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Fixed point results and Pachpatte theorem

Mumtaz Ali①*, Muhammad Arshad②

① Department of Mathematics, International Islamic University, H-10, Islamabad - 44000, Pakistan

② Department of Mathematics, International Islamic University, H-10, Islamabad-44000, Pakistan
E-mail: mumtaz6767@yahoo.com

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Abstract The aim of this paper is to present some fixed point results in the setting of compact metric spaces satisfying new contractive conditions. We prove some general fixed point theorems for self maps satisfying new contractive conditions involving iterative techniques due Edelstein for the existence and uniqueness of fixed points. Our main theorem extends and generalize corresponding results due to Pachpatte.

Key Words Continuous mapping, Fixed point, Contractive mapping, Compact Metric Space

MSC 2010 54H25, 53D12

1 Introduction

The Banach Contraction mapping theorem of 1922 popularly known as the Banach contraction mapping principle is a rewarding result in analysis and fixed point theory. Due to its widespread applications in both pure and applied mathematics it has attracted the attention of several great mathematicians for research in fixed point theory. According to Banach [2] a self map $T : X \to X$ on a complete metric space (X, d) is called a contraction mapping if for some $0 \leqslant k < 1$, it satisfies the inequality $d(Tx, Ty) \leqslant kd(x, y)$, then the mapping T has a unique fixed point in X.The study of fixed points mappings satisfying different contractive conditions have been actively investigated by several mathematicians.

In 1961, Edelstein [9] introduced new concept of contractive mapping defined on compact metric space as a generalization of Banach contraction mapping. According to Edelstein if T is a continuous mapping of a compact metric space X into itself satisfying the inequality $d(Tx, Ty) < d(x, y)$ for all $x, y \in X, x \neq y$, then T has a unique fixed point in X. Edelstein's contractive mapping theorem has been generalized and improved by several mathematicians in several different ways and obtained fixed points and fixed point results viz, Bailey [1], Ciric [7] , Iseki [8], Kannan and Sharma [12], Fisher [13], Pachpatte[16], Popa [17], Sahu M. K. [18] and Soni G. K[19] extensively investigated fixed point results for continuous self maps on compact metric spaces and established interesting results. Inspired by the ideas

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of Edelstein[9], and Pachpatte[16] the aim of the present paper is to prove the existence of unique fixed point results for continuous self-maps in compact metric space satisfying new contractive type conditions

2 Main Result

Theorem *Let f be a continuous self-map defined on a compact metric space (X,d). Further, let f satisfy the following conditions:*

$$\begin{aligned} &\{d(Tx,Ty)\}^2 \\ &< \alpha\left[d(x,Tx)\,d(y,Ty)+d(x,Ty)\,d(y,Tx)\right]+ \\ &\beta\left[d(x,Tx)\,d(y,Tx)+d(x,Ty)\,d(y,Ty)\right]+ \\ &\gamma\left[\{d(y,Tx)\}^2+\{d(y,Ty)\}^2\right]+ \\ &\delta\ max\begin{bmatrix} d(x,Tx)\,d(Tx,Ty)\,,d(x,Ty)\,d(y,Ty)\,, \\ d(y,Tx)\,d(x,y)\,,\{d(y,Ty)\}^2 \end{bmatrix} \end{aligned} \tag{2. 1}$$

for all $x,y\in X, x\neq y$ and $\alpha,\beta,\gamma,\delta$ are non negative real numbers such that $\alpha+2\beta+\gamma+2\delta\leqslant 1$ then T has a unique fixed point.

Proof. First we define a function F on X as follows

$$F(x)=d(x,T(x))\ \text{ for all } x\in X$$

Since d and T are continuous on X, from the compactness of X there exists a point $a\in X$ such that

$$F(a)=\inf\{F(x):x\in X\} \tag{2.2}$$

If $F(a)\neq 0$, it follows that $a\neq T(a)$ and so $F(T(a))=d\left(T(a)\,,T^2(a)\right)$

$$\begin{aligned} &\{d(T(a)\,,T(T(a)))\}^2 \\ &< \alpha\left[d(a,T(a))\,d(T(a)\,,T(T(a)))+d(a,T(T(a)))\,d(T(a)\,,T(a))\right]+ \\ &\beta\left[d(a,T(a))+d(T(a)\,,T(T(a)))\right]d(T(a)\,,T(T(a)))+ \\ &\gamma\left[\{d(T(a)\,,T(a))\}^2+\{d(T(a)\,,T(T(a)))\}^2\right]+ \\ &\delta\max\begin{bmatrix} d(a,T(a))\,d(T(a)\,,T(T(a)))\,,d(a,T(T(a)))\,d(T(a)\,,T(T(a)))\,, \\ d(T(a)\,,T(a))\,d(a,T(a))\,,\{d(T(a)\,,T(T(a)))\}^2 \end{bmatrix} \end{aligned}$$

Case(i)-If

$$\begin{aligned} &\max\begin{bmatrix} d(a,T(a))\,d(T(a)\,,T(T(a)))\,, \\ d(a,T(T(a)))\,d(T(a)\,,T(T(a)))\,, \\ d(T(a)\,,T(a))\,d(a,T(a))\,,\{d(T(a)\,,T(T(a)))\}^2 \end{bmatrix} \\ =\ &\{d(T(a)\,,T(T(a)))\}^2 \end{aligned}$$

Then

$$\begin{gathered}\{d(T(a),T(T(a)))\}^2\\ <\alpha\left[d(a,T(a))+d(T(a),T(T(a)))\right]+\\ \beta\left[d(a,T(a))+d(T(a),T(T(a)))\right]d(T(a),T(T(a)))+\\ \gamma\{d(T(a),T(T(a)))\}^2+\delta\{d(T(a),T(T(a)))\}^2\end{gathered}$$

$$\begin{gathered}d(T(a),T(T(a)))\\ <\alpha d(a,T(a))+\beta\left[d(a,T(a))+d(T(a),T(T(a)))\right]+\\ \gamma d(T(a),T(T(a)))+\delta d(T(a),T(T(a)))\end{gathered}$$

or

$$\begin{gathered}\left[1-(\beta+\gamma+\delta)\right]d(T(a),T(T(a)))<(\alpha+\beta)\,d(a,T(a))\\ d(T(a),T(T(a)))=\tfrac{(\alpha+\beta)}{[1-(\beta+\gamma+\delta)]}d(a,T(a))\\ d(T(a),T(T(a)))=\eta_1 d(a,T(a))\end{gathered}$$

Where

$$\eta_1=\frac{(\alpha+\beta)}{[1-(\beta+\gamma+\delta)]}$$

But $\alpha+2\beta+\gamma+2\delta\leqslant 1$, which is a contradiction.

Case-(ii) If

$$\max\begin{bmatrix} d(a,T(a))\,d(T(a),T(T(a))),\,d(a,T(T(a)))\,d(T(a),T(T(a))),\\ d(T(a),T(a))\,d(a,T(a)),\,\{d(T(a),T(T(a)))\}^2\end{bmatrix}$$
$$=\{d(a,T(T(a)))\,d(T(a),T(T(a)))\}$$

Then

$$\begin{gathered}\{d(T(a),T(T(a)))\}^2\\ <\alpha\left[d(a,T(a))\,d(T(a),T(T(a)))\right]+\\ \beta\left[d(a,T(a))+d(T(a),T(T(a)))\right]d(T(a),T(T(a)))+\\ \gamma\{d(T(a),T(T(a)))\}^2+\delta\left[d(a,T(a))+d(T(a),T(T(a)))\right]d(T(a),T(T(a)))\end{gathered}$$

$$\begin{gathered}d(T(a),T(T(a)))<\alpha d(a,T(a))+\\ \beta\left[d(a,T(a))+d(T(a),T(T(a)))\right]+\\ \gamma d(T(a),T(T(a)))+\delta\left[d(a,T(a))+d(T(a),T(T(a)))\right]\end{gathered}$$

$$\begin{gathered}[1-(\beta+\gamma+\delta)]d(T(a),T(T(a)))\\ <(\alpha+\beta+\delta)\,d(a,T(a))\end{gathered}$$

$$\begin{gathered}d(T(a),T(T(a)))=\frac{[1-(\beta+\gamma+\delta)]}{(\alpha+\beta+\delta)}d(a,T(a))\\ d(T(a),T(T(a)))=\eta_2 d(a,T(a))\end{gathered}$$

Where

$$\eta_2 = \frac{[1-(\beta+\gamma+\delta)]}{(\alpha+\beta+\delta)}$$

But $\alpha+2\beta+\gamma+2\delta \leqslant 1$, which is a contradiction.

Case-(iii) If

$$\max\left[\begin{array}{c} d(a,T(a))\,d(T(a),T(T(a))),\, d(a,T(T(a)))\,d(T(a),T(T(a))),\\ d(T(a),T(a))\,d(a,T(a)),\, \{d(T(a),T(T(a)))\}^2 \\ = d(a,T(a))\,d(T(a),T(T(a))) \end{array}\right]$$

Then

$$\begin{gathered} \{d(T(a),T(T(a)))\}^2 \\ < \alpha\left[d(a,T(a))\,d(T(a),T(T(a)))\right] + \\ \beta\left[d(a,T(a)) + d(T(a),T(T(a)))\right] d(T(a),T(T(a))) + \\ \gamma\{d(T(a),T(T(a)))\}^2 + d(a,T(a))\,d(T(a),T(T(a))) \end{gathered}$$

$$\begin{gathered} [1-(\beta+\gamma)]\,d(T(a),T(T(a))) < (\alpha+\beta+\delta)\,d(a,T(a)) \\ d(T(a),T(T(a))) = \frac{(\alpha+\beta+\delta)}{[1-(\beta+\gamma)]} d(a,T(a)) \\ d(T(a),T(T(a))) = \eta_3 d(a,T(a)) \end{gathered}$$

Where

$$\eta_3 = \frac{(\alpha+\beta+\delta)}{[1-(\beta+\gamma)]}$$

But $\alpha+2\beta+\gamma+2\delta \leqslant 1$, which is a contradiction.

Here

$$\eta = \max\{\eta_1,\eta_2,\eta_3\}$$

$$\eta = \max\left\{\frac{(\alpha+\beta)}{[1-(\beta+\gamma+\delta)]}, \frac{[1-(\beta+\gamma+\delta)]}{(\alpha+\beta+\delta)}, \frac{(\alpha+\beta+\delta)}{[1-(\beta+\gamma)]}\right\}$$

Hence, $T(a) = a$ is a fixed point of T.

Uniqueness: Let, if possible $a \neq b$ be another fixed pioint of T. Then

$$\begin{gathered} \{d(T(a),T(b))\}^2 \\ = \{d(a,b)\}^2 < \alpha\left[d(a,T(a))\,d(b,T(b)) + d(a,T(b))\,d(b,T(a))\right] + \\ \beta\left[d(a,T(a))\,d(b,T(a)) + d(a,T(b))\,d(b,T(b))\right] + \\ \gamma\left[\{d(b,T(a))\}^2 + \{d(b,T(b))\}^2\right] + \\ \delta\max\left[\begin{array}{c} d(a,T(a))\,d(T(a),T(b)), \\ d(a,T(b))\,d(b,T(b)), \\ d(b,T(a))\,d(a,b),\, \{d(b,T(b))\}^2 \end{array}\right] \end{gathered}$$

$$\Rightarrow \quad d(a,b) < \alpha d(a,b) + \gamma d(a,b) + \delta d(a,b)$$
$$\Rightarrow \quad [1-(\alpha+\gamma+\delta)]\, d(a,b) < 0$$

This is a contradiction because $\alpha+\gamma+\delta<1$. Hence a is unique fixed point of T.

Remark: If we put $\delta = 0$, then we get the result of Pachpatte[16]

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