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Cross*-regular semigroups and cross*-completely regular semigroups

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Abstract We introduce the concept of cross*-regular semigroups which is generalization of regular semigroups. By applying the cross*-regularity of semigroups, we re-define Green's relations on the cross*-regular semigroups. Then we study and characterize the congruences on the cross*-regular semigroups, in particular, the existence of the idempotents in the $\mathcal{J}^*$-class of a cross*-regular semigroup is discussed. Some characterizations of cross*-completely regular semigroups are obtained.

Key Words Cross*-regular semigroups; Green's relations; Cross*-completely regular semigroups
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1 Introduction

Let S be a semigroup. An element $a \in S$ is called regular if there exists an element $x \in S$ such that $a = axa$. If all elements of a semigroup S are regular, then we call S a regular semigroup. Obviously, a group must be a regular semigroup. In view of this observation, there is a need to explore some new methods and techniques to study various kinds of generalized regular semigroups. In the literature, the most interesting result of regular semigroups is the celebrated Clifford theorem for completely regular semigroups which was first proved by A. H. Clifford dated back to 1944 (see [10-11]). In fact, A. H. Clifford proved that a semigroup is a completely regular semigroup if and only if the semigroup is a semi-lattice of completely simple semigroups, moreover, he also proved some interesting results concerning the $\mathcal{D}$ classes of a semigroup S (see [2,10]). Later on, some special subclasses of the class of regular semigroups such as the class of ordered regular semigroups, the class of eventually regular semigroups and the class of quasi-regular semigroups were further considered and extensively investigated by many authors. We notice that the power semigroups were particularly studied by S. Bogdanovic et al. (see[2-9]). Power regular semigroups were also investigated by S. Bogdanovic (see[1]). It is noticed that J. Hanumnthachari et al (see[13]) also considered the congruences on ordered regular semigroups and obtained several interesting results. In this paper, we will describe the cross*-regular semigroups and the cross* -completely regular

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semigroups. Although some results of S. Bogdanovic were given more than two decades ago, these results up to now are still important results in generalized regular semigroups. Another kind of non-regular semigroups which are closely related to regular semigroups is the legal semigroups studied and investigated by P. Y. Zhu and K. P. Shum in 2000 (see[21]). A semigroup S is called legal if for all a and b in S, either $aba = ba$ and $bab = ab$ or $aba = ab$ and $bab = ba$ holds. In fact, P. Y. Zhu and K. P. Shum proved that a legal semigroup is a semi-lattice Y of semigroups S_α, where each $\alpha \in Y$, and each $E(S_\alpha)$ is a right zero or a left zero semigroup with $a \in E(S_\alpha)$, for every $a \in S_\alpha$. Thus, the well known Clifford theorem for completely regular semigroups were extended to the legal semigroups. In this paper, as inspired by the ideas of legal semigroups, we will extend and generalize the power semigroups described by S.Bogdanovic ([see 2-6]) to cross*-regular semigroups. We discuss the existence of some of the subclasses of power semigroups so that some known results of completely regular semigroups will be established in a cross*-completely regular semigroups and the structure theory of prime semigroups will be enriched and strengthened. For notations and terminology not given in this paper, the reader is referred to the monograph of A. H. Clifford and G. B. Preston (see[10]), K. P. Shum and Y. Q. Guo in (see[20]).

2 Cross*-regularity

Definition 2.1. *Let S_1 and S_2 be two nonempty regular semigroups and S a nonempty subset of $S_1 \times S_2$. An element $(a,b) \in S$ is called cross*-regular if there exists a positive integer m such that $(a,b)^m \in (a,b)^m S (a,b)^m$. The subset S is said to be a cross*-regular semigroup if all elements of S are cross*-regular.*

Definition 2.2. *A cross*-regular semigroup S is said to be cross*-completely regular if for every $(a,b) \in S$, there exists a positive integer m and $(x,y) \in S$ such that $(a,b)^m(x,y) = (x,y)(a,b)^m$.*

Definition 2.3. *A cross*-regular semigroup S is called a right (resp., left) cross*-inverse semigroup if $(a,b) = (a,b)(x,y)(a,b) = (a,b)(u,v)(a,b)$ implies $(x,y)(a,b) = (u,v)(a,b)$ (resp., $(a,b)(x,y) = (a,b)(u,v)$)for any (a,b), (x,y) and (u,v) in S.*

We now consider a finite cross*-regular semigroup S, it is well known that S contains at least one idempotent of S. We now define the Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ on S by the following on the semigroup S.

(1) $(a,b)\mathcal{L}(c,d)$ if and only if $S(a,b) = S(c,d)$

(2) $(a,b)\mathcal{R}(c,d)$ if and only if $(a,b)S = (c,d)S$

(3) $(a,b)\mathcal{J}(c,d)$ if and only if $S(a,b)S = S(c,d)S$.

For the class of cross*-regular semigroups, we have the following lemma.

Lemma 2.4. *Let S be a semigroup with $(a,b) \in S$. Then the following statements are equivalent.*

(1) *(a,b) is cross*-regular;*

(2) *There exists $m \in \mathbb{Z}^+$ such that $\mathcal{R}(a,b)^m(\mathcal{L}(a,b)^m)$ is generated by an idempotent of S;*

(3) *There exists $m \in \mathbb{Z}^+$ such that there exists a left (or a right) identity in $\mathcal{R}(a,b)^m(\mathcal{L}(a,b)^m)$.*

Proof. (1) $\Rightarrow$ (2). If $(a,b) \in S$ is a cross*-regular element of S. Then for any element $(x,y) \in S$ and $m \in \mathbb{Z}^+$, we have $(a,b)^m = (a,b)^m(x,y)(a,b)^m$. This implies $(e,e) = (a,b)^m(x,y)$ is an idempotent and $(e,e)(a,b)^m = (a,b)^m$ holds in S. Thus (2) is proved.

(2) $\Rightarrow$ (3). Suppose that $(e,e)(a,b)^m = (a,b)^m$. Then for any $(x,y) \in S$, there exists $m \in \mathbb{Z}^+$ such that for any $(u,v) \in \mathcal{R}(a,b)^m$, $(u,v) = (a,b)^m(c,d)$, where $(c,d) \in S$. Consequently, we have $(a,b)^m(x,y)(u,v) = (a,b)^m(x,y)(a,b)^m(c,d) = (a,b)^m(c,d) = (u,v)$. Thus that $(a.b)^m(x,y)$ is indeed a left identity of $\mathcal{R}(a,b)^m$.

(3) $\Rightarrow$ (1). Let $(a,b)^m(x,y)$ be a left identity of $R(a,b)^m$. Then, we take $(e,e) = (a,b)^m(x,y)$ for any $(x,y) \in S$. Hence, $(e,e)(a,b)^m = (a,b)^m(x,y)(a,b)^m$, that is, $(a,b)^m = (a,b)^m(x,y)(a,b)^m$. This proves that (a,b) is a cross*-regular element of S. The proofs of (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) are obvious and hence omitted. □

By using the above lemma, we obtain the following two corollaries.

Corollary 2.5. *Let S be a semigroup. Then the following statements are equivalent.*

(1) *S is a cross*-regular semigroup ;*

(2) *For any $(a,b) \in S$, there exists an $m \in Z^+$ such that $\mathcal{R}(a,b)^m = (e,e)S$ $(\mathcal{L}(a,b)^m = S(e,e))$;*

(3) *For every $(a,b) \in S$, there exists an $m \in Z^+$ such that $\mathcal{R}(a,b)^m(\mathcal{L}(a,b)^m)$ contains a left identity (or a right identity).*

Corollary 2.6. *Let (a,b) be an arbitrary element of the semigroup S. Then (a,b) is a cross*-regular element of S if and only if there exists an idempotent $(e,e) \in S$ such that $(a,b)S^1 = (e,e)S$.*

We now consider the congruences on a cross*-regular semigroup S. We start with the following lemma.

Lemma 2.7. *Let σ be a congruence on a cross*-regular semigroup S with I, $\Lambda \in S/\sigma$ such that $I = I\Lambda I$ and $\Lambda = \Lambda I \Lambda$ in the quotient semigroup S/σ. Then there exists a pair of elements $(a,b),(c,d) \in S$ such that $(a,b) \in I$, $(c,d) \in \Lambda$ so that $(a,b) = (a,b)(c,d)(a,b)$ and $(c,d)(a,b)(c,d) \in S$ hold in S.*

Proof. Let $(x,y) \in I$, $(u,v) \in \Lambda$ and (h,l) be the inverse element of $((x,y)(u,v))^{2m}$. Then for any $m \in \mathbb{Z}^+$, we deduce that

$$(a,b) = (x,y)(u,v)(h,l)((x,y)(u,v))^{2m-1}(x,y)$$

and

$$(c,d) = (u,v)(h,l)((x,y)(u,v))^{2m-1}.$$

Moreover, we have

$$(a,b) = (a,b)(c,d)(a,b), (c,d) = (c,d)(a,b)(c,d).$$

Because $I = I\Lambda I$, we have $\Lambda = \Lambda I \Lambda$. Now we deduce that

$$(x,y)\sigma(x,y)(u,v))(x,y) \;\; and \;\; (u,v)\sigma(u,v)(x,y)(u,v).$$

Consequently, $(x,y)(u,v)\sigma((x,y)(u,v))^k$, $k\in\mathbb{Z}^+$, we have

$$(x,y)(u,v)(h,l)\sigma((x,y)(u,v))^{2m}(h,l)$$

and

$$((x,y)(u,v))^{2m-1}(x,y)\sigma((x,y)(u,v))^{2m}(x,y).$$

Thus, we have proved the following equality.

$$(x,y)(u,v)(h,l)((x,y)(u,v))^{2m-1}(x,y)\sigma((x,y)(u,v))^{2m}(h,l)((x,y)(u,v)))^{2m}=((x,y)(u,v))^{2m}(x,y).$$

According to $(x,y)(u,v)\sigma((x,y)(u,v))^k$, have $((x,y)(u,v))^{2m}(x,y)\sigma(x,y)$ and so $(a,b)\sigma(x,y)$. Similarly, we can also proved that $(c,d)\sigma(u,v)$, and hence we have $(a,b)\in I$ and $(c,d)\in\Lambda$. □

Correspondingly, we immediately obtain the following corollary.

Corollary 2.8. *Let σ be a congruence on a cross*-regular semigroup S. Then, the idempotent of all the σ-classes of the quotient semigroup S/σ is contained in S.*

We now define the following equivalent relation K on a cross*-regular semigroup S. We denote the set of all idempotents of S by $E(S)$. Then, for any $(e,e)\in E(S)$ and any $(x,y),(u,v)\in K(e,e)$, we define $(x,y)K(u,v)$ if and only if there exists $(e,e)\in E(S)$ such that $(x,y),(u,v)\in K(e,e)$. For the above equivalent relation K on a cross*-regular semigroups, we prove again the following lemma stated by J.M.Howie in ([14]).

Lemma 2.9. *Let S be a cross*-regular semigroup with $J\subseteq K$. Then the following statements hold.*
(1) *For any $(a,b),(c,d)\in S$ and $(e,e)\in E(S)$, $(a,b)(c,d)\in K(e,e)$ if and only if $(c,d)(a,b)\in K(e,e)$;*
(2) $(a,b)(c,d)K(a,b)^2(c,d)$, $(a,b)(c,d)K(a,b)(c,d)^2$.

Proof. (1) Since $J\subseteq K$, $(a,b),(c,d)\in S$, and $(e,e)\in E(S)$, we have $(a,b)J(c,d)$ if and only if $S(a,b)S=S(c,d)S$. Thus, we deduce that $(a,b)(c,d)\in K(e,e)$ if and only if $(c,d)(a,b)\in K(e,e)$.
(2) By (1), it follows that $(a,b)(c,d)K(a,b)^2(c,d)$. Also ,$(a,b)(c,d)K(a,b)(c,d)^2$ is obvious. □

3 Properties of Green*-relations on Cross*-regular Semigroups

By the same method as in the study of regular semigroups, we first define the Green*-relations L^*, R^*, H^*, and J^* on a cross*-regular semigroup S by

$$(i)\quad (a,b)L^*(x,y) \text{ if and only if } S(a,b)^m=S(x,y)^n;$$
$$(ii)\quad (a,b)R^*(x,y) \text{ if and only if } (a,b)^mS=(x,y)^nS;$$
$$(iii)\quad (a,b)H^*(x,y) \text{ if and only if } (a,b)L^*\cap R^*(x,y);$$
$$(iv)\quad (a,b)J^*(x,y) \text{ if and only if } S(a,b)^mS=S(x,y)^nS.$$

where the integers m,n are the minimal integers such that $(a,b)^m,(x,y)^n$ is a regular element of S. Since $(a,b)^m,(x,y)^n\in S$, we denote the classes $L^*_{(a,b)},R^*_{(a,b)},H^*_{(a,b)},J^*_{(a,b)}$ simply by L^*-,R^*-,H^*-,J^*-, where $(a,b)\in S$, respectively. We get below some useful lemmas for cross*-regular semigroups.

Lemma 3.1. *Let S be a cross*-regular semigroup. Then every $L^*(R^*)$ class of S contains at least one idempotent of S.*

Lemma 3.2. *Let S be a cross*-regular semigroup. Then every idempotent $(e,e) \in S$ is a right (left, two sided) identity of the regular class $L^*_{(e,e)}(R^*_{(e,e)}, J^*_{(e,e)})$.*

As a consequence of the above results in a cross*-regular semigroup, we have the following lemmas which are analogous to the corresponding results in a usual regular semigroup.

Lemma 3.3. *Let S be a cross*-regular semigroup. Then every H^* -class of S contains at most one idempotent of S.*

Lemma 3.4. *Let S be a cross*-regular semigroup with $(a,b) \in S$. If p is the smallest integer such that $(a,b)^p \in S$, then $(a,b)^p \in L^*_{(a,b)} \bigcap R^*_{(a,b)} = H^*_{(a,b)}$.*

Lemma 3.5. *Let S be a cross*-regular semigroup. Then the following statements hold.*

(1) *Every J^*-class of S contains at least one idempotent of S;*

(2) *$G_{(e,e)} \subseteq H_{(e,e)} \subseteq J_{(e,e)}$ for any $(e,e) \in E(S)$, $(f,f) \in E(S)$.*

In the following lemmas, we consider the J^*-class of the cross*-regular semigroups.

Lemma 3.6. *Let S be a cross*-regular semigroup. Then for any $(e,e) \in E(S)$, and $(f,f) \in E(S)$,$J^*_{(e,e)} = J^*_{(f,f)}$ so that $(e,e)(f,f) = (f,f)(e,e) = (f,f) \Rightarrow (e,e) = (f,f)$.*

Proof. By $S(e,e)S = S(f,f)S$, the equality $(e,e) = (u,v)(f,f)(s,t)$ holds if $(u,v),(s,t) \in S$. Suppose that $(a,b) = (e,e)(u,v)(e,e)$. Then, we have the following equality.

$$(a,b) = (e,e)(f,f)(s,t)(e,e)(u,v)(e,e) = (a,b)(f,f)(s,t)(a,b).$$

By hypothesis, there exists $(a,b)' \in S$ such that $(a,b) = (a,b)(a,b)'(a,b) = (a,b)(a,b)'$. Now, we suppose that $(x,y) = (e,e)(s,t)(e,e)$. Then we deduce that

$$(a,b)(f,f)(x,y) = (e,e)(u,v)(e,e)(f,f)(e,e)(s,t)(e,e) = (e,e)(u,v)(f,f)(s,t)(e,e) = (e,e).$$

Hence, we get

$$(e,e) = (a,b)(f,f)(x,y) = (a,b)(a,b)'(a,b)(f,f)(x,y) = (a,b)(a,b)'(e,e) = (a,b)'(a,b).$$

Consequently, we have

$$(e,e) = (a,b)'(a,b) = (a,b)'(e,e)(a,b) = (a,b)'(a,b)(f,f)(x,y)(a,b)(f,f)(x,y)(a,b).$$

Thus $(e,e) = (f,f)(e,e)$ and $(e,e) = (f,f)$. □

Lemma 3.7. *Let S be a cross*-regular semigroup. For any $(u,v) \in S$, $(e,e) \in E(S)$, if $J^*_{(e,e)} = J^*_{(e,e)(u,v)(e,e)}$ then, $(e,e)(u,v)(e,e) \in G_{(e,e)}$.*

Lemma 3.8. *Let S be a cross*-regular semigroup. For all $(e,e),(f,f) \in E(S)$, then the following equalities hold. $J^*_{(e,e)} = J^*_{(f,f)} = J^*_{(e,e)(f,f)} = J^*_{(e,e)}$.*

Proof. Let $S(e,e)S = S(f,f)S$. Then we have

$$(e,e) = (a,b)(f,f)(e,e) = (a,b)(f,f)(u,v)(e,e)((a,b)(u,v) \in S).$$

Now, by Lemma 3.6, we have

$$J^*_{(f,f)} = J^*_{(e,e)} = J^*_{(a,b)(f,f)(u,v)(e,e)} = J^*_{(a,b)(f,f)((f,f)(u,v)(e,e))} = J^*_{(f,f)(u,v)(e,e)(a,b)(f,f)}.$$

and also

$$(f,f)(u,v)(e,e)(a,b)(f,f) \in E(S).$$

According to Lemma 3.6, we further deduce that

$$(f,f) = (f,f)(u,v)(e,e)(a,b)(f,f),\ (f,f)(u,v)(f,f) \in G_{(f,f)}.$$

Consequently, we

$$((f,f)(e,e)(f,f))^n = ((f,f)(a,b)(f,f)(f,f)(u,v)(f,f))^p \in G_{(f,f)} \subseteq J^*_{(f,f)},$$

where $n, p \in Z^+$ and hence, we have

$$S(f,f)S = S((f,f)(e,e)(f,f))^p S \subseteq S((e,e)(f,f))^p S.$$

It is clear that the reverse containment holds on S and we have $S(f,f)S = S((e,e)(f,f))^p S$. Thus, we have proved that

$$J^*_{(e,e)} = J^*_{(f,f)} = J^*_{(e,e)(f,f)} = J^*_{(e,e)}$$

. □

We state below a theorem of cross*-regular semigroups containing an idempotent.

Theorem 3.9. *Let (e,e) be an idempotent of a cross*-regular semigroup S. Then $G_{(e,e)} \subseteq H^*_{(e,e)}$. Moreover, if $(u,v) \in H^*_{(e,e)}$ and p is the smallest integer such that $(u,v)^p \in S$, then for any $q \geqslant p$, $(u,v)^q \in G_{(e,e)}$.*

Proof. Let $(a,b) \in G_{(e,e)}$ and suppose that (s,t) is the inverse element of (a,b). Then, we deduce that $(s,t)(a,b) = (e,e) = (a,b)(s,t)$. This implies $(s,t) \in V$ and hence $G_{(e,e)} \subseteq H^*_{(e,e)}$. On the other hand, we may assume that $(u,v) \in H^*_{(e,e)}$ and let p be the smallest integer such that $(u,v)^p \in S$. Then there exists $(u,v)' \in V$ such that $(u,v)'(u,v)^p = (e,e) = (u,v)^p(u,v)'$. This shows that $(u,v)^p$ is a completely regular element of S, that is, $(u,v)^p \in G_{(f,f)}$. It is easy to see that $(e,e) = (f,f)$ and hence, $(u,v)^q \in G_{(e,e)}$, with $q \leqslant p$. □

For inverse elements of a cross*-regular semigroup S, we have the following theorem for a cross*-regular semigroup.

Theorem 3.10. *Let S be a cross*-regular semigroup. Denote the set of all inverse elements of S by V. Let $(a,b),(x,y)\in S$. Then the following statements hold.*

(1) $(a,b)L^*(x,y)$ if and only if there exists $(a,b)',(x,y)'\in V$ such that $(a,b)'(a,b)^m=(x,y)'(x,y)^n$;

(2) $(a,b)R^*(x,y)$ if and only if there exists $(a,b)',(x,y)'\in V$ such that $(a,b)^m(a,b)'=(x,y)^n(x,y)'$;

(3) $(a,b)H^*(x,y)$ if and only if there exist $(a,b)',(x,y)'\in V$ such that $(a,b)'(a,b)^m=(x,y)'(x,y)^n$ and $(a,b)^m(a,b)'=(x,y)^n(x,y)'$.

It can be verified that the congruences on cross*-regular semigroups are different from the usual regular semigroups. Hence, we concentrate on the congruences on a cross*-regular semigroup and and to generalize some results of prime semigroups given by S.Bogdanovich in ([2-6]).

Now, we consider the congruences on a cross*-regular semigroup. We have the following theorem.

Theorem 3.11. *Let σ be a congruence on a cross *-regular semigroup S with $n\in Z^+$. If $A,S_1,S_2,...,S_n\in S/\sigma$ and $A=AS_iA$,$S_i=S_iAS_i$. Then there exist $(a_{01},a_{02}),(s_{11},s_{12}),(s_{21},s_{22}),...,(s_{n1},s_{n2})\in S$ such that $(a_{01},a_{02})\in A,(s_{i1},s_{i2})\in S_i$, $(a_{01},a_{02})=(a_{01},a_{02})(s_{i1},s_{i2})(a_{01},a_{02})$ and $(s_{i1},s_{i2})=(s_{i1},s_{i2})(a_{01},a_{02})$ (s_{i1},s_{i2}),where $i=1,2,..,n$.*

Proof. We proceed by induction. Clearly, the theorem holds for $n=1$. Now, we assume that the theorem is true for all integers $k<n$. Then there exist elements

$$(x_{01},x_{02}),(y_{11},y_{12}),(y_{21},y_{22}),...,(y_{k1},y_{k2})\in S$$

so that $(x_{01},x_{02})\in A$, $(y_{i1},y_{i2})\in S_i$,

$$(x_{01},x_{02})=(x_{01},x_{02})(y_{i1},y_{i2})(x_{01},x_{02}),\ (y_{i1},y_{i2})=(y_{i1},y_{i2})(x_{01},x_{02})(y_{i1},y_{i2}),$$

for all $i=1,2,...,k$. Now, we let an arbitrary element $(y_{k+11},y_{k+12})\in S_{k+1}$. Since S is a cross*-regular semigroup, there exists an integer $n\in Z^+$ such that $((x_{01},x_{02})(y_{k+11},y_{k+12}))^{2n}$ is a cross *-regular element. Now, we suppose that (h,l) is an inverse element of $((x_{01},x_{02})(y_{k+11},y_{k+12}))^{2n}$. Then, we again let

$$\alpha{=}(x_{01},x_{02})(y_{k+11},y_{k+12})(h,l)((x_{01},x_{02})(y_{k+11},y_{k+12}))^{2n-1}(x_{01},x_{02});$$
$$\beta{=}(y_{k+11},y_{k+12})(h,l)((x_{01},x_{02})(y_{k+11},y_{k+12}))^{2n-1};$$
$$\gamma_i{=}(y_{i1},y_{i2})(x_{01},x_{02})(y_{k+11},y_{k+12})(h,l)\bullet((x_{01},x_{02})(y_{k+11},y_{k+12}))^{2n-1}(x_{01},x_{02})(y_{i1},y_{i2}),\ i=1,2,...,k,$$

By routine checking, we can easily verify that $\alpha\in A$, $\gamma_i\in S_i$ and $\alpha=\alpha\gamma_i\alpha$, and $\gamma_i=\gamma_i\alpha\gamma_i$, for all $i=1,2,...,k,k+1$. □

By Definitions 2.2 and 2.3 and the above result, we can easily prove the following theorem.

Theorem 3.12. *Let S be a cross *-regular semigroup. Then the following conditions are equivalent.*

(1) S is a right cross *-inverse semigroup;

(2) S is a cross*-regular semigroup and for any $(e,e),(f,f)\in E(S)$, there exists $m\in Z^+$ such that $((e,e)(f,f))^m=((f,f)(e,e)(f,f))^m$;

(3) *S is a cross *-regular semigroup and for any $(e,e),(f,f)\in E(S)$, there exists $m\in Z^+$ such that $((e,e)(f,f))^m R^*((f,f)(e,e))^m$;*

(4) *For every ideal I of S, there exists a right cross*-inverse quotient semigroup S/I.*

4 Theorems of Cross*-regular Semigroups

Theorem 4.1. *A cross*-regular semigroup S is a cross*-completely regular semigroup if and only if every H^*- class of S contains at least one idempotent of S.*

Proof. To proceed the proof, we first let $(a,b),(x,y)\in H^*$- and $(a,b)',(x,y)'\in V$. Then, we have

$$(e,e)=(a,b)'(a,b)^m\in L^*_{(a,b)}\cap E(S){=}L^*_{(x,y)}\cap E(S);(f,f)=(a,b)^m(a,b)'\in R^*_{(a,b)}\cap E(S)=R^*_{(x,y)}\cap E(S);$$

consequently, we deduce the following

$$(x,y)^n=(x,y)^n(e,e)=(f,f)(x,y)^n \text{ and } (f,f)=(x,y)^n(u,v),\ (e,e)=(s,t)(x,y)^n((u,v),(s,t)\in S).$$

Suppose that $(x,y)'=(e,e)(u,v)(f,f)$. Then we have $(x,y)'\in V$ and so, we establish the following

$$(x,y)^n(x,y)'(x,y)^n=(x,y)^n(e,e)(u,v)(f,f)(x,y)^n=(x,y)^n(u,v)(x,y)^n=(f,f)(x,y)^n=(x,y)^n;$$
$$(x,y)'(x,y)^n(x,y)=(e,e)(u,v)(f,f)(x,y)^n(e,e)(u,v)(f,f)=(e,e)(u,v)(x,y)^n(u,v)(f,f)=$$
$$(e,e)(u,v)(f,f)=(x,y)',$$

Now, by using the above equalities, we conclude the following

$$(x,y)^n(x,y)'=(x,y)^n(e,e)(u,v)(f,f)=(x,y)^n(u,v)(f,f)=(f,f)=(a,b)^m(a,b)';$$
$$(x,y)'(x,y)^n=(e,e)(u,v)(f,f)(x,y)^n=(s,t)(x,y)^n(u,v)(f,f)(x,y)^n=(s,t)(f,f)(x,y)^n=$$
$$(s,t)(x,y)^n=(e,e)=(a,b)'(a,b)^m.$$

For the converse part, suppose we have

$$(a,b)'(a,b)^m=(x,y)'(x,y)^n \text{ and } (a,b)^m(a,b)'=(x,y)^n(x,y)',$$

for some $(a,b)',(x,y)'\in V$. Then we can easily deduce that

$$(a,b)^m=(a,b)^m(a,b)'(a,b)^m=(a,b)^m(x,y)'(x,y)^n=(x,y)^n(x,y)'(a,b)^m; \text{ and}$$
$$(x,y)^n=(x,y)^n(x,y)(x,y)^n=(x,y)^n(a,b)'(a,b)^m=(a,b)^n(a,b)'(x,y)^n.$$

Hence, we have

$$S(a,b)^m=S(x,y)^n, \text{ and } (a,b)^mS=(x,y)^nS \text{ and}$$

consequently $(a,b)H^*(x,y)$. □

We now state a theorem of a cross*-regular semigroup S related to J^*-class of S.

Theorem 4.2. *Let S be a cross*-regular semigroup. Then, for any $(a,b),(x,y)\in S$, $J^*_{(a,b)(x,y)}=J^*_{(x,y)(a,b)}$.*

Proof. Let p and q be the smallest positive integers such that

$$((a,b)(x,y))^p, ((x,y)(a,b))^q \in S.$$

Then we have

$$((a,b)(x,y))^p \in G_{(e,e)} \subseteq J^*_{(e,e)}, ((x,y)(a,b))^q \in G_{(f,f)} \subseteq J^*_{(f,f)}.$$

According to ([5]), we have

$$((a,b)(x,y))^{p+m} \in G_{(e,e)} \subseteq J^*_{(e,e)},\ ((x,y)(a,b))^{q+n} \in G_{(f,f)} \subseteq J^*_{(f,f)},$$

for every $p, q \geqslant 0$, and so, we get

$$((a,b)(x,y))^m J^*((a,b)(x,y))^{p+m}, ((x,y)(a,b))^n J^*((x,y)(a,b)^{q+n}.$$

When $k = max(m,n)$, we obtain the following equation.

$$S((a,b)(x,y))^m)S = S((a,b)(x,y))^{k+1}S = S(a,b)((x,y)(a,b))^k(x,y)S \subseteq S((x,y)(a,b))^k S \subseteq S((x,y)(a,b))^n S.$$

Similarly, we deduce that

$$S((x,y)(a,b))^n \subseteq S((a,b)(x,y))^m S.$$

Consequently, we have the following

$$S((a,b))(x,y))^m S = S((x,y)(a,b))^n S.$$

So, we have proved that

$$J^*_{(a,b)(x,y)} = J^*_{(x,y)(a,b)}.$$

□

Finally, by Theorems 4.1 and 4.2, we establish the following theorem for cross*-regular semigroups.

Theorem 4.3. *Let S be a cross*-regular semigroup. Then for any $(a,b),(x,y) \in S$, $J^*_{(a,b)(x,y)} = J^*_{(e,e)(f,f)}$, for all $m,n \in \mathbb{Z}^+$.*

Proof. Let r be the smallest positive integer such that $((a,b)(x,y))^r$ is regular. Then we have the following relations:

$$(1) \quad ((a,b)(x,y))^r \in J^*_{(a,b)(x,y)}, ((a,b)(x,y))^r \in G_{(h,h)} \subseteq J^*_{(h,h)},$$

and

$$(2) \quad (h,h)(a,b)(x,y) = (a,b)(x,y)(h,h) \in G_{(h,h)},$$

where $(h,h) \in E(S)$. We proceed to prove the theorem for p by mathematical induction. Consider(2)

$$(h,h)(a,b)^p(h,h) \in G_{(h,h)},$$

and

$$(h,h)(x,y)^p(h,h) \in G_{(h,h)}.$$

Suppose that for every $p \geqslant 0$, $(h,h)(a,b)^p(h,h) \in G_{(h,h)}$. Then we have the following equation

$$((h,h)(a,b)^p(h,h))(h,h)(a,b)(x,y) = (h,h)(a,b)^{p+1}(x,y)(h,h) \in G_{(h,h)}.$$

Let (u,v) be an inverse element of $(h,h)(a,b)^{p+1}(x,y)(h,h) \in G_{(h,h)}$. Then, we deduce the equality $(h,h) = (h,h)(a,b)^{p+1}(x,y)(u,v)(h,h)$. Consequently, it is easy to see that $(h,h)(a,b)^{p+1}$ is a regular element of S. Thus, by theorem 4.1, we have proved that

$$J^*_{(h,h)} = J^*_{(h,h)(a,b)^{p+1}} = J^*_{(h,h)(a,b)^{p+1}(h,h)}.$$

Observe that $(h,h)(a,b)^{p+1}(h,h) \in G_{(h,h)}$ and $(h,h)(a,b)^0(h,h) \in G_{(h,h)}$, we can easily prove the first part of the theorem. By using similar method, we can also prove the second part of the theorem. By the fact $(a,b)^m \in G_{(e,e)}$ and by one part of (2), we obtain $(h,h)(e,e)(a,b)^m(h,h) \in G_{(h,h)}$. Let (u,v) be the inverse element of $(h,h)(e,e)(a,b)^m(h,h) \in G_{(h,h)}$. Then, we have the following equality.

$$(h,h) = (h,h)(e,e)(a,b)^m(u,v)(h,h),$$

and hence, $(h,h)(e,e)$ is a regular element of S. Finally, we deduce the following equality

$$J^*_{(h,h)} = J^*_{(h,h)(e,e)} = J^*_{(h,h)(e,e)(h,h)}.$$

and hence,$(h,h)(e,e)(h,h) \in G_{(h,h)}$. Similarly, we have

$$(3) \quad (h,h)(f,f)(h,h) \in G_{(h,h)}.$$

We now proceed to show that

$$(4) \quad (h,h)(e,e)(f,f)^p(h,h) \in G_{(h,h)},$$

for all $p \geqslant 0$. In fact, by $(h,h)(f,f)(h,h) \in G_{(h,h)}$, we immediately deduce that

$$(h,h) = (h,h)(e,e)(u,v)(f,f)(h,h) = (h,h)(f,f)(h,h)(e,e)(s,t).$$

Since $(f,f)(h,h)(e,e)$ and $(e,e)(f,f)(h,h)$ are regular elements of S and

$$J^*_{(h,h)} = J^*_{(e,e)(f,f)(h,h)} = J^*_{(f,f)(h,h)(e,e)} = J^*_{(h,h)(e,e)(f,f)(h,h)}.$$

We deduce that $(h,h)(e,e)(f,f)(h,h) \in G_{(h,h)}$, and so $(h,h)((e,e)(f,f))^p(h,h) \in G_{(h,h)}$ holds when $p = 1$. Now, we suppose that $p = 2$, then we have $S(h,h)S \subseteq S((e,e)(f,f))^p(h,h)(e,e)(f,f)S$. On the other hand, we have $S((e,e)(f,f))^p(h,h)(e,e)(f,f)S \subseteq S(h,h)S$. Hence, we derive the equality

$$S(h,h)S = S((e,e)(f,f))^p(h,h)(e,e)(f,f)S.$$

and consequently, we have

$$J^*_{(h,h)} = J^*_{((e,e)(f,f))^p(h,h)(e,e)(f,f)} = J^*_{(h,h)((e,e)(f,f))^{p+1}(h,h)}.$$

Now, and by induction hypothesis, we have $(h,h)((e,e)(f,f))^p(h,h) \in G_{(h,h)}$. Let q be the smallest positive integer such that $((e,e)(f,f))^q$ is a regular element of S. Then, we have

$$(5) \quad ((e,e)(f,f))^q \in J^*_{(e,e)(f,f)},$$

$$((e,e)(f,f))^q \in G_{(k,k)} \subseteq J^*_{(k,k)}.((k,k) \in E(S))$$

Now, we prove the following fact.

$$(6) \quad ((k,k)((a,b)(x,y))^p \in G_{(k,k)},$$

for every $p \geqslant 0$. If $p=1$, then by induction hypothesis, $((e,e)(f,f))^{2q} \in G_{(k,k)} \subseteq J^*_{(k,k)}$, and therefore we have the following

$$((e,e)(f,f))^{2q} = ((e,e)(f,f))^{2q}(u,v)((e,e)(f,f))^{2q}.((u,v) \in G_{(k,k)}$$

Furthermore, we deduce that

$$J^*_{(k,k)} = J^*_{(a,b)((e,e)(f,f))^{2q}(x,y)} = J^*_{(x,y)(a,b)((e,e)(f,f))^{2q}} = J^*_{(k,k)((e,e)(f,f))^q(x,y)(a,b)((e,e)(f,f))^q}.$$

Also we have

$$((e,e)(f,f))^q(a,b)(x,y)((e,e)(f,f))^q = (k,k)((e,e)(f,f))^q(x,y)(a,b)((e,e)(f,f))^q(k,k) \in G_{(k,k)}.$$

By using (1),(4) (5),(6) and let w be the smallest positive integer, then we deduce the following .

$$S((a,b)(x,y))^rS = S(h,h)S = S(h,h)((e,e)(f,f))^w(h,h)S \subseteq S((e,e)(f,f))^qS;$$

$$S((e,e)(f,f))^qS = S(k,k)S = S(k,k)((x,y)(a,b))^r(k,k)S \subseteq S((x,y)(a,b))^rS.$$

Recall that if

$$J^*_{(a,b)(x,y)} = J^*_{(x,y)(a,b)},$$

then $S((a,b)(x,y))^rS = S((e,e)(f,f))^qS$. Thus,

$$J^*_{(a,b)(x,y)} = J^*_{(e,e)(f,f)}.$$

Hence, our proof is completed. □

5 Characterizations of Cross*-completely Regular Semigroups

In this section, we characterize the cross*-completely regular semigroups.

Theorem 5.1. *The following statements are equivalent .*

(1) *S is a cross*-completely regular semigroup;*

(2) *S is a left cross*-regular semigroup and a right cross *-regular semigroup ;*

(3) *Every left(right) ideal of S is cross*-regular;*

(4) *The power of every element in S belongs to a sub-semigroup of S;*

(5) *For every element $(a,b) \in S$, there exists an $m \in Z^+$ such that $(a,b)^m \in (a,b)^m S(a,b)^{m+1}$.*

Proof. (1) ⇒ (2). It is correct by Lemma 2.4 and Corollary 2.5.

(2) ⇒ (3)is obvious. We only need prove (5)⇒(4) because (3)⇒4), (4)⇒(5) are obvious. Suppose that (5) holds. Then, for any element $(a,b) \in S$, there exists $m \in Z^+$ and $(x,y) \in S$ such that

$$(a)\ (a,b)^m = (a,b)^m(x,y)(a,b)^{2m}.$$

Now, we further deduce that

$$\text{(b) } (a,b)^m = (a,b)^m((x,y)(a,b)^m))^2(a,b)^{2m} = ... = (a,b)^m((x,y)(a,b)^m)^n(a,b)^{nm} = ...$$

□

By using the above equalities, we state the another theorem of cross*-completely regular semigroups which is related with their $J^* - classes$.

Theorem 5.2. *Let S be a cross*-completely regular semigroup. Then, for any $(a,b),(x,y) \in S$, we have* $J^*_{(a,b)(x,y)} = J^*_{(x,y)(a,b)}$.

Proof. The proof of the theorem is similar to Theorem4.2. □

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