A^{'_1} = \{a, d\}$ and $A^{'_2} = \{a, c\}$

(3) Let $(X,'_1)$ and $(Y,'_2)$ be sets with involution. Then $' : X \times Y \to X \times Y$ which is defined by

$$(x, y)' = (x^{'_1}, y^{'_2}), \quad \forall (x, y) \in X \times Y$$

is an involution on $X \times Y$.

Definition 2.4. Let $(X,')$ and $(Y,'')$ be sets with involution and $f : X \to Y$ be a point function.

(a) If $f \circ' =''\circ f$ then f is called involution preserving.

(b) The complement function $f' : X \to Y$ is defined by $f'(x) = (f(x'))''$.

(c) f is called complemented if $f = f'$.

Lemma 2.5.

(i) f is involution preserving if and only if f is complemented.

(ii) f is complemented if and only if $f(x') = f(x)''$ for all $x \in X$.

Proof. (i)

$$\begin{aligned} f \text{ is involution preserving} &\Longleftrightarrow f \circ' =''\circ f \\ &\Longleftrightarrow f(x') = (f(x))'',\ \forall x \in X \\ &\Longleftrightarrow f(x) = (f(x'))'' = f'(x) \\ &\Longleftrightarrow f \text{ is complemented} \end{aligned}$$

(ii)

$$\begin{aligned} f \text{ is complemented} &\Longleftrightarrow f(x) = f'(x),\ \forall x \in X \\ &\Longleftrightarrow f(x') = f'(x'),\ \forall x \in X \\ &\Longleftrightarrow f(x') = f'(x') = (f(x')')'' = (f(x))'' \end{aligned}$$

□

We will denote by **SetInv** the category of whose objects are sets with involution and whose morphisms are point functions.

Proposition 2.6. *The category of sets and functions which is **Set** is isomorphic to a full subcategory of **SetInv**.*

Proof. Let $SetInv_{id}$ be the category of sets with identity involution and point functions. Clearly, it is a full subcategory of **SetInv**. Now consider the mapping $F : Set \to SetInv_{id}$ which is defined by

$$F(X) = (X,'), \quad F(X \xrightarrow{f} Y) = (X,') \xrightarrow{f} (Y,'), \quad ' : X \to X,\ x \to x' = x$$

for every morphism in **Set**.

We observe that $(X,')$ is a object and f is a morphism in the category $SetInv_{id}$. Clearly F maps the identity function on X to the identity function on $(X,')$, while composition of morphisms in **Set** corresponds to composition of relations in sets with involution, so $F(f \circ g) = F(f) \circ F(g)$. This establishes that F is a functor. Obviously, F is full and faithful and bijective on objects and so it is an isomorphism functor. □

Proposition 2.7. *The mapping $F : SetInv \to SetInv$, where*

$$F(X, ') = (X, ') \quad \textit{and} \quad F(f) = f'$$

is an isomorphism functor, where f' is complement function of f.

Proof. Let $(X,')$ be **SetInv**-object and 1_X be the identity morphism. Then $F(1_X) = 1'_F(X) = 1_F(X)$. Now let f and g be **SetInv**-morphisms. So,

$$F(f \circ g) = (f \circ g)' = f' \circ g' = F(f) \circ F(g).$$

Hence, F is a functor. We take $(X_1, '_1), (X_2, '_2) \in \mathrm{Ob}\, SetInv$. It is easy to see that the hom-set restriction,

$$F : \hom((X_1, '_1), (X_2, '_2)) \to \hom((X_1, '_1), (X_2, '_2))$$

is full and faithful. Further, the mapping F is bijective on objects. □

Let $(X_j,'_j)$, $j \in J$ be a family of sets with involution and $X = \prod_{j \in J} X_j$. Then for $x = (x_j)_{j \in J} \in X$

$$' : X \to X_j, \quad x \to x' = (x_j^{'_j})_{j \in J}$$

is an involution on X. Now let $p_j : X \to X_j$ be $jth-$ projection functions. Then we have:

Proposition 2.8. *$(X,', (p_j))$ is product of the family $(X_j,'_j)$ in the category* ***SetInv****.*

Proof. Let $(Z,'')$ be a **SetInv**-object and $q_j : Z \to X_j$ be **SetInv**-morphisms. We must establish the existence of a unique difunction p which makes the diagram below commutative for each $j \in J$.

$$\begin{tikzcd}
(Z,'') \arrow[r, "p"] \arrow[dr, "q_j"'] & (X,') \arrow[d, "p_j"] \\
 & (X_j,'_j)
\end{tikzcd}$$

Define $p : Z \to X$ as

$$q_j(z) = p_j(p(z)), \quad \forall z \in Z$$

Clearly, it is a **SetInv**-morphism and the above diagram is commutative. It remains to show the uniqueness of p. Suppose that the function $k : (Z,'') \to (X,')$ also satisfies $q_j = p_j \circ k$ for all $j \in J$. Then for all $x_j \in X_j$

$$p^{-1}(p_j^{-1}(x_j)) = (p_j \circ p)^{-1}(x_j) = (p_j \circ k)^{-1}(x_j) = k^{-1}(p_j^{-1}(x_j))$$

and we deduce that $p^{-1}(x) = k^{-1}(x)$ for all $x \in X$, and so $p = k$. □

Lemma 2.9. *The composition of complemented functions is complemented.*

Proof. Let $(X_i, '_i)$ be sets with involution for $i = 1, 2, 3$. Suppose that $f : X_1 \to X_2$ and $g : X_2 \to X_3$ are complemented function.

Then we observe that for all $x \in X$

$$(g \circ f)(x^{'_1}) = g(f(x^{'_1})) = g((f(x))^{'_2}) = (g(f(x)))^{'_3}$$

Hence the proof is completed. □

We will denote by **cSetInv** the category of whose objects are sets with involution and whose morphisms are complemented point functions.

It is noted that **cSetInv** is subcategory of **SetInv**.

Lemma 2.10. *Let $(X, ')$ and $(Y, '')$ be sets with involution and $f : X \to Y$ be a point function. Then we have the following properties:*

(i) $(f')' = f$.

(ii) $f'(A) = (f(A'))''$, $A \subseteq X$.

(iii) $(f')^{-1}(B) = (f^{-1}(B''))'$, $B \subseteq Y$.

(iv) *f is complemented iff* $f(A') = f(A)'$, $\forall A \subseteq X$.

(v) *If f is complemented then* $f^{-1}(B'') = (f^{-1}B)'$, $\forall B \subseteq Y$.

Proof. (i) Let $x \in X$. Then $(f')'(x) = (f'(x'))'' = ((f((x')'))'')'' = (f(x)'')'' = f(x)$.

(ii) Let $A \subseteq X$. Then

$$\begin{aligned} y'' \in f(A')'' &\Longleftrightarrow y \in f(A') \\ &\Longleftrightarrow \exists x' \in A' \text{ such that } y = f(x') \\ &\Longleftrightarrow \exists x' \in A' \text{ such that } y'' = (f(x'))'' \\ &\Longleftrightarrow \exists x \in A \text{ such that } y'' = f'(x) \\ &\Longleftrightarrow y'' \in f'(A). \end{aligned}$$

(iii) Let $B \subseteq Y$. Then

$$x' \in (f^{-1}(B''))' \Longleftrightarrow x \in f^{-1}(B'') \Longleftrightarrow f(x) \in B'' \Longleftrightarrow f(x)'' = f'(x') \in B \Longleftrightarrow x' \in (f')^{-1}(B).$$

Since $f' = f$, the results (iv) and (v) are consequences of the results (ii) and (iii). □

Let $(X, ')$ be product of the family of sets with involution $(X_j, '_j)$, $j \in J$ and $p_j : X \to X_j$ be *jth*-projection functions. Then for $x = (x_j)_{j \in J} \in X$,

$$p_j(x') = p_j((x_j)') = p_j((x_j^{'_j})) = x_j^{'_j} = (p_j(x))^{'_j}$$

and so projection functions are complemented. Hence, from Proposition 2.8, we have:

Corollary 2.11. *The category **cSetInv** has products.*

3 Topology and Involution

Lemma 3.1. *Let $(X,')$ be a set with involution. Let J be an index set, $A_j \in P(X)$. Then:*

(i) $(\bigcap_{j\in J} A_j)' = \bigcap_{j\in J} A'_j$.

(ii) $(\bigcup_{j\in J} A_j)' = \bigcup_{j\in J} A'_j$.

Proof. (i) $x' \in (\bigcap_{j\in J} A_j)' \iff x \in (\bigcap_{j\in J} A_j) \iff \forall j \in J, x \in A_j \iff \forall j \in J, x' \in A'_j \iff x' \in \bigcap_{j\in J} A'_j$.

(ii) $x' \in (\bigcup_{j\in J} A_j)' \iff x \in (\bigcup_{j\in J} A_j) \iff \exists j \in J, x \in A_j \iff \exists j \in J, x' \in A'_j \iff x \in \bigcup_{j\in J} A'_j$. □

Corollary 3.2. *Let $(X,')$ be a set with involution and T is a topology on X. Then*

$$T' = \{G' \mid G \in T\}$$

is a topology on X.

Further, if $F \subseteq X$ is T-closed then F' is T'-closed.

Proof. Since $X' = X$ and $\emptyset' = \emptyset$, we have $X, \emptyset \in T'$. Further, T' is closed under finite intersections and arbitrary unions by Lemma 3.1. Hence, T' is a topology on X.

On the other hand, for $F \subseteq X$:

$$\begin{aligned} F \quad \text{is} \quad T-\text{ closed} &\iff X \setminus F \quad \text{is} \quad T-\text{ open} \\ &\iff (X \setminus F)' \quad \text{is} \quad T'- \quad \text{open} \\ &\iff X \setminus F' \quad \text{is} \quad T'-\text{ open} \\ &\iff F' \quad \text{is} \quad T'-\text{ closed} \end{aligned}$$

□

Example 3.3. The identity involution is defined by $x \to x' = x$. Then we have $T = T'$.

Example 3.4. Let $X = \{a, b, c, d\}$. Then $T = \{\emptyset, \{a\}, X\}$ is a topology on X. Now we consider the involution,

$$a \to b, \quad b \to a, \quad c \to d, \quad d \to c$$

Then the involution topology is $T' = \{\emptyset, \{b\}, X\}$.

Let $(X, ', T)$ be topological spaces with involution. Then identity function on $(X,')$ is continuous, and the composition of continuous function is continuous. We will denote by **TopInv** the category of whose objects are topological spaces with involution and whose morphisms are continuous point functions.

Corollary 3.5. *The category topological spaces and continuous functions which is* ***Top*** *is isomorphic to a full subcategory of* ***TopInv****.*

Proof. Let $TopInv_{id}$ be the category of topological spaces with identity involution and continuous point functions. Clearly, it is a full subcategory of **TopInv**. Now consider the mapping $F : Top \to TopInv_{id}$ which is defined by

$$F(X,T) = (X,' ,T), \quad F((X,T) \xrightarrow{f} (Y,V)) = (X,' ,T) \xrightarrow{f} (Y,' ,V), \quad ' : X \to X,\, x \to x' = x$$

for every morphism in **Top**. By Proposition 2.6, F is isomorphism functor. $\square$

Lemma 3.6. *Let $(X, ', T)$ and $(Y, '', V)$ be topological spaces with involution. $f : (X, ', T) \to (Y, '', V)$ is continuous if and only if $f' : (X, ', T') \to (Y, '', V'')$ is continuous.*

Proof. The proof is obvious since $(f^{-1}(B))' = (f')^{-1}(B'') = (f^{-1}(B))' \in T'$ for $B'' \in V''$ from Lemma 2.10 (iii). $\square$

Corollary 3.7. *The mapping $F : TopInv \to TopInv$, where*

$$F(X, ', T) = (X, ', T') \quad \text{and} \quad F(f) = f'$$

is an isomorphism functor.

Proof. The proof is obtained easily from Lemma 3.6 and Proposition 2.7. $\square$

4 Rough Set with Involution

In this section, the concept of rough set with involution are presented. Further some characterizations of lower and upper approximation operators are given in approximation spaces with involution.

Let $(X,')$ be a set with involution and r be a relation on X. Now we set

$$r' = \{(x', y') \mid (x, y) \in r\}$$

Clearly, r' is a relation on X. Then we have:

Lemma 4.1. *For $x \in X$ and $A \subseteq X$,*

(i) $\big(r(x)\big)' = r'(x')$, *where* $r(x) = \{y \mid (x, y) \in r\}$

(ii) $(r')^{-1}A = (r^{-1}A')'$

Proof. (i) Let $x \in X$. For $y \in X$,

$$\begin{aligned} y' \in \big(r(x)\big)' &\iff y \in r(x) \iff (x, y) \in r \\ &\iff (x', y') \in r' \iff y' \in r'(x'). \end{aligned}$$

(ii) Let $A \subseteq X$. For $y \in X$,

$$\begin{aligned} y' \in (r^{-1}A')' &\iff y \in r^{-1}A' \iff r(y) \subseteq A' \\ &\iff r'(y') \subseteq A \iff y' \in (r')^{-1}A. \end{aligned}$$

$\square$

Let r be a relation on a universe X. Then the pair (X, r) is called an approximation space [7]. The operators $\underline{app}_r, \overline{app}_r : P(X) \to P(X)$ defined by

$$\begin{aligned} \underline{app}_r A =& \{x \mid r(x) \subseteq A\} \\ \overline{app}_r A =& \{x \mid r(x) \cap A \neq \emptyset\}, \quad \forall A \subseteq X. \end{aligned}$$

are called *lower approximation operator* and *upper approximation operator*, respectively [7]. Then the pair $(\underline{app}_r A, \overline{app}_r A)$ is called a rough set of A. From [2, Theorem 6.2], we have

$$\underline{app}_r A = X \setminus r^{-1}(X \setminus A) \quad \text{and} \quad \overline{app}_r A = r^{-1} A.$$

Lemma 4.2. *Let $(X,', r)$ be an approximation space with involution and $A \subseteq X$. Then the following equalities are satisfied for the rough set $(\underline{app}_r A, \overline{app}_r A)$ of A.*

$$(\underline{app}_r A)' = \underline{app}_{r'} A' \text{ and } \quad (\overline{app}_r A)' = \overline{app}_{r'} A'$$

Proof. For $x \in X$,

$$\begin{aligned} x \in (\underline{app}_r A)' &\iff x' \in \underline{app}_r A \\ &\iff r(x') \subseteq A \\ &\iff (r(x'))' = r'(x) \subseteq A' \\ &\iff x \in \underline{app}_{r'} A' \end{aligned}$$

On the other hand, it is observe that

$$\begin{aligned} x \in (\overline{app}_r A)' &\iff x' \in \overline{app}_r A \\ &\iff r(x') \cap A \neq \emptyset \\ &\iff (r(x') \cap A)' = r'(x) \cap A' \neq \emptyset \\ &\iff x \in \overline{app}_{r'} A' \end{aligned}$$

□

Proposition 4.3. *Let $(X,', r)$ be an approximation space with involution and $A \subseteq X$. Then*

$$\underline{app}_{r'} A = X \setminus (r')^{-1}(X \setminus A) \quad \text{and} \quad \overline{app}_{r'} A = (r')^{-1}(A)$$

Proof. Let $A \subseteq X$. Then, by [2, Theorem 6.2.], we can write

$$\underline{app}_r A' = X \setminus r^{-1}(X \setminus A').$$

From Lemma 4.2,

$$\left(\underline{app}_r A'\right)' = \left(X \setminus r^{-1}(X \setminus A')\right)' \Longrightarrow \underline{app}_{r'} A = X \setminus (r^{-1}(X \setminus A'))' = X \setminus (r')^{-1}(X \setminus A).$$

Likewise, we write

$$\overline{app}_r A' = r^{-1}(A')$$

from [2, Theorem 6.2.]. Then we have

$$(\overline{app}_r A')' = ((r)^{-1}(A'))' \Longrightarrow \overline{app}_{r'} A = (r')^{-1}(A).$$

□

Let (X, r) and (Y, s) be approximation spaces and $f : X \to Y$ be a point function. Now we recall that [3] if

$$f^{-1}(\underline{app}_s(B)) \subseteq \underline{app}_r(f^{-1}(B)), \quad \forall B \subseteq Y$$

then f is called *continuous*. Equivalently,

$$f \quad \text{is continuous} \Longleftrightarrow \overline{app}_r(f^{-1}(B)) \subseteq f^{-1}(\overline{app}_s B), \quad \forall B \subseteq Y.$$

Further the category of approximation spaces and continuous functions was denoted by **App** in [3].

Proposition 4.4. *Let $(X,', r)$ and $(Y,'', s)$ be approximation spaces with involution and $f : X \to Y$ be a function. Then $f : (X,', r) \to (Y,'', s)$ is continuous function if and only if $f' : (X,', r') \to (Y,'', s'')$ is continuous.*

Proof. Let $B \subseteq Y$. Then $B'' \subseteq Y$. Hence,

$$\begin{aligned} f \text{ is continuous} &\Longleftrightarrow f^{-1}(\underline{app}_s(B'')) \subseteq \underline{app}_r(f^{-1}(B'')) \\ &\Longleftrightarrow (f^{-1}(\underline{app}_s(B''))' \subseteq (\underline{app}_r(f^{-1}(B'')))' \\ &\Longleftrightarrow (f')^{-1}(\underline{app}_s(B''))'' \subseteq \underline{app}_{r'}(f^{-1}(B''))' \\ &\Longleftrightarrow (f')^{-1}(\underline{app}_{s''}B) \subseteq \underline{app}_{r'}(f')^{-1}(B) \\ &\Longleftrightarrow f' \text{ is continuous.} \end{aligned}$$

□

Corollary 4.5. *Let $f : (X,', r) \to (Y,'', s)$ be a point function. Then f is continuous if and only if $\overline{app}_{r'}((f')^{-1}(B)) \subseteq (f')^{-1}(\overline{app}_{s''}B)$ for all $B \subseteq Y$.*

Proof. Let $B \subseteq Y$. Then $B \subseteq Y$. Hence,

$$\begin{aligned} f \text{ is continuous} &\Longleftrightarrow f' \text{ is continuous} \\ &\Longleftrightarrow (f')^{-1}(\underline{app}_{s''}(Y \setminus B) \subseteq \underline{app}_{r'}(f')^{-1}(Y \setminus B) \\ &\Longleftrightarrow X \setminus (f')^{-1}(\overline{app}_{s''}B) \subseteq X \setminus \overline{app}_{r'}((f')^{-1}(B)) \\ &\Longleftrightarrow \overline{app}_{r'}((f')^{-1}(B)) \subseteq (f')^{-1}(\overline{app}_{s''}B) \end{aligned}$$

□

We will denote by **AppInv** the category of whose objects are approximation spaces with involution and whose morphisms are continuous point functions.

Proposition 4.6. *The mapping $F : AppInv \to AppInv$, where*

$$F(X, {}', r) = (X, {}', r') \quad and \quad F(f) = f'$$

is an isomorphism functor.

Proof. Let $(X,', r)$ be **AppInv**-object and 1_X be the identity morphism. Then $F(1_X) = 1'_F(X) = 1_F(X)$. Now let f and g be **AppInv**-morphisms. So,

$$F(f \circ g) = (f \circ g)' = f' \circ g' = F(f) \circ F(g).$$

Hence, F is a functor. We take $(X, {}', r), (Y, {}'', s) \in \mathrm{Ob}\, AppInv$. It is easy to see that the hom-set restriction,

$$F : \hom((X, {}', r), (Y, {}'', s)) \to \hom((X, {}', r'), (Y, {}'', s''))$$

is full and faithful. Further, the mapping F is bijective on objects. □

Proposition 4.7. *The category **App** is isomorphic to a full subcategory of **AppInv**.*

Proof. Let $AppInv_{id}$ be the category of approximation spaces with identity involution and continuous-point functions. Clearly, it is a full subcategory of **AppInv**. Now consider the mapping $F : App \to AppInv_{id}$ which is defined by

$$F(X, r) = (X,', r), \quad F((X, r) \xrightarrow{f} (Y, s)) = (X,', r) \xrightarrow{f} (Y,', s), \quad ' : X \to X,\, x \to x' = x$$

for every morphism in **App**.

We observe that $(X,', r)$ is an object and f is a morphism in the category $AppInv_{id}$. Clearly F maps the identity function on X to the identity function on $(X,', r)$, while composition of morphisms in **App** corresponds to composition of relations in sets with involution, so $F(f \circ g) = F(f) \circ F(g)$. This establishes that F is a functor. Obviously, F is full and faithful and bijective on objects and so it is an isomorphism functor. □

Recall that if r is a reflexive relation on X then the family

$$T_r = \{G \subseteq X \mid \underline{app}_r G = G\}$$

is a topology on X by Proposition 2 in [3]. Further if $'$ is an involution on X, then r' is also reflexive relation on $(X,')$, and the family

$$T_{r'} = \{G \subseteq X \mid \underline{app}_{r'} G = G\}$$

is also topology on X.

Note that for $F \subseteq X$,

$$\begin{aligned} F \quad \text{is} \quad T_r - \text{closed} &\iff X \setminus F \quad \text{is} \quad T_r - \text{open} \\ &\iff \underline{app}_r (X \setminus F) = X \setminus F \\ &\iff X \setminus \big(\underline{app}_r (X \setminus F)\big) = X \setminus \big(X \setminus F\big) \\ &\iff \overline{app}_r (F) = F \end{aligned}$$

Then we have the following:

Proposition 4.8. *Let $(X,', r)$ be approximation space with involution and r be a reflexive relation on X. Then*

(i) $T'_r = \{G' \subseteq X \mid G \in T_r\}$ *is a topology on* X.

(ii) $T'_r = T_{r'}$.

Further, if $F \subseteq X$ is T_r-closed then F' is T'_r-closed.

Proof. (i) Firstly, $X, \emptyset \in T'_r$, since $X' = X$ and $\emptyset' = \emptyset$. Further, T'_r is closed under finite intersections and arbitrary unions by Lemma 3.1. Hence, T'_r is a topology on X.

(ii) Let $G \subseteq X$. Then $G' \in T'_r \Longleftrightarrow G \in T_r \Longleftrightarrow \underline{app}_r G = G \Longleftrightarrow (\underline{app}_r G)' = G' \Longleftrightarrow \underline{app}_{r'} G' = G' \Longleftrightarrow G' \in T_{r'}$.

On the other hand, for $F \subseteq X$:

$$\begin{aligned} F \quad \text{is} \quad T_r - \text{closed} &\Longleftrightarrow \overline{app}_r(F) = F \\ &\Longleftrightarrow \left(\overline{app}_r(F)\right)' = F' \\ &\Longleftrightarrow \overline{app}_{r'}(F') = F' \\ &\Longleftrightarrow F' \quad \text{is} \quad T'_r - \text{closed} \end{aligned}$$

□

Proposition 4.9. *Let $(X,', r)$and $(Y,'', s)$ be approximation spaces with involution and r, s be reflexive relations. Let $f : X \to Y$ be a point function. Then f is $T_r - T_s$ continuous if and only if f' is $T_{r'} - T_{s''}$ continuous.*

Proof. ($\Longrightarrow$:) Suppose that f be a $T_r - T_s$ continuous function. Let $B \in T_{s''}$. We must show that $(f')^{-1}B \in T_{r'}$. Then

$$\begin{aligned} B \in T_{s''} &\Longrightarrow \underline{app}_{s''} B = B \Longrightarrow \underline{app}_s B'' = B'' \\ &\Longrightarrow B'' \in T_s \Longrightarrow f^{-1}(B'') \in T_r \\ &\Longrightarrow \underline{app}_r f^{-1}(B'') = f^{-1}(B'') \\ &\Longrightarrow \underline{app}_{r'} (f^{-1}(B''))' = (f^{-1}(B''))' \\ &\Longrightarrow \underline{app}_{r'} (f')^{-1} B = (f')^{-1} B \\ &\Longrightarrow (f')^{-1} B \in T_{r'} \end{aligned}$$

($\Longleftarrow$:) Suppose that f' be a $T_{r'} - T_{s'}$ continuous function. Let $B \in T_s$. We must show that $f^{-1}B \in T_r$.

Then

$$
\begin{aligned}
B \in T_s &\Longrightarrow \underline{app}_s B = B \Longrightarrow \underline{app}_{s''} B'' = B'' \\
&\Longrightarrow B'' \in T_{s''} \Longrightarrow (f')^{-1}(B'') \in T_{r'} \\
&\Longrightarrow \underline{app}_{r'} (f')^{-1}(B'') = (f')^{-1}(B'') \\
&\Longrightarrow \underline{app}_{r} ((f')^{-1}(B''))' = ((f')^{-1}(B''))' \\
&\Longrightarrow \underline{app}_{r} f^{-1}B = f^{-1}B \\
&\Longrightarrow f^{-1}B \in T_r
\end{aligned}
$$

□

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