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Power series and hypergeometric representations with positive integral powers of arctangent function

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Abstract In this paper, we obtain power series representations and corresponding hypergeometric forms of positive integral powers of arctangent functions, associated with Gauss, Kampé de Fériet's general double, Srivastava's general triple, Saigo's general quadruple and Srivastava-Daoust multi-variable hypergeometric functions.

Key Words Kampé de Fériet's function; Srivastava's general triple hypergeometric function

MSC 2010 33C20, 33B99

1 Introduction

For definitions of Pochhammer symbol, generalized hypergeometric function ${}_pF_q$, Kampé de Fériet's general double hypergeometric function $F^{\ell:m;n}_{r:s;t}$, Srivastava's general triple hypergeometric function $F^{(3)}$, Srivastava-Daoust multi-variable hypergeometric function $F^{A\,:\,B^{(1)}\,;\ldots;\,B^{(n)}}_{C\,:\,D^{(1)}\,;\ldots;\,D^{(n)}}$, we refer monumental work of Srivastava and Manocha [3] and other notation have their usual meanings.

$$\left(\frac{\tan^{-1} z}{z}\right) = {}_2F_1\left[\begin{array}{c} 1, \frac{1}{2}; \\ \qquad -z^2 \\ \frac{3}{2}; \end{array}\right] \quad ; \quad |z| < 1. \tag{1}$$

$$\tan^{-1}(iz) = i\tanh^{-1}(z). \tag{2}$$

$$\frac{1}{2}\ell n\left[\frac{1+z}{1-z}\right] = \tanh^{-1} z = z\;{}_2F_1\left[\begin{array}{c} 1, \frac{1}{2}; \\ \qquad z^2 \\ \frac{3}{2}; \end{array}\right] \quad ; \quad |z| < 1. \tag{3}$$

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Saigo's general quadruple hypergeometric function $F_M^{(4)}$

In 1988, M. Saigo defined a more general quadruple hypergeometric function [1, p.15, Equation 17]; [2, pp.455–456, Equation 16] $F_M^{(4)}$ (slightly modified notation) in the following form:

$$F_M^{(4)}\left[\begin{array}{l}(a_A):::(b_B)\,;(d_D)\,;(e_E)\,;(g_G)::(h_H)\,;(m_M)\,;(n_N)\,;(p_P);(q_Q)\,;(r_R):\\ (a'_{A'}):::(b'_{B'});(d'_{D'});(e'_{E'});(g'_{G'})::(h'_{H'});(m'_{M'});(n'_{N'});(p'_{P'});(q'_{Q'});(r'_{R'}):\end{array}\right.$$

$$\left.\begin{array}{l}(s_S)\,;(t_T)\,;(u_U)\,;(w_W)\,;\\ (s'_{S'});(t'_{T'});(u'_{U'});(w'_{W'});\end{array}\; x,\,y,\,z,\,c\right]$$

$$= \sum_{i,j,k,\ell=0}^{\infty} \frac{[(a_A)]_{i+j+k+\ell}[(b_B)]_{i+j+k}[(d_D)]_{j+k+\ell}[(e_E)]_{k+\ell+i}[(g_G)]_{\ell+i+j}[(h_H)]_{i+j}}{[(a'_{A'})]_{i+j+k+\ell}[(b'_{B'})]_{i+j+k}[(d'_{D'})]_{j+k+\ell}[(e'_{E'})]_{k+\ell+i}[(g'_{G'})]_{\ell+i+j}[(h'_{H'})]_{i+j}}\times$$

$$\times\frac{[(m_M)]_{i+k}[(n_N)]_{i+\ell}[(p_P)]_{j+k}[(q_Q)]_{j+\ell}[(r_R)]_{k+\ell}[(s_S)]_i[(t_T)]_j[(u_U)]_k[(w_W)]_\ell\; x^i\; y^j\; z^k\; c^\ell}{[(m'_{M'})]_{i+k}[(n'_{N'})]_{i+\ell}[(p'_{P'})]_{j+k}[(q'_{Q'})]_{j+\ell}[(r'_{R'})]_{k+\ell}[(s'_{S'})]_i[(t'_{T'})]_j[(u'_{U'})]_k[(w'_{W'})]_\ell\; i!\; j!\; k!\; \ell!}, \quad (4)$$

provided that the above multiple series is absolutely convergent.

Multiple Series Identities [3, pp.100–103 and p.161]

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\mathbf{\Phi}(m,n) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\mathbf{\Phi}(m+n,n) = \sum_{m,n=0}^{\infty}\mathbf{\Phi}(m+n,n), \quad (5)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{p=0}^{n}\mathbf{\Phi}(m,n,p) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\mathbf{\Phi}(m+n+p,n+p,p) = \sum_{m,n.p=0}^{\infty}\mathbf{\Phi}(m+n+p,n+p,p), \quad (6)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\sum_{p=0}^{n}\sum_{q=0}^{p}\mathbf{\Phi}(m,n,p,q) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{p=0}^{\infty}\mathbf{\Phi}(m+n+p+q,n+p+q,p+q,q)$$

$$= \sum_{m,n,p,q=0}^{\infty}\mathbf{\Phi}(m+n+p+q,n+p+q,p+q,q), \quad (7)$$

$$\sum_{P_1=0}^{\infty}\sum_{P_2=0}^{P_1}\sum_{P_3=0}^{P_2}\cdots\sum_{P_m=0}^{P_{m-1}}\mathbf{\Phi}(P_1,P_2,P_3,\ldots,P_m)$$

$$= \sum_{P_1=0}^{\infty}\sum_{P_2=0}^{\infty}\cdots\sum_{P_m=0}^{\infty}\mathbf{\Phi}(P_1+P_2+\cdots+P_m,P_2+P_3+\cdots+P_m,\ldots,P_m)$$

$$= \sum_{P_1,P_2,P_3,\ldots,P_m=0}^{\infty}\mathbf{\Phi}(P_1+P_2+\cdots+P_m,P_2+P_3+\cdots+P_m,\ldots,P_m), \quad (8)$$

provided that the above multiple series is absolutely convergent.

2 Power Series Representations of Positive Integral Powers of $\tan^{-1}(z)$

In this section we obtain power series representations of some positive integral powers of $\tan^{-1}(z)$ when $|z|<1$,

$$\frac{(\tan^{-1}z)^2}{2!}=\sum_{k=0}^{\infty}\left[\left\{\sum_{n=0}^{k}\frac{1}{(2n+1)}\right\}\frac{(-1)^k z^{2k+2}}{(2k+2)}\right], \tag{9}$$

$$\frac{(\tan^{-1}z)^3}{3!}=\sum_{k=0}^{\infty}\left[\left\{\sum_{n=0}^{k}\frac{1}{(2n+2)}\sum_{m=0}^{n}\frac{1}{(2m+1)}\right\}\frac{(-1)^k z^{2k+3}}{(2k+3)}\right], \tag{10}$$

$$\frac{(\tan^{-1}z)^4}{4!}=\sum_{k=0}^{\infty}\left[\left\{\sum_{n=0}^{k}\frac{1}{(2n+3)}\sum_{m=0}^{n}\frac{1}{(2m+2)}\sum_{\ell=0}^{m}\frac{1}{(2\ell+1)}\right\}\frac{(-1)^k z^{2k+4}}{(2k+4)}\right] \tag{11}$$

and

$$\frac{(\tan^{-1}z)^m}{m!}=\sum_{n_1=0}^{\infty}\left[\prod_{j=2}^{m}\left\{\sum_{n_j=0}^{n_{j-1}}\left(\frac{1}{1+m-j+2n_j}\right)\right\}\frac{(-1)^{n_1}z^{m+2n_1}}{2n_1+m}\right]. \tag{12}$$

By replacing z by iz in the above power series forms we can obtain power series forms of positive integral powers of $(\tanh^{-1}z)^m$ for $m=2,3,4,\ldots$.

3 Hypergeometric Forms of Positive Integral Powers of $\tan^{-1}(z)$

When $|z|<1$, then hypergeometric forms of some positive integral powers of $\tan^{-1}(z)$ are given by

$$\left(\frac{\tan^{-1}z}{z}\right)^2 = F^{1:1;2}_{1:0;1}\left[\begin{array}{ll} 1:1;\frac{1}{2},1; & -z^2,-z^2 \\ 2:-;\ \frac{3}{2}\ ; & \end{array}\right],$$

$$= F^{1:1;2}_{1:0;1}\left(\begin{array}{ll} [1:1,1]:[1:1];[\frac{1}{2},1],[1:1]; & \\ & -z^2,-z^2 \\ [2:1,1]:\ \text{---}\ ;\quad [\frac{3}{2}:1]\ ; & \end{array}\right), \tag{13}$$

$$\left(\frac{\tan^{-1}z}{z}\right)^3 = F^{(3)}\left[\begin{array}{ll} \frac{3}{2}::-;1;-:1;1;\frac{1}{2},1; & \\ & -z^2,-z^2,-z^2 \\ \frac{5}{2}::-;2;-:\text{---};-;\frac{3}{2}\ ; & \end{array}\right]$$

$$= F^{2:1;1;2}_{2:0;0;1}\left(\begin{array}{ll} [\frac{3}{2}:1,1,1],[1:0,1,1]:[1:1];[1:1];[\frac{1}{2}:1],[1:1]; & \\ & -z^2,-z^2,-z^2 \\ [\frac{5}{2}:1,1,1],[2:0,1,1]:\ -\ ;\ -\ ;\quad [\frac{3}{2}:1]\ ; & \end{array}\right), \tag{14}$$

$$\left(\frac{\tan^{-1} z}{z}\right)^4 = F_M^{(4)}\left[\begin{array}{l} 2 ::: -; \tfrac{3}{2}; -; - :: -; -; -; -; - ; 1 : 1; 1; 1; \tfrac{1}{2}, 1; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -z^2, -z^2, -z^2, -z^2 \\ 3 ::: -; \tfrac{5}{2}; -; - :: -; -; -; -; - ; 2 : -; -; -; \tfrac{3}{2}; \end{array}\right]$$

$$= F^{3:1;1;1;2}_{3:0;0;0;1}\left(\begin{array}{l} [2 : 1, 1, 1, 1], [\tfrac{3}{2} : 0, 1, 1, 1], [1 : 0, 0, 1, 1] : [1 : 1]; [1 : 1]; [1 : 1]; [1 : 1], [\tfrac{1}{2} : 1]; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -z^2, -z^2, -z^2, -z^2 \\ [3 : 1, 1, 1, 1], [\tfrac{5}{2} : 0, 1, 1, 1], [2 : 0, 0, 1, 1] : \text{——} ; \text{——} ; \text{——} ; \quad [\tfrac{3}{2} : 1] \quad ; \end{array}\right), \tag{15}$$

$$\left(\frac{\tan^{-1} z}{z}\right)^m =$$

$$= F^{m-1:\overbrace{1;1;\ldots;1;1}^{m-1};2}_{m-1:\underbrace{0;0;\ldots;0;0}_{m-1};1}\left(\begin{array}{l} [\tfrac{m}{2} : \overbrace{1, 1, \ldots, 1, 1}^{m}], [\tfrac{m-1}{2} : 0, \overbrace{1, 1, \ldots, 1, 1}^{m-1}], [\tfrac{m-3}{2} : 0, 0, \overbrace{1, 1, \ldots, 1, 1}^{m-2}], [\tfrac{m-4}{2} : 0, 0, 0, \overbrace{1, 1, \ldots, 1, 1}^{m-3}], \\ [\tfrac{m+2}{2} : \underbrace{1, 1, \ldots, 1, 1}_{m}], [\tfrac{m+1}{2} : 0, \underbrace{1, 1, \ldots, 1, 1}_{m-1}], [\tfrac{m}{2} : 0, 0, \underbrace{1, 1, \ldots, 1, 1}_{m-2}], [\tfrac{m-1}{2} : 0, 0, 0, \underbrace{1, 1, \ldots, 1, 1}_{m-3}], \end{array}\right.$$

$$\ldots, [3 : \overbrace{0, 0, \ldots, 0, 0}^{m-6}, 1, 1, 1, 1, 1, 1], [\tfrac{5}{2} : \overbrace{0, 0, \ldots, 0, 0}^{m-5}, 1, 1, 1, 1, 1], [2 : \overbrace{0, 0, \ldots, 0, 0}^{m-4}, 1, 1, 1, 1],$$

$$\ldots, [4 : \underbrace{0, 0, \ldots, 0, 0}_{m-6}, 1, 1, 1, 1, 1, 1], [\tfrac{7}{2} : \underbrace{0, 0, \ldots, 0, 0}_{m-5}, 1, 1, 1, 1, 1], [3 : \underbrace{0, 0, \ldots, 0, 0}_{m-4}, 1, 1, 1, 1],$$

$$\left.\begin{array}{l} [\tfrac{3}{2} : \overbrace{0, 0, \ldots, 0, 0}^{m-3}, 1, 1, 1], [1 : \overbrace{0, 0, \ldots, 0, 0}^{m-2}, 1, 1] : \overbrace{[1 : 1]; [1 : 1]; \ldots; [1 : 1]; [1 : 1]}^{m-1} ; [1 : 1], [\tfrac{1}{2} : 1]; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underbrace{-z^2, -z^2, \ldots, -z^2, -z^2}_{m} \\ [\tfrac{5}{2} : \underbrace{0, 0, \ldots, 0, 0}_{m-3}, 1, 1, 1], [2 : \underbrace{0, 0, \ldots, 0, 0}_{m-2}, 1, 1] : \underbrace{- ; - ; \ldots ; - ; -}_{m-1} ; \quad [\tfrac{3}{2} : 1] \quad ; \end{array}\right). \tag{16}$$

By replacing z by iz in the above hypergeometric forms we can obtain hypergeometric forms of positive integral powers of $[\tanh^{-1} z]^m$ for $m = 2, 3, 4, \ldots$.

4 Proofs of Power Series Forms and Hypergeometric Forms

To obtain power series representations of positive integral powers of $\tan^{-1} z$, we shall expand $exp(a \tan^{-1} z)$ with the help of Leibnitz's theorem (for the successive differentiation of the product of two functions) in powers of z.

$$exp(a \tan^{-1} z) = 1 + az + \frac{a^2 z^2}{2!} + \frac{(a^3 - 2a)z^3}{3!} + \frac{(a^4 - 8a^2)z^4}{4!} + \frac{(a^5 - 20a^3 + 24a)z^5}{5!} +$$

$$+ \frac{(a^6 - 40a^4 + 184a^2)z^6}{6!} + \frac{(a^7 - 70a^5 + 784a^3 - 720a)z^7}{7!} + \frac{(a^8 - 112a^6 + 2464a^4 - 8448a^2)z^8}{8!} +$$

$$+\frac{(a^9-168a^7+6384a^5-52352a^3+40320a)z^9}{9!}+$$
$$+\frac{(a^{10}-240a^8+14448a^6-229760a^4+648576a^2)z^{10}}{10!}+$$
$$+\frac{(a^{11}-330a^9+29568a^7-804320a^5+5360256a^3-3628800a)z^{11}}{11!}+$$
$$+\frac{(a^{12}-440a^{10}+55968a^8-2393600a^6+30633856a^4-74972160a^2)z^{12}}{12!}+$$
$$+\frac{(a^{13}-572a^{11}+99528a^9-6296576a^7+136804096a^5-782525952a^3+479001600a)z^{13}}{13!}+$$
$$+\frac{(a^{14}-728a^{12}+168168a^{10}-15027584a^8+510205696a^6-5561407488a^4+12174658560a^2)z^{14}}{14!}+\dots \tag{17}$$

Further we can expand $exp(a\tan^{-1}z)$ in power of $a\tan^{-1}z$, using well known exponential series

$$exp(a\tan^{-1}z)=1+a\tan^{-1}z+\frac{(a\tan^{-1}z)^2}{2!}+\frac{(a\tan^{-1}z)^3}{3!}+\frac{(a\tan^{-1}z)^4}{4!}+\frac{(a\tan^{-1}z)^5}{5!}$$
$$+\frac{(a\tan^{-1}z)^6}{6!}+\frac{(a\tan^{-1}z)^7}{7!}+\dots+\frac{(a\tan^{-1}z)^k}{k!}+\dots \tag{18}$$

Now equating the coefficients of various powers of a, we get when $|z|<1$

$$\tan^{-1}z=z-\frac{2z^3}{3!}+\frac{24z^5}{5!}-\frac{720z^7}{7!}+\frac{40320z^9}{9!}-\frac{3628800z^{11}}{11!}+\frac{479001600z^{13}}{13!}-\dots$$
$$=z-\frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+\frac{z^9}{9}-\frac{z^{11}}{11}+\frac{z^{13}}{13}-\dots=\sum_{k=0}^{\infty}\left[\frac{(-1)^k z^{2k+1}}{(2k+1)}\right]. \tag{19}$$

It is also known as Gregory series

$$\frac{(\tan^{-1}z)^2}{2!}=\frac{z^2}{2!}-\frac{8z^4}{4!}+\frac{184z^6}{6!}-\frac{8448z^8}{8!}+\frac{648576z^{10}}{10!}-\frac{74972160z^{12}}{12!}+\frac{12174658560z^{14}}{14!}-\dots$$
$$=\frac{z^2}{2}-\frac{z^4}{3}+\frac{23z^6}{90}-\frac{22z^8}{105}+\frac{563z^{10}}{3150}-\frac{1627z^{12}}{10395}+\frac{88069z^{14}}{630630}-\dots$$
$$=\frac{z^2}{2}-\left(1+\frac{1}{3}\right)\frac{z^4}{4}+\left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{z^6}{6}-\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right)\frac{z^8}{8}+$$
$$+\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\right)\frac{z^{10}}{10}-\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}\right)\frac{z^{12}}{12}+$$
$$+\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}\right)\frac{z^{14}}{14}-\dots=\sum_{k=0}^{\infty}\left[\left\{\sum_{n=0}^{k}\frac{1}{(2n+1)}\right\}\frac{(-1)^k z^{2k+2}}{(2k+2)}\right], \tag{20}$$

$$\frac{(\tan^{-1}z)^3}{3!} = \frac{z^3}{3!} - \frac{20z^5}{5!} + \frac{784z^7}{7!} - \frac{52352z^9}{9!} + \frac{5360256z^{11}}{11!} - \frac{782525952z^{13}}{13!} + \dots$$

$$= \frac{z^3}{6} - \frac{z^5}{6} + \frac{7z^7}{45} - \frac{409z^9}{2835} + \frac{47z^{11}}{350} - \frac{13063z^{13}}{103950} + \dots$$

$$= \frac{z^3}{3!} - \left[\frac{1}{2} + \frac{1}{4}\left(1+\frac{1}{3}\right)\right]\frac{z^5}{5} + \left[\frac{1}{2} + \frac{1}{4}\left(1+\frac{1}{3}\right) + \frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\right]\frac{z^7}{7} -$$

$$- \left[\frac{1}{2} + \frac{1}{4}\left(1+\frac{1}{3}\right) + \frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right) + \frac{1}{8}\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right)\right]\frac{z^9}{9} +$$

$$+ \left[\frac{1}{2} + \frac{1}{4}\left(1+\frac{1}{3}\right) + \frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right) + \frac{1}{8}\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right) + \frac{1}{10}\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}\right)\right]\frac{z^{11}}{11} - \dots$$

$$= \sum_{k=0}^{\infty}\left[\left\{\sum_{n=0}^{k}\frac{1}{(2n+2)}\sum_{m=0}^{n}\frac{1}{(2m+1)}\right\}\frac{(-1)^k z^{2k+3}}{(2k+3)}\right], \tag{21}$$

$$\frac{(\tan^{-1}z)^4}{4!} = \frac{z^4}{4!} - \frac{40z^6}{6!} + \frac{2464z^8}{8!} - \frac{229760z^{10}}{10!} + \frac{30633856z^{12}}{12!} - \frac{5561407488z^{14}}{14!} + \dots$$

$$= \frac{z^4}{24} - \frac{z^6}{18} + \frac{11z^8}{180} - \frac{259z^{10}}{5670} + \frac{21757z^{12}}{340200} - \frac{1421z^{14}}{22275} + \dots$$

$$= \frac{z^4}{4!} - \left[\frac{1}{6} + \frac{1}{5}\left\{\frac{1}{2} + \frac{1}{4}\left(1+\frac{1}{3}\right)\right\}\right]\frac{z^6}{6} +$$

$$+ \left[\frac{1}{6} + \frac{1}{5}\left\{\frac{1}{2} + \frac{1}{4}\left(1+\frac{1}{3}\right)\right\} + \frac{1}{7}\left\{\frac{1}{2} + \frac{1}{4}\left(1+\frac{1}{3}\right) + \frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\right\}\right]\frac{z^8}{8} -$$

$$- \left[\frac{1}{6} + \frac{1}{5}\left\{\frac{1}{2} + \frac{1}{4}\left(1+\frac{1}{3}\right)\right\} + \frac{1}{7}\left\{\frac{1}{2} + \frac{1}{4}\left(1+\frac{1}{3}\right) + \frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)\right\}\right.$$

$$\left. + \frac{1}{9}\left\{\frac{1}{2} + \frac{1}{4}\left(1+\frac{1}{3}\right) + \frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right) + \frac{1}{8}\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right)\right\}\right]\frac{z^{10}}{10} + \dots$$

$$= \sum_{k=0}^{\infty}\left[\left\{\sum_{n=0}^{k}\frac{1}{(2n+3)}\sum_{m=0}^{n}\frac{1}{(2m+2)}\sum_{\ell=0}^{m}\frac{1}{(2\ell+1)}\right\}\frac{(-1)^k z^{2k+4}}{(2k+4)}\right]. \tag{22}$$

In the same manner we can derive the power series forms (12).

Proof of general hypergeometric form (16): By the equation (12), we have

$$\frac{(\tan^{-1}z)^m}{m!} = \sum_{n_1=0}^{\infty}\left[\prod_{j=2}^{m}\left\{\sum_{n_j=0}^{n_{j-1}}\left(\frac{1}{1+m-j+2n_j}\right)\right\}\frac{(-1)^{n_1}z^{m+2n_1}}{2n_1+m}\right]$$

$$= \sum_{n_1=0}^{\infty}\left\{\left[\sum_{n_2=0}^{n_1}\frac{1}{m-1+2n_2}\sum_{n_3=0}^{n_2}\frac{1}{m-2+2n_3}\cdots\sum_{n_{m-1}=0}^{n_{m-2}}\frac{1}{2+2n_{m-1}}\sum_{n_m=0}^{n_{m-1}}\frac{1}{1+2n_m}\right]\frac{(-1)^{n_1}z^{m+2n_1}}{2n_1+m}\right\}.$$

Now applying the multiple series identity (8), we get

$$\frac{[\tan^{-1} z]^m}{m!} = \sum_{n_1,n_2,n_3,\ldots,n_m=0}^{\infty} \left\{ \left(\frac{(-1)^{n_1+n_2+n_3+\cdots+n_m} z^{m+2n_1+2n_2+2n_3+\cdots+2n_m}}{m+2n_1+2n_2+2n_3+\cdots+2n_m} \right) \right.$$

$$\left(\frac{1}{m-1+2n_2+2n_3+\cdots+2n_m} \right) \left(\frac{1}{m-2+2n_3+\cdots+2n_m} \right) \left(\frac{1}{m-3+2n_4+2n_5+\cdots+2n_m} \right) \cdots$$

$$\left. \cdots \left(\frac{1}{3+2n_{m-2}+2n_{m-1}+2n_m} \right) \left(\frac{1}{2+2n_{m-1}+2n_m} \right) \left(\frac{1}{1+2n_m} \right) \right\}.$$

$$= \frac{z^m}{m!} \sum_{n_1,n_2,n_3,\ldots,n_m=0}^{\infty} \left\{ \left(\frac{\left(\frac{m}{2}\right)_{n_1+n_2+\cdots+n_m}}{\left(\frac{m+2}{2}\right)_{n_1+n_2+\cdots+n_m}} \right) \right.$$

$$\left(\frac{\left(\frac{m-1}{2}\right)_{n_2+n_3+\cdots+n_m}}{\left(\frac{m+1}{2}\right)_{n_2+n_3+\cdots+n_m}} \right) \left(\frac{\left(\frac{m-2}{2}\right)_{n_3+n_4+\cdots+n_m}}{\left(\frac{m}{2}\right)_{n_3+n_4+\cdots+n_m}} \right) \left(\frac{\left(\frac{m-3}{2}\right)_{n_4+n_5+\cdots+n_m}}{\left(\frac{m-1}{2}\right)_{n_4+n_5+\cdots+n_m}} \right) \cdots$$

$$\cdots \left(\frac{\left(\frac{3}{2}\right)_{n_{m-2}+n_{m-1}+n_m}}{\left(\frac{5}{2}\right)_{n_{m-2}+n_{m-1}+n_m}} \right) \left(\frac{(1)_{n_{m-1}+n_m}}{(2)_{n_{m-1}+n_m}} \right) \left(\frac{\left(\frac{1}{2}\right)_{n_m}}{\left(\frac{3}{2}\right)_{n_m}} \right)$$

$$\left. \left[(1)_{n_1} (1)_{n_2} \ldots (1)_{n_{m-2}} (1)_{n_{m-1}} (1)_{n_m} \right] \frac{(-z^2)^{n_1+n_2+n_3+\cdots+n_m}}{n_1! n_2! \ldots n_{m-2}! n_{m-1}! n_m!} \right\}.$$

$$= \frac{z^m}{m!} \sum_{n_1,n_2,n_3,\ldots,n_m=0}^{\infty} \left\{ \left(\frac{\left(\frac{m}{2}\right)_{n_1+n_2+\cdots+n_m}}{\left(\frac{m+2}{2}\right)_{n_1+n_2+\cdots+n_m}} \right) \right.$$

$$\left(\frac{\left(\frac{m-1}{2}\right)_{0n_1+n_2+n_3+\cdots+n_m}}{\left(\frac{m+1}{2}\right)_{0n_1+n_2+n_3+\cdots+n_m}} \right) \left(\frac{\left(\frac{m-2}{2}\right)_{0n_1+0n_2+n_3+n_4+\cdots+n_m}}{\left(\frac{m}{2}\right)_{0n_1+0n_2+n_3+n_4+\cdots+n_m}} \right) \left(\frac{\left(\frac{m-3}{2}\right)_{0n_1+0n_2+0n_3+n_4+n_5+\cdots+n_m}}{\left(\frac{m-1}{2}\right)_{0n_1+0n_2+0n_3+n_4+n_5+\cdots+n_m}} \right) \cdots$$

$$\cdots \left(\frac{\left(\frac{3}{2}\right)_{0n_1+\cdots+0n_{m-3}+n_{m-2}+n_{m-1}+n_m}}{\left(\frac{5}{2}\right)_{0n_1+\cdots+0n_{m-3}+n_{m-2}+n_{m-1}+n_m}} \right) \left(\frac{(1)_{0n_1+\cdots+0n_{m-2}+n_{m-1}+n_m}}{(2)_{0n_1+\cdots+0n_{m-2}+n_{m-1}+n_m}} \right)$$

$$\left. \left[\frac{(1)_{n_1} (1)_{n_2} \ldots (1)_{n_{m-2}} (1)_{n_{m-1}} (1)_{n_m} \left(\frac{1}{2}\right)_{n_m}}{\left(\frac{3}{2}\right)_{n_m}} \right] \frac{(-z^2)^{n_1+n_2+n_3+\cdots+n_m}}{n_1! n_2! \ldots n_{m-2}! n_{m-1}! n_m!} \right\}.$$

In view of the definition of Srivastava-Daoust multi-variable hypergeometric function, we get the hypergeometric form (16). Similarly when we apply series identities (5), (6) and (7) in power series forms (9), (10) and (11) respectively we can obtain the hypergeometric forms (13), (14) and (15) in view of the definitions of Kampé de Fériet's general double hypergeometric function, Srivastava's general triple hypergeometric function, Saigo's general quadruple hypergeometric function.

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