South Asian Journal of Mathematics 2018, Vol. 8 (1): 39 ∼ 47
www.sajm-online.com ISSN 2251-1512 RESEARCH ARTICLE

Noether symmetries and conserved quantities of constrained Hamilton systems with quasi coordinates

Mingliang Zheng[①*]

① School of mechanical engineering, Zhejiang Sci-Tech University, Hangzhou, Zhengjiang, 310018, China
E-mail: zhmlwxcstu@163.com

Received: Dec-5-2017; Accepted: Feb-20-2018 *Corresponding author
This work was supported by the China National Natural Science Fund (11472247).

Abstract The constraint mechanical systems with quasi coordinates are more universal than generalized coordinates, in this paper, we study the Noether symmetries and conserved quantities of non-conservative singular systems in phase space. Firstly, the internal constraints induced by singularity are equivalent considered as extrinsic nonholonomic constraints, the canonical equations of constrained Hamilton systems with quasi coordinates are obtained by using transform to the Euler-Lagrange equations. Secondly, the infinitesimal transformations of time, quasi coordinates and generalized momentum are introduced, the definition, criterion and Noether theorem are obtained according to the regular action quantity keep generalized quasi invariance under the transformation, meanwhile, the inverse problem of the Noehter symmetry is also studied. Finally, an example is given to illustrate the application of the content. The results found that the rational use of quasi coordinates will make the constraints caused by the singularity of the system do not affect the standard form of the regular equations and avoid the emergence of constrained multipliers, the conservation is more concise.

Key Words Heisenberg group, Quasi coordinates, constrained Hamilton systems, Noether symmetries
MSC 2010 53B30, 53B20

1 Introduction

Study on about the symmetries and conserved quantities of dynamical systems are a main branch of intersection in modern mathematics and physics mechanics. The systems described by singular Lagrange function are called singular systems and expressed in Hamilton form in phase space, it is also called constrained Hamilton systems [1].Many important dynamic problems of mathematical physics and engineering technology are in accord with the model of constraint Hamilton systems, such as electromagnetic field light traversing phenomena quantum electrodynamics superstring theory etc.. In recent years,

Citation: M. Zheng, Noether symmetries and conserved quantities of constrained Hamilton systems with quasi coordinates, South Asian J Math, 2018, 8(1), 39-47.

the research on constrained Hamilton systems's symmetries and the conserved quantities are paid more and more attention, and have made some progress. Dirac [2]and Li [3, 4, 5]firstly studied the Hamilton canonical equations for singular systems with the Noether symmetry and conserved quantity and many physical applications. Mei [6]and Zhang [7]respectively study the Lie symmetries and conserved quantities of singular systems in the form space and phase space. Luo [8] study Mei symmetries and conserved quantities of Hamilton canonical equations of Singular systems, the relation between Mei symmetry and Noether symmetry and Lie symmetry are also illustrated. Li [9] also studied the theory of regular symmetry of constrained Hamilton systems under external constraints and compared it with the nonsingular systems. The present study to the singular systems' theory is all based on the generalized coordinates, however in analytical mechanics, the use of quasi coordinates is more universal, if the quasi coordinates is rational, some problems will become great convenient. In document [10], the Noether symmetries and Noether conserved quantities of holonomic systems with quasi coordinates are discussed. In document [11], the Lie symmetries and Hojman type conserved quantities of holonomic systems with quasi coordinates are given. In document [12], they study the form invariance of holonomic systems with quasi coordinates, that is Mei symmetries and conserved quantities. In document [13], the Lie symmetries and conserved quantities of nonholonomic nonsingular mechanical systems with quasi coordinates are studied. The research literatures about the quasi coordinates have [14, 15].

Through literature discovery, the current research of constraint mechanics systems with quasi coordinates are all in the form space, that is, the equations of motion of the systems are in the form of Lagrange instead of Hamilton. As everyone knows, Hamilton systems have a set of perfect mechanical characteristics (such as integral theory and canonical transformation theory), therefore, the symmetries and conserved quantities of the singular systems with quasi coordinates in phase space are of great significance for the constrained dynamics and engineering science. According to this idea the equations of motion expressed by quasi coordinates and quasi speed have completely single structure, and it do not depend on constraints or not . By introducing infinite transformations of time, quasi coordinates and generalized momenta, the Noehter symmetry theory (including positive and inverse problems) with quasi coordinates of singular systems in phase space are firstly obtained.

2 Canonical equations of constrained Hamilton systems with quasi coordinates

The form of mechanical systems is determined by n generalized coordinates $q_s(s = 1, \cdots, n)$ and the Lagrange function of the systems is $L(t, q, \dot{q}) = T - V$, the nonpotential generalized force is $Q_s(t, q, \dot{q})$, according to Euler-Lagrange principle, the differential equations of motion for nonconservative systems is:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_S} - \frac{\partial L}{\partial q_S} = Q_s \ , (s = 1, 2, \cdots, n) \tag{1}$$

For constrained Hamilton systems, since the rank of L's Hess matrix $[\frac{\partial^2 L}{\partial \dot{q}_S \partial \dot{q}_k}]$is $n - r < n$, after the (1) expansion, it is impossible to completely solve all generalized accelerations $\ddot{q}_S$, and only a part generalized acceleration can be solved, so exists some relations that the number is r:

$$f_\beta(t,q,\dot{q}) = 0 \ ,(\beta = 1,\cdots,r) \tag{2}$$

Take quasi velocities:

$$\omega_k = \omega_k(q_s,\dot{q}_s,t) = \alpha_{ks}\dot{q}_s \ ,(s,k = 1,2,\cdots,n) \tag{3}$$

It can be solved by formula (3):

$$\dot{q}_s = \dot{q}_s(q_k,\omega_k,t) = b_{sk}\omega_k \tag{4}$$

Theis inverse matrix elements of the matrix $[\alpha_{ks}]$, now introducing quasi coordinates π_s:

$$\omega_s = \dot{\pi}_s \tag{5}$$

The internal constraints (2) can be shown as the form with quasi coordinates:

$$\omega_{\varepsilon+\beta} = 0, \delta\pi_{\varepsilon+\beta} = 0 \ ,(\varepsilon = n-r) \tag{6}$$

The relation between quasi coordinates and generalized coordinates are [16]:

$$\delta q_s = \frac{\partial \dot{q}_s}{\partial \omega_k}\delta\pi_k, \delta\pi_k = \frac{\partial \omega_k}{\partial \dot{q}_s}\delta q_s \tag{7}$$

The system's Lagrange function L^* and nonconservative force Q_S^* are represented by quasi velocities:

$$\begin{aligned} &L^* = L^*(t,q_s,\omega_s) = L(t,q_s,b_{sk},\omega_k) \\ &Q_S^* = Q^*(t,q_s,\omega_s) = Q_s b_{ks} \qquad ,(s,k = 1,2,\cdots,n) \end{aligned} \tag{8}$$

Introducing the generalized momentums

$$p_s = \frac{\partial L^*}{\partial \omega_s} \Rightarrow \omega_s = d_{sk}p_k + e_s \tag{9}$$

According to the inverse theorem of Legendre transform, The Hamilotn function $H(t,q,p)$ with the quasi coordinates is:

$$H(t,q,p) = (p_s\omega_s - L^*)|_{p_s\to\omega_s} \tag{10}$$

The generalized Boltzmann-Hamel equations for constrained Hamilton systems with quasi coordinates are [17]:

$$\frac{d}{dt}\frac{\partial L^*}{\partial \omega_\sigma} - \frac{\partial L^*}{\partial \omega_\sigma} + \sum_{r=1}^{n}\sum_{v=1}^{\varepsilon}\frac{\partial L^*}{\partial \omega_r}\gamma_{v\sigma}^r w_v = Q_\sigma^* \ (\sigma = 1,2,\cdots,\varepsilon) \tag{11}$$

$\gamma_{v\sigma}^r$ is three mark for Boltzmann:

$$\gamma_{v\sigma}^r = \sum_{m=1}^{n}\sum_{t=1}^{n}(\frac{\partial \alpha_{rt}}{\partial q_m} - \frac{\partial \alpha_{rm}}{\partial q_t})b_{mv}b_{t\sigma} \tag{12}$$

Form (10) has relations toπ_l, p_l:

$$\begin{aligned} &\dot{\pi}_l = \frac{\partial H}{\partial p_l} \\ &\frac{\partial H}{\partial \pi_l} = -\frac{\partial L^*}{\partial \pi_l} \end{aligned} \tag{13}$$

By (11) and (13) , we obtain the canonical equations of Hamilton systems in phase space with quasi Coordinates are:

$$\begin{cases} \dot{\pi}_l = -\dfrac{\partial H}{\partial p_l} \\ \dot{p}_l = -\dfrac{\partial H}{\partial \pi_l} - \displaystyle\sum_{r=1}^{n}\sum_{v=1}^{\varepsilon} \gamma_{vl}^{r} p_r \dot{\pi}_v + Q_l^* \end{cases} \quad (l = 1, 2, \cdots, n-r) \tag{14}$$

The form is completely similar to the holonomic nonconservative nonsingular systems, but there are more internal constraint equations (6). It shows that the singularity of the systems does not affect the single structure of the canonical equation with quasi coordinates, but it reduces the number of differential equations.

3 Noether symmetries and conserved quantities of the systems

The regular action of the constrained Hamilotn systems with quasi coordinates in phase space is:

$$S = \int_{t_1}^{t_2} [p_s\omega_s - H(t, p, q)]dt \tag{15}$$

The infinitesimal transformation of a continuous Lie group involving a single parameter is:

$$\begin{cases} t^* = t + \Delta t = t + \varepsilon\xi_0(t, q, p) \\ \pi_s^*(t^*) = \pi_s(t) + \Delta\pi_s = \pi_s(t) + \varepsilon\xi_s(t, q, p) \\ p_s^*(t^*) = p_s(t) + \Delta p_s = p_s(t) + \varepsilon\eta_s(t, q, p) \end{cases} \tag{16}$$

Under the infinitesimal transformations (16), the total variation of the (15) is:

$$\begin{aligned} \Delta S = \delta S + \dot{S}\Delta(t) &= \textstyle\int_{t_1}^{t_2} (p_s\delta\omega_s + \omega_s\delta p_s - \dfrac{\partial H}{\partial \pi_s}\delta\pi_s - \dfrac{\partial H}{\partial p_s}\delta p_s)dt + (p_s\omega_s - H)\Delta t \\ &= \textstyle\int_{t_1}^{t_2} \{\dfrac{d}{dt}[p_s\delta\pi_s + (p_s\omega_s - H)\Delta t] + (-\dot{p}_s - \dfrac{\partial H}{\partial \pi_s})\delta\pi_s + (\omega_s - \dfrac{\partial H}{\partial p_s})\delta p_s\}dt \end{aligned} \tag{17}$$

In formula (17), $\dfrac{\partial H}{\partial \pi_s}\delta\pi_s = \dfrac{\partial H}{\partial q_s}\delta q_s$, and it also because:

$$\begin{aligned} \delta\pi_s &= \Delta\pi_s - \omega_s\Delta t = \varepsilon(\xi_s - \omega_s\xi_0) = \varepsilon\overline{\xi}_s \\ \delta p_s &= \Delta p_s - \dot{p}_s\Delta t = \varepsilon(\eta_s - \dot{p}_s\xi_0) = \varepsilon\overline{\eta}_s \end{aligned} \tag{18}$$

So formula (17) may be changed as:

$$\begin{aligned} \Delta S &= \textstyle\int_{t_1}^{t_2} \varepsilon\{\dfrac{d}{dt}[p_s\overline{\xi}_s + (p_s\omega_s - H)\xi_0] + (-\dot{p}_s - \dfrac{\partial H}{\partial \pi_s})\overline{\xi}_s + (\omega_s - \dfrac{\partial H}{\partial p_s})\overline{\eta}_s\}dt \\ &= \textstyle\int_{t_1}^{t_2} \varepsilon\{\dfrac{d}{dt}[p_s\overline{\xi}_s + (p_s\omega_s - H)\xi_0] + (-\dot{p}_s - \dfrac{\partial H}{\partial \pi_s} - \gamma_{vs}^{r} p_r \dfrac{\partial H}{\partial p_v})\overline{\xi}_s \\ &\quad + (\omega_s - \dfrac{\partial H}{\partial p_s})\overline{\eta}_s + \gamma_{vs}^{r} p_r \dfrac{\partial H}{\partial p_v}\overline{\xi}_s\}dt \\ &\qquad (s, v, r = 1, 2, \cdots, n) \end{aligned} \tag{19}$$

By using the internal constraint conditions (6), the (19) can be transformed into:

$$\begin{aligned} \Delta S &= \int_{t_1}^{t_2} \varepsilon\{\frac{d}{dt}[p_s\overline{\xi}_s + (p_\sigma\omega_\sigma - H)\xi_0] + (-\dot{p}_\sigma - \frac{\partial H}{\partial \pi_\sigma})\overline{\xi}_\sigma + (\omega_\sigma - \frac{\partial H}{\partial p_\sigma})\overline{\eta}_\sigma\}dt \\ &= \int_{t_1}^{t_2} \varepsilon\{\frac{d}{dt}[p_s\overline{\xi}_s + (p_\sigma\omega_\sigma - H)\overline{\xi}_0] + (-\dot{p}_\sigma - \frac{\partial H}{\partial \pi_\sigma} - \gamma_{v\sigma}^r p_r\frac{\partial H}{\partial p_v})\overline{\xi}_\sigma \\ &\quad +(\omega_\sigma - \frac{\partial H}{\partial p_\sigma})\overline{\eta}_\sigma + \gamma_{v\sigma}^r p_r\frac{\partial H}{\partial p_v}\overline{\xi}_\sigma\}dt \\ &\qquad (\sigma = 1, 2, \cdots, \varepsilon; s, r, v = 1, 2, \cdots, n) \end{aligned} \tag{20}$$

If the regular action quantity (15) of systems is the invariant under the infinitesimal transformation (16) in phase space, that is, for each transformation, it is always established

$$\Delta S = -\int_{t_1}^{t_2}\{\frac{d}{dt}(\Delta G + Q_s^*\delta\pi_s)\}dt \tag{21}$$

So the infinitesimal transformation (16) is a generalized quasi Noether symmetric transformation for constrained Hamilton systems.

Be careful, the singularity of constrained Hamilton systems with quasi coordinates need to be considered, so the variational principle of mechanics should be changed into a constrained variational principle with additional conditions, thus, the generalized quasi Noether symmetries need to be satisfied for the equations:

$$\begin{cases} p_\sigma\dot{\xi}_\sigma - \dfrac{\partial H}{\partial t}\xi_0 - \dfrac{\partial H}{\partial \pi_s}\xi_s - H\dot{\xi}_0 + (Q_\sigma^* - \gamma_{v\sigma}^r p_r\dfrac{\partial H}{\partial p_v})(\xi_\sigma - \omega_\sigma\xi_0) + \dot{G} = 0 \\ \xi_{\varepsilon+\beta} - \omega_{\varepsilon+\beta}\xi_0 = 0 \end{cases} \tag{22}$$

Theorem it is assumed that the infinitesimal transformation (16) of a given Lie group is the generalized quasi Noether symmetric transformation (21) for constrained Hamilton systems in phase space, so, the systems have the first integral that is linear independence:

$$I = p_s\xi_s - H\xi_0 + G = L^*\xi_0 + \frac{\partial L^*}{\partial \omega_\sigma}(\xi_\sigma - \omega_\sigma\xi_0) + G = const \tag{23}$$

The proof of the above theorem is very simple, it is only necessary to substitute the canonical equation (14) into the generalized quasi Noether symmetries formula(21) (20), and simplify by using (18) is all right.

4 The inverse problem of Noether symmetry

The inverse problem of Noether symmetry for constrained Hamilton systems is known the first integral (conserved quantity of motion) of holonomic systems in phase space, the generating element of infinitesimal transformations and gauge function are obtained by using the invariance condition of the regular action quantity and the Noether theorem [18]. The conserved quantities of singular systems generally do not necessarily have corresponding Noether symmetries, therefore, the inverse problem of Noether symmetry for onstrained Hamilton systems may be solvable, and may not be solvable.

Suppose the systems have first integral

$$I = I(t, q, p) = const \tag{24}$$

The second equation in the canonical equations (14) multiply $\overline{\xi}_\sigma = \xi_\sigma - \omega_\sigma \xi_0$, it can obtained:

$$(-\dot{p}_\sigma - \frac{\partial H}{\partial \pi_\sigma} - \gamma^r_{v\sigma} p_r \frac{\partial H}{\partial p_v} + Q^*_\sigma)(\xi_\sigma - \frac{\partial H}{\partial p_\sigma}\xi_0) = 0 \tag{25}$$

Because of the rule of total derivatives:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial \pi_s}\omega_s + \frac{\partial I}{\partial p_\sigma}\dot{p}_\sigma = 0 \tag{26}$$

Make formula (25) and formula (26) add up, and make the coefficient of the item $\dot{p}_\sigma$ is equal to zero:

$$\xi_\sigma = \frac{\partial H}{\partial p_\sigma}\xi_0 + \frac{\partial I}{\partial p_\sigma} \tag{27}$$

By formula (23) and (27), the all generators of infinitesimal transform ξ_0, ξ_scan be obtained, the generator η_s can be given by the following formula:

$$\eta_s = \frac{\partial p_s}{\partial t}\xi_0 + \frac{\partial p_s}{\partial \pi_k}\xi_s + \frac{\partial p_s}{\partial \omega_s}(\xi_s - \omega_s \xi_0) \tag{28}$$

The form(28) is easy to prove , because:

$$\Delta p_s = \varepsilon \eta_s = \frac{\partial p_s}{\partial t}\Delta t + \frac{\partial p_s}{\partial \pi_k}\Delta \pi_s + \frac{\partial p_s}{\partial \omega_s}\Delta \omega_s \tag{29}$$

Combined with formula (16), it can be proved.

5 Example

The Lagrange function and the nonpotential generalized force of the holonomic mechanical systems with two degrees of freedom are established:

$$\begin{aligned} L &= \frac{1}{2}\dot{q}_1^2 + q_1 \dot{q}_2 \\ Q_1 &= -\frac{q_1}{\dot{q}_1}, Q_2 = q_1 \dot{q}_2 \end{aligned} \tag{30}$$

Try to investigate the Noether symmetry and inverse problem of the systems with quasi coordinates.

It is obvious that the L's Hess matrix rank is $1 < 2$, so it is constrained Hamilton systems, then the Euler-Lagrange equation and the internal constraint equation of the systems are:

$$\begin{aligned} &\ddot{q}_1 - \dot{q}_2 + \frac{q_1}{\dot{q}_1} = 0 \\ &\dot{q}_1 = q_1 \dot{q}_2 \end{aligned} \tag{31}$$

Take the quasi velocities are $\omega_1 = \dot{q}_1, \omega_2 = \dot{q}_1 - q_1 \dot{q}_2$, it can be solved by reverse:$\dot{q}_1 = \omega_1, \dot{q}_2 = (\omega_1 - \omega_2)\frac{1}{q_1}$, then the internal constraint becomes:

$$\omega_2 = 0 \tag{32}$$

After calculating the systems's three mark of Boltzmann, it can obtain $r_{11}^1 = r_{11}^2 = 0$, then the canonical equations of the systems are:

$$\begin{cases} \dot{\pi}_1 = p_1 - 1 \\ \dot{\pi}_2 = 0 \\ \dot{p}_1 = \dfrac{\dot{\pi}_1}{\pi_1} - \dfrac{\pi_1}{\dot{\pi}_1} \\ \dot{p}_2 = 0 \end{cases} \tag{33}$$

According to the generalized quasi Noether equation (22), the system's infinitesimal generator ξ_0, ξ_s satisfies the following equations:

$$\begin{cases} p_1\dot{\xi}_1 - \dfrac{(\omega_2 - \omega_1)}{q_1}\xi_1 - H\dot{\xi}_0 - \dfrac{q_1}{\omega_1}(\xi_1 - \omega_1\xi_0) + \dot{G} = 0 \\ \xi_2 - \omega_2\xi_0 = 0 \\ \omega_2 = 0 \end{cases} \tag{34}$$

The corresponding generating elements of infinitesimal transformations and gauge function are preferable to:

$$\begin{aligned} &\xi_0 = 0, \xi_1 = 1, \xi_2 = 0, G = -p_1 \\ &\xi_0 = 1, \xi_1 = \xi_2 = 0, G = -\textstyle\int q_1 \end{aligned} \tag{35}$$

It can obtained by formula (28):

$$\eta_1 = \eta_2 = 0 \tag{36}$$

Take formula (35) and (36) into the system's Noether conserved quantities formula (23), it can obtained:

$$\begin{aligned} &I_1 = const \\ &I_2 = -p_1\omega_1 - \frac{1}{2}\omega_1^2 - \omega_1 - \int q_1 \end{aligned} \tag{37}$$

It can be seen that the first set of Noether symmetry generators leads to a ordinary conserved quantity, and the second set of generators leads to the conservation of the energy of nonconservative systems.

Secondly, we study the inverse problem of Noether symmetry. Suppose the systems have first integral

$$I = -p_1\omega_1 - \frac{1}{2}\omega_1^2 - \omega_1 - \int q_1 \tag{38}$$

Given by equations (27), (6) and (28):

$$\begin{aligned} &\xi_0 = \omega_1\xi_0 - \omega_1 \\ &\xi_2 - \omega_2\xi_0 = 0 \\ &\eta_1 = (\dot{\xi}_1 - \omega_1\dot{\xi}_0)\eta_2 = 0 \end{aligned} \tag{39}$$

To (39), a set of solutions is preferred:

$$\xi_0 = 1, \xi_1 = \xi_2 = 0, \eta_1 = \eta_2 = 0 \tag{40}$$

Then, according to the conserved quantities of the Noether symmetry (23), the gauge function is:

$$G = -\int q_1 \tag{41}$$

An example of Noether symmetry for constrained Hamilton systems with quasi coordinates in phase space is given.

6 Conclusion

In summary, the Noether symmetry theory of nonconservative constrained Hamilton systems in phase space with quasi coordinates is studied in this paper. By constructing quasi velocities, the canonical equations of dimension reduction are obtained by transforming the Euler-Lagrange equations, the existence of bound multipliers are avoided, the equations preserves the single structure form of the canonical equations in the complete with nonsingular mechanical systems. Based on the generalized quasi invariance of regular function under infinitesimal transformations of regular coordinates, the generating elements of constrained Hamilton systems satisfying the generalized quasi Noether equation are obtained by variational computation. The corresponding conserved quantity of Noether type is also found, at the same time, the inverse problem of Noether symmetry of constrained Hamilton systems with quasi coordinates in phase space is studied. With quasi coordinates, the form of the constrained Hamilton systems is more concise and convenient and the conservation quantity has multiple expression forms. However, this article does not consider the existence of external constraints, when external nonholonomic constraints and singularities exist at the same time, how to construct quasi coordinates is a big problem, this is a question that can be further studied.

References

1 Xing, S.Y., Deng, F.Q. (2017). Delay-dependent dissipative filtering for nonlinear stochastic singular systems with time-varying delays[J]. SCIENCE CHINA-INFORMATION SCIENCES, 60(12):2-8.

2 Dirac, P. A. M. (1950). Quantization of Singular systems[J]. Can J Math, 2,122-129.

3 Li, Z. P. (1992). Generalized Noether theorem of regular form of nonholonomic singular systems and its inverse theorem[J]. HuangHuai Journal, 3(1),8-16. (in chinese)

4 Li, Z. P. (1991). Noether theorem of regular form and its application[J]. Science Bulletin, 36(12),954- 958. (in chinese)

5 Li, Z. P. (1999). Constrained Hamiltonian systems and their symmetry properties[M]. Beijing: Beijing University of Technology press, 6-14. (in chinese)

6 Mei, F. X., Zhu, H. P. (2000). Lie symmetries and conserved quantities of singular Lagrange systems[J]. Journal of Beijing Institute of Technology, 9(1),11-14.

7 Zhang, Y., Xue, Y. (2001). Lie symmetry constraint Hamilton systems with the second type of constraints[J]. Acta phys Sinica, 50(5), 816-819.

8 Luo, S. K. (2004). Mei symmetries, Noether symmetries and Lie symmetries of Hamilton canonical equations of singular systems[J]. Acta phys Sinica, 53(1),5-11.

9 Li, A. M., Jiang, Z. H. (2013). Canonical symmetry of nonholonomic constraint systems[J]. Journal of Wuhan Technical College of Communication, 15(3),1-3. (in chinese)

10 Mei, F. X. (1985). Fundamentals of nonholonomic systems mechanics[M]. Beijing: Beijing Institute of Technology Press, 12-14. (in chinese)

11 Qiao, Y. F., Zhao, S. H. (2001). Lie symmetric theorem and inverse theorem of generalized mechanical systems in quasi coordinates[J]. Acta phys Sinica, 50(1),1-7.

12 Xu, Z. X., Mei, F. X. (2005). Unified symmetry of general holonomic systems under quasi coordinates[J]. Acta phys Sinica, 54(12),5521-5524.

13 Fu, J. L., Liu, R. W. (2000). Lie symmetries and conserved quantities of nonholonomic mechanical systems in quasi coordinates[J]. Journal of Mathematical Physics, 20(1),63-69.

14 Mahmoudkhani, S. (2017). Dynamics of a mass-spring-beam with 0:1:1 internal resonance using the analytical and continuation method[J]. INTERNATIONAL JOURNAL OF NON-LINEAR MECHANICS, 97:48-67.

15 Jahromi, A. F., Bhat, R. B., Xie, W. F. (2015). Integrated ride and handling vehicle model using Lagrangian quasi-coordinates[J]. INTERNATIONAL JOURNAL OF AUTOMOTIVE TECHNOLOGY, 16(2):239-251.

16 Fasso, F., Sansonetto, N. (2016). Conservation of 'Moving' Energy in Nonholonomic Systems with Affine Constraints and Integrability of Spheres on Rotating Surfaces[J], JOURNAL OF NONLINEAR SCIENCE, 26(2):519-544

17 Wang, X. J., Zhao, X. L., Fu, J. L. (2004). Noether symmetry of nonholonomic systems in quasi coordinates in phase space[J]. Journal of Henan Institute of Education, 13(2),21 -23. (in chinese)

18 Mei, F. X. (2013). Analytical mechanics[M](2) . Beijing: Beijing Institute of Technology press, 145- 162. (in chinese)