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Generalized Fixed point results in compact metric spaces

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Abstract The goal of this research study is to investigate some generalized fixed point results in compact metric space. In this paper our main focus is to prove some general fixed point theorems for the existence and uniqueness of a fixed point using self map in the setting of compact metric spaces.In this paper iterative techniques due to Edelstein are used to show existence of a unique fixed point for a self map satisfying generalized contractive conditions.Our results generalize the corresponding results of Edelstein, Fisher and Bailey.

Key Words Contractive mapping, Compact metric space, Fixed point
MSC 2010 54H25, 54H20

1 Introduction

The Banach contraction mapping theorem of 1922 popularly known as the Banach contraction mapping principle is a rewarding result in fixed point theory and analysis. Due to its widespread applications in both pure and applied mathematics it has attracted the attention of several great mathematicians for research in fixed point theory. According to Banach [1] a self map $T : X \to X$ on a complete metric space (X, d) is called a contraction mapping if for some $0 \leqslant k < 1$,it satisfies the inequality $d(Tx, Ty) \leqslant kd(x, y)$, then the mapping T has a unique fixed point in X, where he used to it to establish the existence of a solution for an integral equation.The study of fixed points mappings satisfying different contractive conditions have been actively investigated by several mathematicians. In 1961, Edelstein [9] introduced new concept of contractive mapping defined on compact metric space as a generalization of Banach contraction mapping which runs as follows. If T is a continuous mapping of a compact metric space X into itself satisfying the inequality $d(Tx, Ty) < d(x, y)$ for all $x, y \in X, x \neq y$,then T has a unique fixed point in X.Edelstein's contractive mapping theorem has been generalized and improved by several mathematicians in several different ways and obtained fixed points and fixed point results

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viz,Bailey [2], Ciric [8] ,Iseki [7],Kannan and Sharma [13],Fisher [14],Jungck [16], Sahu M.K.[17] and Soni G.K[18] extensively investigated fixed point results for continuous self maps on compact metric spaces and established interesting results. Recently, Bhatnagar et al [4] Bhardwaj et al [5] and Choudhary et al [6] also generalized Edelstein type contractive mappings on compact mectric spaces and obtained interesting results. Inspired by the ideas of Bailey[2], Edelstein[9], and Fisher[10] the aim of the present paper is to prove the existence of unique fixed point results for continuous self-maps in compact metric spaces satisfying new contractive type conditions.

2 Preliminaries

Definition 2.1 *Let (X, d) be a metric space.A self map $T : X \to X$ is called contractive if $d(Tx, Ty) < d(x, y)$ for all $x, y \in X, x \neq y$.*

Definition 2.2 *A class $\{G_\alpha\}$ of open subset of X is said to be an open cover of X,if each point in X belongs to one G_α such that $\cup_\alpha G_\alpha = X$. A subclass of an open cover which is at least an open cover is called a subcover.A compact space is a space in which every open cover has a finite subcover.*

Before starting the main results first we are giving some fundamental results.

Theorem 2.3. (Edelstein,1961) *Let (X, d) be a compact metric space and let $T : X \to X$ is a self map of X with itself satisfying*

$$d(Tx, Ty) < d(x, y) \tag{2.1}$$

$\forall x, y \in X, x \neq y$,then T has a unique fixed point in X.

Theorem 2.4.(Bailey,1966) *If T is a continuous mapping on a compact metric space X such that for every $x, y \in X, x \neq y$, there exists a positive integer $n = n(x, y)$ such that*

$$d\left(T^{n(x,y)}, T^{n(x,y)}\right) < d(x, y) \tag{2.2}$$

then T has a unique fixed point in X.

Theorem 2.5.(Fisher,1976) *If T is a continuous mapping of a compact metric space X in itself satisfying the inequality:*

$$d(Tx, Ty) < \frac{1}{2}\left[d(x, Tx) + d(y, Ty)\right] \tag{2.3}$$

$\forall x, y \in X, x \neq y$,then T has a unique fixed point.

Theorem 2.6.(Fisher,1976) *If T is a continuous mapping of a compact metric space X in itself satisfying the inequality:*

$$d(Tx, Ty) < \frac{1}{2}\left[d(x, Ty) + d(y, Tx)\right] \tag{2.4}$$

$\forall x, y \in X, x \neq y$,then T has a unique fixed point.

3 Main Results

Theorem 3.1. *If T is a continuous mapping of a compact metric space X into itself satisfying the inequality:*

$$d\left(T\left(x\right),T\left(y\right)\right)<\alpha\left[\tfrac{d(x,T(x)).d(y,T(y))}{d(x,y)}\right]+$$
$$\gamma\left[d\left(y,T\left(x\right)\right)+d\left(y,T\left(y\right)\right)\right]+\xi\left[d\left(x,T\left(x\right)\right)+d\left(y,T\left(y\right)\right)\right]$$
$$+\eta\left[d\left(x,T\left(y\right)\right)+d\left(y,T\left(x\right)\right)\right]+\delta d\left(x,y\right)$$

for all $x,y\in X,x\neq y$,where $\alpha,\gamma,\xi,\eta,\delta$ are nonnegative real numbers such that $\alpha+\gamma+2\left(\zeta+\eta\right)+\delta<1$.then T has unique fixed point.

Proof. First we define a function F on X, such that

$$F\left(x\right)=d\left(x,Tx\right),\text{for all } x\in X,$$

since d and T are continuous on X,therefore F is also continuous on X.From compactness of X,there exists a point $a\in X$ such that

$$F\left(a\right)=\inf\left\{F\left(x\right):x\in X\right\} \tag{3.1}$$

If $F\left(a\right)\neq 0$,it follows that $Ta\neq a$,and then

$$F\left(Ta\right)=d\left(T\left(a\right),T\left(T\left(a\right)\right)\right)$$

$$d\left(T\left(a\right),T\left(T\left(a\right)\right)\right)<\alpha\left[\tfrac{d(a,T(a)).d(T(a),T(T(a)))}{d(a,T(a))}\right]$$
$$+\gamma\left[d\left(T\left(a\right),T\left(a\right)\right)+d\left(T\left(a\right),T\left(T\left(a\right)\right)\right)\right]$$
$$+\xi\left[d\left(a,T\left(a\right)\right)+d\left(T\left(a\right),T\left(T\left(a\right)\right)\right)\right]$$
$$+\eta\left[d\left(T\left(a\right),T\left(T\left(a\right)\right)\right)+d\left(T\left(a\right),T\left(a\right)\right)\right]$$
$$+\delta d\left(a,T\left(a\right)\right)$$

$$d\left(T\left(a\right),T\left(T\left(a\right)\right)\right)<\left(\alpha+\gamma+\xi+\eta\right)d\left(T\left(a\right),T\left(T\left(a\right)\right)\right)+\left(\xi+\eta+\delta\right)d\left(a,T\left(a\right)\right)$$
$$\left[1-\left(\alpha+\gamma+\xi+\eta\right)\right]d\left(T\left(a\right),T\left(T\left(a\right)\right)\right)<\left(\xi+\eta+\delta\right)d\left(a,T\left(a\right)\right)$$
$$d\left(T\left(a\right),T\left(T\left(a\right)\right)\right)<\frac{\left(\xi+\eta+\delta\right)}{\left[1-\left(\alpha+\gamma+\xi+\eta\right)\right]}d\left(a,T\left(a\right)\right)$$
$$d\left(T\left(a\right),T\left(T\left(a\right)\right)\right)<d\left(a,T\left(a\right)\right)$$

because $\alpha+\gamma+2\left(\xi+\eta\right)+\delta<1$,that is

$$F\left(T\left(a\right)\right)<F\left(a\right)$$

this contradicts the definition of a and condition (3.1) so $a=T\left(a\right)$therefore a is fixed point of T

Uniqueness. If possible suppose $a\neq b$ is another fixed point of T Then

$$d\left(a,b\right)=d\left(T\left(a\right),T\left(b\right)\right)$$

$$d(a,b) < \alpha\left[\frac{.d(a,T(a)).d(b,T(b))}{d(a,b)}\right]$$
$$+\gamma\left[d(b,T(a))+d(b,T(b))\right]+\xi\left[d(a,T(a))+d(b,T(b))\right]$$
$$+\eta\left[d(a,T(a))+d(b,T(b))\right]+\delta d(a,b)$$
$$d(a,b) < \gamma d(a,b)+\delta d(a,b)$$
$$\Rightarrow d(a,b) < 0$$

This gives us a contradiction, because $\alpha+\gamma+2(\xi+\eta)+\delta<1$.It follows that a is a unique fixed point of T.

Theorem 3.2. *Let T be a continuous mapping of a compact metric space X into itself satisfying the inequality:*

$$d(Tx,Ty) < \beta\left[\frac{d(x,T(x)).d(y,T(y))}{d(x,y)}\right]+\gamma\left[d(y,T(x))+d(y,T(y))\right]$$
$$+\xi\left[d(x,T(x))+d(y,T(y))\right]$$
$$+\eta\left[d(x,T(y))+d(y,T(x))\right]+\delta d(x,y)$$

for all $x,y\in X, x\neq y$,where $\beta,\gamma,\xi,\eta,\delta$ are non negative real numbers such that $\beta+\gamma+2(\xi+\eta)+\delta<1$.Then T has a unique fixed point.

Proof. We apply the proof as in theorem (3.1) ,utilizing the condition in (3.2) we get

$$d(T(a),T(T(a))) < \frac{(\xi+\eta+\delta)}{[1-(\beta+\gamma+\xi+\eta)]}d(a,T(a))$$
$$d(T(a),T(T(a))) < d(a,T(a))$$

because

$$\beta+\gamma+2(\xi+\eta)+\delta<1$$

Thus T has unique fixed point.

Theorem 3.3. *Let T be a continuous mapping of a compact metric space X into itself satisfying the inequality:*

$$d(Tx,Ty) < \alpha\left[\frac{d(x,T(x)).d(y,T(y))}{d(x,y)}\right]+\beta\left[\frac{.d(x,T(x))d(y,T(y))}{d(x,y)}\right]$$
$$+\gamma\left[d(y,T(x))+d(y,T(y))\right]+\xi\left[d(x,T(x))+d(y,T(y))\right]$$
$$+\eta\left[d(x,T(y))+d(y,T(x))\right]+\delta d(x,y)$$

for all $\alpha,\beta,\gamma,\xi,\eta,\delta$ are non negative real numbers such that $\alpha+\beta+\gamma+2(\xi+\eta)+\delta<1$.

Proof.. Utilizing the method as in (3.1) and (3.2),we can find that

$$d(T(a),T(T(a))) < \frac{(\xi+\eta+\delta)}{[1-(\alpha+\beta+\gamma+\xi+\eta)]}d(a,T(a))$$
$$d(T(a),T(T(a))) < d(a,T(a))$$

because

$$\alpha+\beta+\gamma+2(\xi+\eta)+\delta<1$$

thus T has a unique fixed point in X.

Remarks.

1.If we put $\alpha=\beta=\gamma=\xi=\eta=0, \delta=1$,we get result of Edelstein [9]

2.If we put $\beta=0$,we get theorem 3.1

3.If we put $\alpha=0$, we get theorem 3.2

4.putting $\alpha=\gamma=\xi=\eta=0, \delta=1$,in theorem4.1we get result of Edelstein[9]

5.putting $\beta=\gamma=\xi=\eta=0, \delta=1$,in theorem4.2we get result of Edelstein[9]

6.If we put $\alpha=\beta=\gamma=\eta=\delta=0, \xi=\frac{1}{2}$ in theorem 3.3,we get theorem 2.5

7.If we put $\alpha=\beta=\gamma=\xi=\delta=0, \eta=\frac{1}{2}$in theorem 3.3, we get theorem 2.6

Corollary 3.4 *If T is a continuous mapping of a compact metric space X into itself satisfying the inequality:*

$$d(T(x), T(y)) < \alpha\left[\frac{d(x,T(x)).d(y,T(y))}{d(x,y)}\right]$$
$$+\gamma\left[d(y,T(x))+d(y,T(y))\right]+\eta\left[d(x,T(y))+d(y,T(x))\right]$$
$$+\delta d(x,y)$$

for all $x, y \in X, x \neq y$,where $\alpha, \gamma, \eta, \delta$ are nonnegative real numbers such that $\alpha+\gamma+2\eta+\delta<1$. then T has a unique fixed point.

Corollary 3.5. *If T is a continuous mapping of a compact metric space X into itself satisfying the inequality:*

$$d(T(x), T(y)) < \alpha\left[\frac{d(x,T(x)).d(y,T(y))}{d(x,y)}\right]$$
$$+\gamma\left[d(y,T(x))+d(y,T(y))\right]+\xi\left[d(x,T(x))+d(y,T(y))\right]$$
$$+\delta d(x,y)$$

for all $x, y \in X, x \neq y$,where $\alpha, \gamma, \xi, \delta$ are nonnegative real numbers such that $\alpha+\gamma+2\zeta+\delta<1$.then T has a unique fixed point.

Corollar 3.6. *If T is a continuous mapping of a compact metric space X into itself satisfying the inequality:*

$$d(T(x), T(y)) < \alpha\left[\frac{d(x,T(x)).d(y,T(y))}{d(x,y)}\right]$$
$$+\gamma[d(y,T(x))+d(y,T(y))]+\delta d(x,y)$$

for all $x, y \in X, x \neq y$,where α, γ, δ are nonnegative real numbers such that $\alpha+\gamma+\delta<1$.then T has a unique fixed point.

Corollary 3.7. *If T is a continuous mapping of a compact metric space X into itself satisfying the inequality:*

$$d(T(x), T(y)) < \alpha\left[\frac{d(x,T(x)).d(y,T(y))}{d(x,y)}\right] + \delta d(x,y)$$

for all $x, y \in X, x \neq y$,where α, δ are nonnegative real numbers such that $\alpha+\delta<1$.then T has a unique fixed point.

Example 3.8. *Let* $X = \{0, 1, 3\}$ *with the usual metric* $d : X \times X \to R^+$ *and define* T *on* X *by*

$$T0 = T3 = 1, T1 = 1$$

Then it is easy to see that example (3.8) satisfy Theorems$(3.1), (3.2)$ and (3.3) along with the hypothesis of Corollary $(3.4) - (3.7)$

Theorem 3.9. *If* T *is a continuous mapping as in Theorem* (4.1) *with* $0 \leqslant \alpha + \gamma + 2(\xi + \eta) + \delta < 1$ *and satisfy the inequality given below*

$$d(T(x), T(y)) < \alpha \left[\frac{d(x,T(x)).d(y,T(y))}{d(x,y)}\right] + \gamma \left[\frac{d(x,T(x))+d(y,T(x))}{1+d(x,T(x))d(y,T(x))}\right] + \xi \left[\frac{d(x,y)+d(x,T(x))+d(T(x),T(y))}{1+d(x,y)d(x,T(x))d(T(x),T(y))}\right]$$
$$+ \eta \left[\frac{d(x,y)+d(x,T(x))d(y,T(x))+d(T(x),T(y))}{1+d(x,y)d(x,T(x))d(y,T(x))d(T(x),T(y))}\right] + \delta d(x, y)$$

then for every $a \in X$,*the sequence* $\{T^n a\}$ *of iterates converges to the unique fixed point of* T.

Proof By Theorem(3.1),T has a unique fixed point a_0 (say) in X.Now for each $n = 0, 1, 2, ...$,define

$$d_n = d(T^n a, a_0)$$

for every $a \in X, a \neq a_0$.We consider the following two cases:

Case 1. If $d_n = 0$ for some n,then $T^m a = a_0$ for some $m \geqslant n$ and hence the sequence $\{T^n a\}$ converges to a_0.

Case 2. If $d_n \neq 0$ for each n,then

$$d_{n+1} = d\left(T^{n+1}a, a_0\right) = d\left(T^{n+1}a, T^{n+1}a_0\right) < \alpha \left[\frac{d\left(T^n(a_0),T^{n+1}(a_0)\right).d\left(T^n(a),T^{n+1}(a)\right)}{d(T^n(a_0),T^n(a))}\right]$$
$$+\gamma \left[\frac{d\left(I^n(a_0),T^{n+1}(a_0)\right)+d\left(T^n(a),T^{n+1}(a_0)\right)}{1+d(T^n(a_0),T^{n+1}(a_0))d(T^n(a),T^{n+1}(a_0))}\right] + \xi \left[\frac{d(T^n(a_0),T^n(a))+d\left(T^n(a_0),T^{n+1}(a_0)\right)+d\left(T^{n+1}(a_0),T^{n+1}(a)\right)}{1+d(T^n(a_0),T^n(a))d(T^n(a_0),T^{n+1}(a_0))d(T^{n+1}(a_0),T^{n+1}(a))}\right]$$
$$+\eta \left[\frac{d(T^n(a_0),T^n(a))+d\left(T^n(a_0),T^{n+1}(a_0)\right)d\left(T^n(a),T^{n+1}(a_0)\right)+d\left(T^{n+1}(a_0),T^{n+1}(a)\right)}{1+d(T^n(a_0),T^n(a))d(T^n(a_0),T^{n+1}(a_0))d(T^n(a),T^{n+1}(a_0))d(T^{n+1}(a_0),T^{n+1}(a))}\right] + \delta d(T^n(a_0), T^n(a))$$

$$\leqslant \alpha(0) + \gamma d_n + \xi(d_n + d_{n+1}) + \eta(d_n + d_{n+1}) + \delta d_n$$

$$\Rightarrow d_{n+1} < (\xi + \eta) d_{n+1} + (\gamma + \xi + \eta + \delta) d_n$$

$$\begin{aligned} &\Rightarrow \quad [1 - (\xi + \eta)] d_{n+1} < (\gamma + \xi + \eta + \delta) d_n \qquad (3.1) \\ &\Rightarrow \quad d_{n+1} < \frac{(\gamma + \xi + \eta + \delta)}{[1 - (\xi + \eta)]} d_n \\ &\Rightarrow \quad d_{n+1} < d_n \end{aligned}$$

because

$$0 \leqslant \alpha + \gamma + 2(\xi + \eta) + \delta < 1$$

Hence $\{d_n\}$ is a non increasing sequence of positive real numbers and hence converges to a real $r \geqslant 0$,which is the greatest lower bound of the sequence $\{d_n\}$.By compactness of X,the sequence $\{T^n(a)\}$ has a convergent subsequence $\{T^{n_k}(a)\}$ which converges to $z \in X$ (say).Since T is continuous as, $k \to \infty$

$$T^{n_{k+1}}(a) = T(T^{n_k}(a)) \to Tz$$

By the continuity of the metric d,letting $k \to \infty$,

$$d_{n_k} = d\left(T^{n_k}(a), a_0\right) \to d\left(Tz, a_0\right) = r$$

where the sequence $\{d_{n_k}\}$ is a subsequence of $\{d_n\}$ .Also by the continuity of the metric d,as $k \to \infty$,

$$d_{n_{k+1}} = d\left(T^{n_{k+1}}(a), a_0\right) \to d\left(z, a_0\right) = r$$

since the sequence $\left\{d_{n_{k+1}}\right\}$ is a subsequence of $\{d_n\}$ .so,

$$r = d\left(z, a_0\right) = d\left(Tz, a_0\right) \tag{3.3}$$

Now,we claim $r = 0$.Suppose $r \neq 0$.Then $z \neq a_0$.By (3.2),we get

$$d\left(Tz, a_0\right) = d\left(Tz, Ta_0\right) < d\left(z, a_0\right)$$

which contradicts (3.3) .Hence, $z = a_0$,which means $r = d\left(z, a_0\right) = 0$. This shows $\{d_n\} \to 0$ as $n \to \infty$, from which the conclusion of the theorem follows.

Theorem 3.10. *Let T be a self map of a compact metric space X into itself such that for some $n \geqslant 1, T^n$ is continuous and satisfies the conditions*

$$\begin{gathered} d\left(T^n(x), T^n(y)\right) < \alpha\left[\tfrac{d(x,T^n(x))d(y,T^n(y))}{d(x,y)}\right] + \\ \gamma\left[d\left(y, T^n(x)\right) + d\left(y, T^n(y)\right)\right] + \xi\left[d\left(x, T^n(x)\right) + d\left(y, T^n(y)\right)\right] + \\ \eta\left[d\left(x, T^n(y)\right) + d\left(y, T^n(x)\right)\right] + \delta d(x, y) \end{gathered}$$

for all $x, y \in X, x \neq y$ and $\alpha, \gamma, \xi, \eta, \delta$ are non negative real numbers such that $\alpha + \gamma + 2(\xi + \eta) + \delta < 1$.Then T has a unique fixed point in X.

Proof. Define $F : X \to [0, \infty)$ by

$$F(x) = d\left(x, T^n(x)\right)$$

for every $x \in X$.Suppose $x \neq T^n x$,then

$$\begin{gathered} F\left(T^n(x)\right) = d\left(T^n(x), T^n\left(T^n(x)\right)\right) < \alpha\left[\tfrac{d(x,T^n(x))d(T^n(x),T^n(T^n(x)))}{d(x,T^n(x))}\right] + \\ \gamma\left[d\left(T^n(x), T^n(x)\right) + d\left(T^n(x), T^n\left(T^n(x)\right)\right)\right] + \\ \xi\left[d\left(x, T^n(x)\right) + d\left(T^n(x), T^n\left(T^n(x)\right)\right)\right] + \\ \eta\left[d\left(x, T^n\left(T^n(x)\right)\right) + d\left(T^n(x), T^n(x)\right)\right] + \delta d\left(x, T^n(x)\right) \end{gathered}$$

$$\begin{aligned} & d\left(T^n(x), T^n\left(T^n(x)\right)\right) \\ < \quad & (\alpha + \gamma + \xi + \eta)\, d\left(T^n(x), T^n\left(T^n(x)\right)\right) + (\xi + \eta + \delta)\, d\left(x, T^n(x)\right) \end{aligned}$$

$$[1 - (\alpha + \gamma + \xi + \eta)]\, d\left(T^n(x), T^n\left(T^n(x)\right)\right) < (\xi + \eta + \delta)\, d\left(x, T^n(x)\right)$$

$$d\left(T^n(x), T^n\left(T^n(x)\right)\right) < \frac{(\xi + \eta + \delta)}{[1 - (\alpha + \gamma + \xi + \eta)]} d\left(x, T^n(x)\right)$$

$$\Rightarrow d\left(T^n(x), T^n\left(T^n(x)\right)\right) < d\left(x, T^n(x)\right)$$

because

$$\alpha + \gamma + 2(\xi + \eta) + \delta < 1$$

so

$$F\left(T^n(x)\right) < F(x), x \neq T^n(x) \tag{3.4}$$

since T^n is continuous,F is also continuous on the compact metric space X,hence it attains its minimum on X at the point (say) $a_{0.}$ Suppose $F(a_0) = d\left(a_0, T^n(a_0)\right) > 0$.Then by (3.4) we obtain

$$F\left(T^n(a_0)\right) < F(a_0)$$

which contradicts minimality of the value of F at a_0. Hence, our supposition $F(a_0) > 0$ is false.Therefore,

$$F(a_0) = d\left(a_0, T^n(a_0)\right) = 0$$

So that a_0 is a fixed point of T^n.

Uniqueness of the fixed point of T^n: Suppose if possible $b \neq a_0$ is another fixed point of T^n.Then

$$\begin{aligned} d(a_0, b) &= d\left(T^n(a_0), T^n(b)\right) \\ &< \alpha\left[\frac{d\left(a_0, T^n(a_0)\right).d\left(b, T^n(b)\right)}{d(a_0, b)}\right] + \\ &\quad \gamma\left[d\left(b, T^n(a_0)\right) + d\left(b, T^n(b)\right)\right] + \xi\left[d\left(a_0, T^n(a_0)\right) + d\left(b, T^n(a_0)\right)\right] + \\ &\quad \eta\left[d\left(a_0, T^n(b)\right) + d\left(b, T^n(a_0)\right)\right] + \delta d(a_0, b) \end{aligned}$$

$$\Rightarrow d(a_0, b) < \gamma d(a_0, b) + \xi d(a_0, b) + 2\eta d(a_0, b) + \delta d(a_0, b)$$

which is a contradiction,because $\alpha + \gamma + 2(\xi + \eta) + \delta < 1$.

Now,let a_0 is a fixed point of T^n and since $T^n(T(a_0)) = T\left(T^n(a_0)\right)$,then

$$T^n(T(a_0)) = T\left(T^n(a_0)\right) = T(a_0) = a_0$$

which shows that any fixed point of T is also a fixed point of T^n.

Remark: Putting $n = 1$ in Theorem 3.10, we obtain Theorem 3.1

Example 3.11. Let $X = \{1, 3, 5, 7\}$ and define $d : X \times X \to [0, +\infty)$ by $d(x, y) = |x - y| \, \forall x, y \in X$.If $T : X \to X$ is a mapping given by

$$T1 = T5 = T7 = 3, T3 = 3$$

Then T satisfy the conditions of both Theorem (3.9) and Theorem (3.10) with 3 as the unique fixed point of T in X.

Theorem 3.12. *If T is a continuous mapping of a compact metric space X into itself satisfying the inequality*

$$d(Tx,Ty) < \max\left\{\begin{array}{c} d(x,y), d(x,T(x)), d(y,T(y)), \\ \frac{1}{2}[d(x,T(x))+d(y,T(y))], \\ \frac{1}{2}[d(x,T(y))+d(y,T(x))] \end{array}\right\}$$

for all $x, y \in X, x \neq y$. Then T has a unique fixed point in X.

Proof. Define a real-valued function F on X by

$$F(x) = d(x,T(x))$$

for all $x \in X$. Since d and T are continuous on X, it follows that F is also continuous on X.Since X is compact, there exists a point $u \in X$ such that

$$F(u) = \inf\{F(x) : x \in X\} \tag{3.5}$$

Assume that $T(u) \neq u$,then we have

$$\begin{aligned} F(T(u)) &= d(T(u), T(T(u))) \\ &< \max\left\{\begin{array}{c} d(u,T(u)), d(u,T(u)), d(T(u),T(T(u))), \\ \frac{1}{2}[d(u,T(u))+d(T(u),T(T(u)))], \frac{1}{2}[d(u,T(T(u)))+0] \end{array}\right\} \\ &< \max\left\{d(u,T(u)), \frac{1}{2}[d(u,T(u))+d(T(u),T(T(u)))]\right\} \\ &= \max\left\{F(u), \frac{1}{2}[F(u)+F(T(u))]\right\} \end{aligned}$$

It follows that either $F(T(u)) < F(u)$ or $F(T(u)) < \frac{1}{2}[F(u)+F(T(u))]$,which gives us a contradiction in both the cases. Hence our assumption was false and so we must have $T(u) = u$.Thus u is a fixed point of T.

Uniqueness: To prove uniqueness. Suppose that T has a second fixed point u' distinct from u.Then, we have

$$\begin{aligned} d(u,u') &= d(T(u),T(u')) \\ &< \max\left\{\begin{array}{c} d(u,u'), d(u,T(u)), d(u',T(u')), \\ \frac{1}{2}[d(u,T(u))+d(u',T(u'))], \\ \frac{1}{2}[d(u,T(u'))+d(u',T(u))] \end{array}\right\} \\ &= d(u,u') \end{aligned}$$

which gives us a contradiction. It follows that our assumption was false and so the fixed point must be unique.

Example 3.13. *Let $X = \{5,6,7\}$ with $d : X \times X \to [0,+\infty)$ defined by $d(x,y) = 0, d(x,y) = d(y,x)$ for all $x, y \in X$ .Clearly,d is a compact metric space on X such that*

$$d(5,6) = d(6,7) = 1, d(5,7) = 2$$

If $T : X \to X$ is a mapping given by $T5 = T7 = 6, T6 = 6$. Then the mapping T satisfy the conditions of Theorem(3.12) with 6 is the only fixed point of T.

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