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Common fixed point theorems for four weakly compatible maps in complete metric spaces

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Abstract In this paper we study and generalize common fixed point theorems for two pairs of weakly compatible mappings using Fisher theorems in complete metric spaces. The main idea is the extension and generalization of some fixed point theorems of Fisher for common fixed point of four weakly compatible mappings.

Key Words Compatible mappings, common fixed points, complete metric spaces
MSC 2010 54H25, 54H20

1 Introduction

The study of common fixed point results began with the pioneering work of Jungck [13, 14], who in 1976, proved his remarkable common fixed point theorem for commuting mappings. Later on in 1996, Jungck also introduced the concept of weakly compatible mappings and proved some common fixed point theorems in ordinary metric spaces. Jungck's idea of commuting and compatible mappings has been equally generalized and extended by several authors in all branches of analysis and topology such as Das and Naik [7], Pant [17], Sessa [20], Singh[21] are a few to name. For more results in this direction, we refer to [1], [6], [18], [19], [21], [22] and references therein.

The present paper aims at proving some common fixed point theorems in the setting of complete metric spaces for pair of weakly compatible mappings. The obtained results are generalization of some fixed point theorems of Fisher [10]. The following fixed point theorems were proved in [3] and [10].

Theorem 1.1. [3] *If $T : X \to X$ is a mapping of the complete metric space X into itself satisfying the inequality*

$$d(Tx, Ty) \leqslant kd(x, y)$$

for all $x, y \in X, x \neq y$, where $0 \leqslant k < 1$, then T has a unique fixed point in X.

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Theorem 1.2. [10] *If $T : X \to X$ is a mapping of the complete metric space (X, d) into itself satisfying the inequality*

$$d(Tx, Ty) \leqslant \max \left\{ \begin{array}{c} 2c_1 d(x, y), \\ c_2 [d(x, Tx) + d(y, Ty)], \\ c_3 [d(x, Ty) + d(y, Tx)] \end{array} \right\}$$

,for all $x, y \in X$ and $0 \leqslant c_1, c_2, c_3 < \frac{1}{2}$,then T has a unique fixed point in X.

Theorem 1.3.[10] *If $T : X \to X$ is a mapping of the complete metric space (X, d) into itself satisfying the inequality*

$$[d(Tx, Ty)]^2 \leqslant \max \left\{ \begin{array}{c} 2c_1 d(x, y) [d(x, Tx) + d(y, Ty)], \\ 2c_2 d(x, y) [d(x, Ty) + d(y, Tx)], \\ c_3 [d(x, Tx) + d(y, Ty)] [d(x, Ty) + d(y, Tx)] \end{array} \right\}$$

,for all $x, y \in X$ and $0 \leqslant c_1, c_2, c_3 < \frac{1}{2}$,then T has a unique fixed point in X.

2 Preliminaries

Definition 2.1. *If X is a non empty set and $d : X \times X \to [0, \infty)$ is a mapping satisfy the conditions:*

(i) $0 \leqslant d(x, y)\ \forall\ x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)\ \forall\ x, y \in X$.

(iii) $d(x, y) \leqslant (d(x, z) + d(z, y))\ \forall\ x, y, z \in X$.

Then d is called a metric on X and the pair (X, d) is called a metric space.

Definition 2.2. *(i) A sequence $\{x_n\}$ in a metric space (X, d) is said to converge to a point $x \in X$, if for every $\in> 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) <\in, \forall n \in N$ denoted by $\lim_{n\to\infty} x_n = x$*

(ii) $\{x_n\}$ is called Cauchy sequence if for some $N \in \mathbb{N}$ there exists $\in> 0$ such that for $m, n \in N$, $m > n$ we have $\lim_{m,n\to\infty} d(x_m, x_n) = 0$.

(iii) A metric space (X, d) is said to be complete if and only if every Cauchy sequence in X converges to a point of X.

Definition 2.3. *Let f and g be two self mappings of a set X into itself. A point $x \in X$ is called a coincident point of the mappings f and g if $fx = gx$.*

Definition 2.4. *A point $x \in X$ is said to be a fixed point of a self-map $f : X \to X$ if $T(x) = x$.*

Definition 2.5.[13] *Let (X, d) be a metric space. Two self maps f, $g : X \to X$ are said to be commuting mappings if $f(g(x)) = g(f(x))$ for all $x \in X$.*

Definition 2.6.[20] *Two self mappings f and g of a metric space (X, d) into itself are said to be weakly commuting if $d(fgx, gfx) \leqslant d(fx, gx)$ for all $x \in X$.*

Definition 2.7.[13] *Two self maps f and g of a metric space (X,d) are said to be compatible if*

$$\lim_{n\to\infty} d\left(fgx_n, gfx_n\right) = 0$$

where $x_n \in X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$.

Lemma 2.8.[13] *Let f and g be compatible mappings of a metric space (X,d) into itself. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$, for some $u \in X$, then*

$$\lim_{n\to\infty} gfx_n = fu, \text{ if } f \text{ is continuous.}$$

Example 2.9. *Let X is a non-empty set and $d : X \times X \to R^+$ is a metric on X given by $d(x,y) = |x-y| \forall x, y \in X$. If f and g on X are given by*

$$f(x) = \frac{1}{10}, \ g(x) = \frac{10x+1}{20}$$

Then f and g commute with each other such that $f(g(x)) = g(f(x)) = \frac{1}{10}$ with $\frac{1}{10}$ as the unique common fixed point of f and g and so are weakly compatible on X.

3 Common fixed point results

In this section we obtain coincidence points and common fixed point theorems for four self mappings in complete metric spaces. In order to start our main results we begin with a simple but useful Lemma that will be used in the sequel.

Lemma 3.1. *Let (X,d) be a complete metric space and let $A, B, S, T : X \to X$ be self mappings of X into itself satisfying the following conditions:*

(i) $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$.

(ii)

$$d(A(x), B(y)) \leqslant \max \left\{ \begin{array}{c} 2c_1 d(S(x), T(y)), \\ c_2 \left[d(A(x), T(y)) + d(B(y), S(x))\right], \\ c_3 \left[d(A(x), S(x)) + d(B(y), T(y))\right] \end{array} \right\}$$

$\forall x, y \in X$ and $c_1, c_2, c_3 \geqslant 0$ are non negative real numbers such that

$$0 \leqslant c_1, c_2, \alpha_3 < \frac{1}{2}$$

then every sequence $\{y_n\}$ with initial point x_0 is a Cauch sequence in X.

Proof. Let $x_0 \in X$ and choose a point $x_1 \in X$ such that $Tx_1 = Ax_0$ and for x_1 there exists $x_2 \in X$ such that $Sx_2 = Bx_1$, continuing this process we construct sequences $\{x_n\}$ and $\{y_n\}$ in X given by

$$\left\{ \begin{array}{c} y_{2n} = Ax_{2n} = Tx_{2n+1} \\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \end{array} \right.$$

Suppose that there exists $k \in [0,1)$ such that

$$d(y_n, y_{n+1}) \leqslant kd(y_{n-1}, y_n) \ \ \forall n \geqslant 1$$

We show that $\{y_n\}$ is a Cauchy sequence in X. Using (i) and (ii), we have

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Tx_{2n+1}, hx_{2n+2}) = d(Ax_{2n}, Bx_{2n+1}) \\
&\leqslant \max \left\{ \begin{array}{c} 2c_1 d(S(x_{2n}), T(x_{2n+1})), \\ c_2 [d(A(x_{2n}), T(x_{2n+1})) + d(B(x_{2n+1}), S(x_{2n}))], \\ c_3 [d(A(x_{2n}), S(x_{2n})) + d(B(x_{2n+1}), T(x_{2n+1}))] \end{array} \right\} \\
&\leqslant \max \left\{ \begin{array}{c} 2c_1 d(y_{2n-1}, y_{2n}), \\ c_2 [d(A(y_{2n}), y_{2n}) + d(y_{2n+1}, y_{2n-1})], \\ c_3 [d(y_{2n}, y_{2n-1}) + d(y_{2n+1}, y_{2n})] \end{array} \right\} \\
&\leqslant \max \{2cd(y_{2n-1}, y_{2n}), c[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]\}
\end{aligned}$$

where

$$c = \max\{c_1, c_2, c_3\} < \frac{1}{2}$$

Hence, ether

$$d(y_{2n}, y_{2n+1}) \leqslant 2cd(y_{2n-1}, y_{2n})$$

or

$$d(y_{2n}, y_{2n+1}) \leqslant \frac{c}{(1-c)} d(y_{2n-1}, y_{2n})$$

In either case, we have

$$d(y_{2n}, y_{2n+1}) \leqslant kd(y_{2n-1}, y_{2n})$$

Similarly,

$$d(y_{2n-1}, y_{2n}) \leqslant k^2 d(y_{2n-2}, y_{2n-1})$$

where $k = \max\left\{2c, \frac{c}{(1-c)}\right\} < 1$ and since, $0 \leqslant c < \frac{1}{2}$ we have $0 \leqslant k < 1$.

Therefore, for all $n \in \mathbb{N}$, we can write

$$d(y_{n+1}, y_{n+2}) \leqslant kd(y_n, y_{n+1}) \leqslant ... \leqslant k^{n+1} d(y_0, y_1)$$

Now, for any $m, n \in \mathbb{N}$, $m > n$, we have

$$\begin{aligned}
d(y_n, y_m) &\leqslant d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + ... + d(y_{m-1}, y_m) \\
&\leqslant k^n d(y_0, y_1) + k^{n+1} d(y_0, y_1) + ... + k^{m-1} d(y_0, y_1) \\
&= \frac{k^n}{(1-k)} d(y_0, y_1) \to 0 \text{ as } n, m \to \infty.
\end{aligned}$$

Thus,

$$d(y_n, y_m) \to 0 \text{ as } n \to \infty.$$

Hence $\{y_n\}$ is a Cauchy sequence in complete metric space X.

The following theorem extends and generalizes Theorem 1.2 of Fisher[10] for four compatible mappings.

Theorem 3.2. *Let (X,d) be a complete metric space and let A, B, S, $T : X \to X$ be self mappings of X into itself satisfying the following conditions*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and

(ii) $T(X)$ or $S(X)$ is a complete subspace of X.

(iii) the pairs $\{A,S\}$ and $\{B,T\}$ are weakly compatible.

(iv) A and B satisfy the inequality

$$d(A(x), B(y)) \leqslant \max \left\{ \begin{array}{c} 2c_1 d(S(x), T(y)), \\ c_2 [d(A(x), T(y)) + d(B(y), S(x))], \\ c_3 [d(A(x), S(x)) + d(B(y), T(y))] \end{array} \right\} \tag{3.1}$$

$\forall x, y \in X$ and $c_1, c_2, c_3 \geqslant 0$ are non negative real numbers such that

$$0 \leqslant c_1, c_2, \alpha_3 < \frac{1}{2} \tag{3.2}$$

then A, B, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X.In view of condition (i), we can inductively define a sequence $\{y_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, $n = 0, 1, 2, 3, ..$ First, we use condition (iv) to show that $\{y_n\}$ is a Cauchy sequence in X. On substituting $x = x_{2n}$ and $y = x_{2n+1}$ in inequality (3.1) gives us

$$d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1}) \leqslant$$
$$\max \left\{ \begin{array}{c} 2c_1 d(S(x_{2n}), T(x_{2n+1})), \\ c_2 [d(A(x_{2n}), T(x_{2n+1})) + d(B(x_{2n+1}), S(x_{2n}))], \\ c_3 [d(A(x_{2n}), S(x_{2n})) + d(B(x_{2n+1}), T(x_{2n+1}))] \end{array} \right\}$$

which is equivalent to

$$d(y_{2n}, y_{2n+1}) \leqslant \max \left\{ \begin{array}{c} 2c_1 d(y_{2n-1}, y_{2n}), \\ c_2 [d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})], \\ c_3 [d(y_{2n}, y_{2n-1}) + d(y_{2n+1}, y_{2n})] \end{array} \right\}$$
$$\leqslant \max \{2cd(y_{2n-1}, y_{2n}), c[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]\}$$

where

$$c = \max\{c_1, c_2, c_3\} < \frac{1}{2}$$

Hence, either

$$d(y_{2n}, y_{2n+1}) \leqslant 2cd(y_{2n-1}, y_{2n})$$

or

$$d(y_{2n}, y_{2n+1}) \leqslant \frac{c}{(1-c)} d(y_{2n-1}, y_{2n})$$

In either case, we have

$$d\left(y_{2n}, y_{2n+1}\right) \leqslant kd\left(y_{2n-1}, y_{2n}\right) \tag{3.3}$$

Similarly, we can write

$$d\left(y_{2n-1}, y_{2n}\right) \leqslant k^{2}d\left(y_{2n-2}, y_{2n-1}\right) \tag{3.4}$$

Where $k = \max\left\{2c, \frac{c}{(1-c)}\right\} < 1$ and since, $0 \leqslant c < \frac{1}{2}$ we have $0 \leqslant k < 1$.

Therefore, for all $n \in N$, we can write

$$d\left(y_{n+1}, y_{n+2}\right) \leqslant kd\left(y_{n}, y_{n+1}\right) \leqslant ... \leqslant k^{n+1}d\left(y_{0}, y_{1}\right) \tag{3.5}$$

So, for all $m > n$,we have by (3.5) and triangle inequality

$$\begin{aligned} d\left(y_{n}, y_{m}\right) &\leqslant d\left(y_{n}, y_{n-1}\right) + d\left(y_{n-1}, y_{n-2}\right) + ... + d\left(y_{m-1}, y_{m}\right) \\ &\leqslant k^{n}d\left(y_{0}, y_{1}\right) + k^{n+1}d\left(y_{0}, y_{1}\right) + ... + k^{m-1}d\left(y_{0}, y_{1}\right) \\ &= \frac{k^{n}}{(1-k)}d\left(y_{0}, y_{1}\right) \to 0 \text{ as } n, m \to \infty. \end{aligned}$$

It follows from Lemma 3.1 that the sequence $\{y_n\}$ is a Cauchy sequence and by the completeness of X, the sequence $\{y_n\}$ converges to some $y \in X$ such that

$$\lim_{n\to\infty} y_{n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\to\infty} Sx_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = y.$$

Now, we assume that $T\left(X\right)$ is a complete subspace of X. Then there exists $v \in X$ such that $Tv = y$.

If $Bv \neq y$ then by using (3.1), we have

$$\begin{aligned} &d\left(A\left(x_{2n}\right), B\left(v\right)\right) \leqslant \\ &\max\left\{\begin{array}{c} 2c_{1}d\left(S\left(x_{2n}\right), T\left(v\right)\right), c_{2}\left[d\left(A\left(x_{2n}\right), T\left(v\right)\right) + d\left(B\left(v\right), S\left(x_{2n}\right)\right)\right], \\ c_{3}\left[d\left(A\left(x_{2n}\right), S\left(x_{2n}\right)\right) + d\left(B\left(v\right), T\left(v\right)\right)\right] \end{array}\right\} \end{aligned}$$

As $n \to \infty$, we obtain

$$\begin{aligned} d\left(y, Bv\right) &\leqslant \max\left\{\begin{array}{c} 2cd\left(y, Tv\right), \\ c\left[d\left(y, Tv\right) + d\left(Bv, y\right)\right], \\ c\left[d\left(y, y\right) + d\left(Bv, Tv\right)\right] \end{array}\right\} \\ &\leqslant \max\left\{\begin{array}{c} 2cd\left(y, y\right), \\ c\left[d\left(y, y\right) + d\left(y, Bv\right)\right] \\ c\left[d\left(y, y\right) + d\left(Bv, y\right)\right], \end{array}\right\} \\ &= \max\left\{0,\ cd\left(y, Bv\right)\right\} \end{aligned}$$

Hence

$$\left(1-c\right)d\left(y, Bv\right) \leqslant 0$$

and the above inequalities is possible only if $d(y, Bv) = 0 \Rightarrow y = Bv = Tv$. In other words, v is a coincidence point of B and T. Since B and T are weakly compatible, they commute at a coincident point. Therefore,

$$BT(v) = TB(v) \text{ and so } By = Ty$$

If $y \neq By$, By using (3.1), we get

$$\lim_{n\to\infty} d(Ax_{2n}, By) \leqslant \max \lim_{n\to\infty} \left\{ \begin{array}{c} 2cd(Sx_{2n}, Ty), \\ c[d(Ax_{2n}, Ty) + d(By, Sx_{2n})], \\ c[d(Ax_{2n}, Sx_{2n}) + d(By, Ty)] \end{array} \right\}$$

Hence,

$$d(y, By) \leqslant \max \left\{ \begin{array}{c} 2cd(y, Ty), \\ c[d(y, Ty) + d(y, By)], \\ d[d(y, y) + d(By, Ty)] \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} 2cd(y, By), \\ 2cd(y, By), \\ 0 \end{array} \right\}$$

$$= 2cd(y, By)$$

and this implies that $(1-2c)\, d(y, By) \leqslant 0$ which is possible only if $d(y, By) = 0 \Rightarrow y = By$

Since $B(X) \subseteq S(X)$, there exists $u \in X$ such that $Su = y$.

If $Au \neq y$,by (3.1), we have

$$d(Au, By) \leqslant \max \left\{ \begin{array}{c} 2cd(Su, Ty), \\ c[d(Au, Ty) + d(By, Su)], \\ c[d(Au, Su) + d(By, Ty)] \end{array} \right\}$$

and this gives us

$$d(Au, y) \leqslant \max \left\{ \begin{array}{c} 2cd(Su, y), \\ c[d(Au, y) + d(y, Su)], \\ c[d(Au, Su) + d(y, y)] \end{array} \right\}$$

$$\begin{aligned} &= cd(Au, By) \\ \Rightarrow\ & (1-c)\, d(Au, By) \leqslant 0 \end{aligned}$$

and the inequality is possible only if $d(Au,y)=0 \Rightarrow Au=y$ and hence, $Au=Su=y$.Since, A and S are weakly compatible, $ASu=SAu$ and so $Ay=Sy$

$$d(Ay,y)=d(Ay,By)$$
$$\leqslant \max\left\{\begin{array}{c} 2cd(Sy,y), \\ c\left[d(Ay,y)+d(y,Sy)\right], \\ c\left[d(Ay,Sy)+d(y,y)\right] \end{array}\right\}$$

If $Ay \neq y$,again by (3.1) we have

$$=\max\left\{\begin{array}{c} 2cd(Ay,y), \\ 0 \end{array}\right\}$$
$$=2cd(Ay,y)$$

This implies that $(1-2c)\,d(Au,y)\leqslant 0$ which is posible only if $d(Au,y)=0$. Hence, $y=Au$. Thus, $Ay=By=Sy=Ty=y$ and so y is a common fixed point of A, B, S and T.

Uniqueness: To claim uniqueness of y, suppose there exists another common fixed point $y^* \in X$ of A, B,S and T such that $A(y^*)=B(y^*)=S(y^*)=T(y^*)=y^*$. Using condition (3.1), we have

$$d(y,y^*)=d(A(y),B(y^*))\leqslant \max\left\{\begin{array}{c} 2cd(S(y),T(y^*)), \\ c\left[d(A(y),T(y^*))+d(B(y^*),S(y))\right], \\ c\left[d(A(y),S(y))+d(B(y^*),T(y^*))\right] \end{array}\right\}$$

$$\leqslant \max\left\{\begin{array}{c} 2cd(y,y^*), \\ c\left[d(y,y^*)+d(y^*,y)\right], \\ c\left[d(y,y)+d(y^*,y^*)\right] \end{array}\right\}$$
$$=2cd(y,y^*)$$

which is possible only if $d(y,y^*)=0$, since $c<\frac{1}{2}$, it follows that $y=y^*$ which gives the uniqueness of the common fixed point y of A, B, S and T in X.

Example 3.3. Let $X=[0,1]$ with the metric $d(x,y)=|x-y|$ and define the self maps A,B,S and T on X by

$$A(x) \quad = \quad B(x)= \quad \left\{\tfrac{1}{3}\right\}$$

$$S(x)=\left\{\begin{array}{ll} \frac{1}{3}, 0\leqslant x<1 \\ 1, & x=1 \end{array}\right. \quad ,T(x)=\left\{\begin{array}{ll} \frac{1}{3}, 0\leqslant x<\frac{1}{3} \\ 1, & \frac{1}{3}<x\leqslant 1 \end{array}\right.$$

Then $A(X)=B(X)=\left\{\frac{1}{3}\right\}$ and $S(X)=T(X)=\left\{\frac{1}{3},1\right\}$. Now, we see that $A(X)=B(X)\subseteq S(X)=T(X)$ with $S(X)$ and $T(X)$ is complete subspace of X.

Also, we have $A(S(x)) = S(A(x))$. Similarly, $B(T(x)) = T(B(x))$ for.So, the pairs $\{A, S\}$ and $\{B, T\}$ commute at coincidence point and are compatible. Hence, these mappings satisfies the conditions of Theorem 3.2 with $\frac{1}{3}$ as the unique common fixed point of A, B, S and T. Theorem 3.2 yields the following Corollaries.

Corollary 3.4. *Let(X, d) be a complete metric space and let A, S, $T : X \to X$ be self mappings of X into itself satisfying the following conditions*

(i) $A(X) \subseteq T(X)$, $A(X) \subseteq S(X)$ and

(ii) either $S(X)$ or $T(X)$ is a complete subspace of X.

(iii) the pairs $\{A, S\}$ and $\{A, T\}$ are weakly compatible and satisfy the inequality

$$d(A(x), A(y)) \leqslant \max \left\{ \begin{array}{c} 2c_1 d(S(x), T(y)), \\ c_2 [d(A(x), T(y)) + d(A(y), S(x))], \\ c_3 [d(A(x), S(x)) + d(A(y), T(y))] \end{array} \right\}$$

$\forall x, y \in X$ and $c_1, c_2, c_3 \geqslant 0$ are non negative real numbers such that

$$0 \leqslant c_1, c_2, \alpha_3 < \frac{1}{2}$$

then A, S and T have a unique common fixed point in X.

Proof The proof follows from Theorem 3.2 by taking $B = A$.

Corollary 3.5. *Let(X, d) be a complete metric space and let A, $T : X \to X$ be commuting self maps of X into itself such that*

(i) $A(X) \subseteq T(X)$

(ii) $T(X)$ is a complete subspace of X

(iii) the pair$\{A, T\}$ is weakly compatible

(iv) A and T satisfy the inequality

$$d(A(x), A(y)) \leqslant \max \left\{ \begin{array}{c} 2c_1 d(T(x), T(y)), \\ c_2 [d(A(x), T(y)) + d(A(y), T(x))], \\ c_3 [d(A(x), T(x)) + d(A(y), T(y))] \end{array} \right\}$$

$\forall x, y \in X$ and $c_1, c_2, c_3 \geqslant 0$ are non negative real numbers such that

$$0 \leqslant c_1, c_2, \alpha_3 < \frac{1}{2}$$

then A and T have a unique common fixed point in X.

Proof. The proof follows from Theorem 3.2 by taking $B = A$ and $S = T$.

Corollary 3.6. If $T : X \to X$ is a mapping of the complete metric space (X, d) into itself satisfying the inequality

$$d(Tx, Ty) \leqslant \max \left\{ \begin{array}{c} 2c_1 d(x, y), \\ c_2 [d(x, Tx) + d(y, Ty)], \\ c_3 [d(x, Ty) + d(y, Tx)] \end{array} \right\}$$

,for all $x, y \in X$ and $0 \leqslant c_1, c_2, c_3 < \frac{1}{2}$, then T has a unique fixed point in X.

Proof The proof follows from Theorem 3.2 by taking $A = B = T$ and $S = I_X$ (Identity mapping).

Our next theorem is an extension of Theorem 1.3 in [10].

Theorem 3.7. *Let (X, d) be a complete metric space and let A, B, S and $T : X \to X$ be self maps of X into itself such that*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$.

(ii) S and T are continuous and

(iii) the pairs $\{A, S\}$ and $\{B, T\}$ are compatible on X and satisfy the inequality

$$[d(A(x), B(y))]^2 \leqslant \max \left\{ \begin{array}{c} 2c_1 d(S(x), T(y))[d(A(x), S(x)) + d(B(y), T(y))], \\ 2c_2 d(S(x), T(y))[d(A(x), T(y)) + d(B(y), S(x))], \\ c_3 [d(A(x), S(x)) + d(B(y), T(y))] \times \\ [d(A(x), T(y)) + d(B(y), S(x))] \end{array} \right\} \tag{3.6}$$

$\forall x, y \in X$ and $0 \leqslant c_1, c_2, c_3 < \frac{1}{4}$, then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. As $A(X) \subseteq T(X)$, we choose $x_1 \in X$ such that $A(x_0) = T(x_1)$. Since $Bx_1 \in S(X)$, we can choose $x_2 \in X$ such that $Bx_1 = Sx_2$. In general x_{2n+1} and x_{2n+2} so that we can define the Picard sequence $\{y_n\}$ in X, given by

$$\left\{ \begin{array}{c} y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and} \\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \forall n \geqslant 0 \end{array} \right. ..$$

Now, from (3.6), we have .

$$[d(y_{2n+1}, y_{2n+2})]^2 = [d(A(x_{2n}), B(x_{2n+1}))]^2 \leqslant$$

$$\max \left\{ \begin{array}{c} 2c_1 d(S(x_{2n}), T(x_{2n+1}))[d(A(x_{2n}), S(x_{2n})) + d(B(x_{2n+1}), T(x_{2n+1}))], \\ 2c_2 d(S(x_{2n}), T(x_{2n+1}))[d(A(x_{2n}), T(x_{2n+1})) + d(B(x_{2n+1}), S(x_{2n}))], \\ c_3 [d(A(x_{2n}), S(x_{2n})) + d(B(x_{2n+1}), T(x_{2n+1}))] \times \\ [d(A(x_{2n}), T(x_{2n+1})) + d(B(x_{2n+1}), S(x_{2n}))] \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} 2c_1 d(y_{2n-1}, y_{2n})[d(y_{2n}, y_{2n-1}) + d(y_{2n+1}, y_{2n})], \\ 2c_2 d(y_{2n-1}, y_{2n})[d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})], \\ c_3 [d(y_{2n}, y_{2n-1}) + d(y_{2n+1}, y_{2n})] \times \\ [d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})] \end{array} \right\}$$

$$\leqslant \max \left\{ \begin{array}{c} 2cd(y_{2n-1}, y_{2n})[d(y_{2n}, y_{2n-1}) + d(y_{2n+1}, y_{2n})], \\ c[d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})]^2 \end{array} \right\}$$

where $c = \max\{c_1, c_2, c_3\}$

Now, since X is a complete metric space, there exist a point $u \in X$ such that

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = u$$

We show that u is a common fixed point of A, B, S and T.

Now, since S is continuous, therefore

$$\lim_{n\to\infty} S^2x_{2n+2} = Su \text{ and } \lim_{n\to\infty} Ax_{2n} = u$$

Since the pair$\{A, S\}$ is compatible on X, so

$$\lim_{n\to\infty} (ASx_{2n}, SAx_{2n}) = 0$$

So, by Lemma 2.8, we have

$$\lim_{n\to\infty} ASx_{2n} = \ Su$$

Put $x = Sx_{2n}$ and $y = x_{2n+1}$ in (3.6) , we obtain

$$[d(A(Sx_{2n}), B(x_{2n+1}))]^2 \leqslant$$
$$\max\left\{\begin{array}{c} 2c_1 d(S(Sx_{2n}), T(x_{2n+1}))[d(A(Sx_{2n}), S(Sx_{2n})) + d(B(x_{2n+1}), T(x_{2n+1}))], \\ 2c_2 d(S(Sx_{2n}), T(x_{2n+1}))[d(A(Sx_{2n}), T(x_{2n+1})) + d(B(x_{2n+1}), S(Sx_{2n}))], \\ c_3[d(A(Sx_{2n}), S(Sx_{2n})) + d(B(x_{2n+1}), T(x_{2n+1}))] \times \\ [d(A(Sx_{2n}), T(x_{2n+1})) + d(B(x_{2n+1}), S(Sx_{2n}))] \end{array}\right\}$$

Taking limit as $n \to \infty$, we get

$$[d(Su, u)]^2 \leqslant$$
$$\max\left\{\begin{array}{c} 2cd(Su, u)[d(Su, Su) + d(u, u)], \\ 2cd(Su, u)[d(Su, u) + d(u, Su)], \\ c[d(Su, Su) + d(u, u)] \times \\ [d(Su, u) + d(u, Su)] \end{array}\right\}$$
$$\leqslant \max\left\{0, 4c[d(u, Su)]^2, 0\right\}$$

$$\Rightarrow [d(Su, u)]^2 \leqslant 4c[d(Su, u)]^2$$
$$\text{where } \ c = \max\{c_1, c_2, c_3\} < \frac{1}{4}$$

and the above inequality is possible only if $[d(Su, u)]^2 = 0 \Rightarrow d(Su, u) = 0 \Rightarrow Su = u$, since $0 \leqslant c < \frac{1}{4}$.

Next we will show that $Su = Tu = u$. Since, T is continuous, so using continuity of T, we have, $\lim_{n\to\infty} T(Tx_{2n+1}) = Tu$ and $\lim_{n\to\infty} TBx_{2n+1} = Tu$. Since B and T are compatible, $\lim_{n\to\infty} d(BTx_{2n}, TBx_{2n}) =$

0.By Lemma 2.8, we have $\lim_{n\to\infty} BTx_{2n} = Tu$.Now, by putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in (3.6), we obtain

$$[d(A(x_{2n}), B(Tx_{2n+1}))]^2 \leqslant$$
$$\max\left\{\begin{array}{c} 2c_1 d(S(x_{2n}), T(Tx_{2n+1}))[d(A(x_{2n}), S(x_{2n})) + d(B(Tx_{2n+1}), T(Tx_{2n+1}))], \\ 2c_2 d(S(x_{2n}), T(Tx_{2n+1}))[d(A(x_{2n}), T(Tx_{2n+1})) + d(B(Tx_{2n+1}), S(x_{2n}))], \\ c_3[d(A(x_{2n}), S(x_{2n})) + d(B(Tx_{2n+1}), T(Tx_{2n+1}))] \times \\ [d(A(x_{2n}), T(Tx_{2n+1})) + d(B(Tx_{2n+1}), S(x_{2n}))] \end{array}\right\}$$

Taking limit as $n \to \infty$, we have

$$[d(u, Tu)]^2 \leqslant \max\left\{\begin{array}{c} 2cd(u, Tu)[d(u, u) + d(Tu, Tu)], \\ 2cd(u, Tu)[d(u, Tu) + d(Tu, u)], \\ c[d(u, u) + d(Tu, Tu)] \times [d(u, Tu) + d(Tu, u)] \end{array}\right\}$$
$$\leqslant \max\left\{0, 4c[d(u, Tu)]^2, 0\right\}$$

which implies

$$[d(u, Tu)]^2 \leqslant 4c[d(u, Tu)]^2$$

which is contradiction, because $0 \leqslant c < \frac{1}{4} \Rightarrow [d(Tu, u)]^2 \leqslant 0$ and the inequality is possible only if $Tu = u$. Hence, $Tu = Su = u$.

Again, utilizing condition (3.6), we obtain

$$[d(A(u), B(x_{2n+1}))]^2 \leqslant$$
$$\max\left\{\begin{array}{c} 2c_1 d(S(u), T(x_{2n+1}))[d(A(u), S(u)) + d(B(x_{2n+1}), T(x_{2n+1}))], \\ 2c_2 d(S(u), T(x_{2n+1}))[d(A(u), T(x_{2n+1})) + d(B(x_{2n+1}), S(u))], \\ c_3[d(A(u), S(u)) + d(B(x_{2n+1}), T(x_{2n+1}))] \times \\ [d(A(u), T(x_{2n+1})) + d(B(x_{2n+1}), S(u))] \end{array}\right\}$$

Taking limit as $n \to \infty$ and $Su = Tu = u$, we have

$$[d(A(u), u)]^2 \leqslant \max\left\{\begin{array}{c} 2cd(S(u), u)[d(A(u), S(u)) + d(u, u)], \\ 2cd(S(u), u)[d(A(u), u) + d(u, S(u))], \\ c[d(A(u), S(u)) + d(u, u)] \times \\ [d(A(u), u) + d(u, S(u))] \end{array}\right\}$$
$$= \max\left\{0, 0, c[d(Au, u)]^2\right\}$$
$$= c[d(u, Au)]^2$$

$$\Rightarrow [d(A(u,u))]^2 \leqslant c[d(u,Au)]^2$$

which implies that $d(Au,u)=0 \Rightarrow Au=u$, since $0 \leqslant c < \frac{1}{4}$.

Finally, by using condition(3.6) and the fact that $Su=Tu=Au=u$, we have

$$[d(u,B(u))]^2 = [d(Au,Bu)]^2 \leqslant$$

$$\max\left\{\begin{array}{c} 2cd(S(u),T(u))[d(A(u),S(u))+d(B(u),T(u))], \\ 2cd(S(u),T(u))[d(A(u),T(u))+d(B(u),S(u))], \\ c[d(A(u),S(u))+d(B(u),T(u))]\times \\ [d(A(u),T(u))+d(B(u),S(u))] \end{array}\right\}$$

$$= \max\left\{0,0,c[d(u,Bu)]^2\right\}$$

$$= c[d(u,Bu)]^2$$

$$\Rightarrow [d(u,Bu)]^2 \leqslant c[d(u,Bu)]^2$$

and the above inequality is possible only, if $d(u,Bu)=0$ and which implies that $Bu=u$.Hence, $Au=Bu=Su=Tu=u$.

Uniqueness: For uniqueness, let $u \neq v$ is another common fixed point of the mappings A, B, S and T . We prove that $u=v$.

Putting $x=u$ and $y=v$ in (3.6), we obtain

$$\begin{aligned} [d(u,v)]^2 &= [d(A(u),B(v))]^2 \leqslant \\ &\max\left\{\begin{array}{c} 2cd(S(u),T(v))[d(A(u),S(u))+d(B(v),T(v))], \\ 2cd(S(u),T(v))[d(A(u),T(v))+d(B(v),S(u))], \\ c[d(A(u),S(u))+d(B(v),T(v))]\times \\ [d(A(u),T(v))+d(B(v),S(u))] \end{array}\right\} \end{aligned}$$

$$= \max\left\{\begin{array}{c} 2cd(u,v)[d(u,u)+d(v,v)], \\ 2cd(u,v)[d(u,v)+d(v,u)], \\ c[d(u,u)+d(v,v)]\times[d(u,v)+d(v,u)] \end{array}\right\}$$

$$= \max\left\{0,4c[d(u,v)]^2,0\right\}$$

$$= 4c[d(u,v)]^2$$

$$\Rightarrow [d(u,v)]^2 \leqslant 4c[d(u,v)]^2$$

or

$$(1-4c)[d(u,v)]^2 \leqslant 0$$

and the inequality is possible only if $[d(u,v)]^2=0$ and this implies $d(u,v)=0$ or $u=v$ which proves the uniqueness of the common fixed point of u of mappings A,B,S and T.

Example 3.8. Let $X = [0,1]$ with the metric $d(x,y) = |x-y|$. Define the self maps A, B, S and T on X as follows

$$A(x) = \begin{cases} \frac{x}{10}, 0 \leqslant x \leqslant 1 \\ 0, x = 1, \end{cases} \qquad B(x) = \begin{cases} \frac{x}{4}, 0 \leqslant x \leqslant \frac{1}{2} \\ 0, \frac{1}{2} < x \leqslant 1 \end{cases}$$

$$S(x) = x \text{ and } T(x) = \frac{x}{2}$$

Clearly, $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ for all $x, y \in X$. Furthermore, the pairs $\{A, S\}$ and $\{B, T\}$ are compatible. Therefore, A, B, S and T satisfy all conditions of Theorem 3.7 with $x = 0$ is the unique common fixed point in X.

Corollary 3.9. Let (X,d) be a complete metric space and let A, S and $T : X \to X$ be self maps of X into itself such that

(i) $A(X) \subseteq T(X)$, $A(X) \subseteq S(X)$.

(ii) S and T are continuous and

(iii) the pairs $\{A, S\}$ and $\{A, T\}$ are compatible on X and satisfy the inequality

$$[d(A(x), A(y))]^2$$
$$\max \left\{ \begin{array}{c} 2c_1 d(S(x), T(y)) [d(A(x), S(x)) + d(A(y), T(y))], \\ 2c_2 d(S(x), T(y)) [d(A(x), T(y)) + d(A(y), S(x))], \\ c_3 [d(A(x), S(x)) + d(A(y), T(y))] \times \\ [d(A(x), T(y)) + d(A(y), S(x))] \end{array} \right\}$$

$\forall x, y \in X$ and $0 \leqslant c_1, c_2, c_3 < \frac{1}{4}$, then A, S and T have a unique common fixed point in X.

Proof The proof follows from Theorem 3.7 by taking $B = A$.

Corollary 3.10. Let (X,d) be a complete metric space and let A, and $T : X \to X$ be self maps of X into itself such that

(i) $A(X) \subseteq T(X)$, $A(X) \subseteq T(X)$.

(ii) A or T is continuous and

(iii) the pairs $\{A, T\}$ is compatible on X and satisfy the inequality

$$[d(A(x), A(y))]^2 \leqslant$$
$$\max \left\{ \begin{array}{c} 2c_1 d(T(x), T(y)) [d(A(x), T(x)) + d(A(y), T(y))], \\ 2c_2 d(T(x), T(y)) [d(A(x), T(y)) + d(A(y), T(x))], \\ c_3 [d(A(x), T(x)) + d(A(y), T(y))] \times \\ [d(A(x), T(y)) + d(A(y), T(x))] \end{array} \right\}$$

$\forall x, y \in X$ and $0 \leqslant c_1, c_2, c_3 < \frac{1}{4}$, then A and T have a unique common fixed point in X.

Proof The proof follows from Theorem 3.7 by taking $B = A$ and $S = T$.

Corollary 3.11. Let (X, d) be a complete metric space and let $T : X \to X$ be self map of X into itself and satisfy the inequality

$$[d(T(x), T(y))]^2 \leqslant \max \left\{ \begin{array}{c} 2c_1 d((x), T(y)) [d(T(x), (x)) + d(T(y), T(y))], \\ 2c_2 d((x), T(y)) [d(T(x), T(y)) + d(T(y), (x))], \\ c_3 [d(T(x), (x)) + d(T(y), T(y))] \times \\ [d(T(x), T(y)) + d(T(y), (x))] \end{array} \right\}$$

$\forall x, y \in X$ and $0 \leqslant c_1, c_2, c_3 < \frac{1}{4}$, then T have a unique fixed point in X.

Proof. The proof follows from Theorem 3.7 by taking $A = B = T$ and $S = I_X$ (Identitymapping).

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