South Asian Journal of Mathematics 2017, Vol. 7 (4): 182 ∼ 196
www.sajm-online.com ISSN 2251-1512 RESEARCH ARTICLE

On semi-invariant submanifolds of a trans-sasakian manifold with a semi-symmetric non-metric connection

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Received: May-10-2017; Accepted: Aug-8-2017 *Corresponding author
I Dedicate this paper to my daughter Diyana Danish on her First Birthday.

Abstract In this paper a semi-symmetric non-metric connection in a nearly trans-Sasakian manifold is defined and semi-invariant submanifolds of a nearly trans-Sasakian manifold endowed with a semi-symmetric non-metric connection is studied. Moreover, Nijenhuis tensor is calculated and integrability conditions of the distributions on semi-invariant submanifolds are discussed.

Key Words semi-invariant submanifolds, semi-symmetric non-metric connection, Weingarten equations

MSC 2010 53D25, 53D12

1 Introduction

The study of differential geometry of semi-invariant or contact CR-sub-manifolds, as a generalization of invariant and anti-invariant submanifolds, of an almost contact metric manifold was initiated by Bejancu ([4], [5]) and was fallowed by Kobayshi [11], Yano [17] and others Geometres (cf. [16], [14], [7], etc.). One has the notion of $\alpha-$Sasakian and $\beta-$Kenmotsu structures also [10]. Oubina introduced a new class of almost contact Riemannian manifolds known as trans-Sasakian manifolds [15]. Both $\alpha-$Sasakian and $\beta-$Kenmotsu belong to this new class. C. Gherghe [9] introduced a nearly trans-Sasakian structure of type (α, β), which generalizes tran-Sasakian structure in the same sense as nearly Sasakian generalizes Sasakian one. A trans-Sasakian structure is always a nearly trans-Sasakian structure. Moreover, nearly trans-Sasakian structure of type (α, β) is nearly-Sasakian ([2], [7]) or nearly Kenmotsu [1] or nearly cosymplectic [6] accordingly as $\beta = 0$ or $\alpha = 0$; or $\alpha = 0 = \beta$. Matsumoto et al. in [14] studied semi-invariant submanifold of a trans-Sasakian manifolds and after that Jeong-Sik Kim and others studied the semi-invariant submanifolds of nearly trans-Sasakian manifolds [12]. In this paper, we study semi-invariant submanifolds of nearly trans-Sasakian manifolds with semi-symmetric non-metric connection.

In 1924, A. Friedmann and J.A. Schouten[8] introduced the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be a semi-symmetric connection if its torsion tensor T is of the form

$$T(X, Y) = \eta(Y)X - \eta(X)Y,$$

Citation: Modhd Danish Siddiqi, Study of semi-invariant submanifolds of a trans-sasakian manifold with a semi-symmetric on metric connection, South Asian J Math, 2017, 7(4), 182-196.

where η is a 1-form.

The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise, it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. Some properties of semi-invariant submanifolds, hypersurfaces and submanifolds with semi-symmetric non-metric connection were studied in ([1], [2], [3]) respectivily.

In this paper, we study semi-invariant submanifolds of nearly trans-Sasakian manifolds with a semi-symmetric non-metric connection. This paper is organized as follows. In section 2, we give a brief introduction of nearly trans-Sasakian manifold. In section 3, we recall some necessary details about semi-invarint submanifolds . In section 4, we derive Nijenhuis tensor for nearly trans-Sasakian manifold with semi-symmetric non-metric connection. In section 5, some basic results on nearly trans-Sasakian manifold with semi-symmetric non-metric connection are obtained . In section 6, 7 and 8, integrability of some distributions on nearly trans-Sasakian manifold are discussed.

2 Nearly trans-Sasakian manifold

Let $\bar{M}$ be an almost contact metric manifold [6] with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a compatiable Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta o \phi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X) \tag{2.3}$$

for all vector fields X, Y on T$\bar{M}$. There are two known classes of almost contact metric, namely Sasakian and Kenmotsu manifolds. Sasakian manifolds are characterized by the tensorial relation

$$(\bar{\bar{\nabla}}_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X,$$

while Kenmotsu manifolds are given by the tensor equation

$$(\bar{\bar{\nabla}}_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a trans-Sasakian structure [15] if $(\bar{M} \times R, J, G)$ belongs to the class W_4 of the Gray-Hervella classification of almost Hermitian manifolds [9], where J is the almost complex structure on $\bar{M} \times R$ defined by

$$J(X, ad/dt) = (\phi X - a\xi, \eta(X)d/dt)$$

for all vector fields X on $\bar{M}$ and smooth function 'a' on $\bar{M} \times R$ and G is the product metric on $\bar{M} \times R$. This may be expressed by the condition [13]

$$(\bar{\bar{\nabla}}_X \phi)(Y) = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.4}$$

for some smooth functions α and β on $\bar{M}$ and we say that the trans-Sasakian structure is of type (α, β). From the formula (2.4), it follows that [4]

$$\bar{\bar{\nabla}}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{2.5}$$

The class $C_6 \oplus C_5$ [15] coincides with the class of trans-Sasakian structures of type (α, β).

We note that trans-Sasakian structures of type (0,0) are Cosymplectic [3], trans-Sasakian structures of tpye $(\alpha, 0)$ are α - Sasakian [10]. Recently, C. Gherghe [9] introduced a nearly trans-Sasakian structure of type (α, β). An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$, is called a nearly trans-Sasakian structure [2, 7] if

$$\begin{aligned} &(\bar{\bar{\nabla}}_X \phi)(Y) + (\bar{\bar{\nabla}}_Y \phi)(X) \\ &= \alpha(2g(X, Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y). \end{aligned} \tag{2.6}$$

A trans-Sasakian structure is always a nearly trans-Sasakian structure. Moreover, a nearly trans-Sasakian structure of type (α, β) is nearly-Sasakian [15] or nearly Kenmotsu [1] or nearly Cosymplectic [6] according as $\beta = 0$, $\alpha = 1$; or $\alpha = 0$, $\beta = 1$; or $\alpha = 0$, $\beta = 0$ respectively.

A nearly trans-Sasakian structure of type (α, β) will be called α-Sasakian (resp. nearly β-Kenmotsu) if $\beta = 0$(resp. $\alpha = 0$). Thus the structural equations for nearly α-Sasakian, naraly Sasakian, nearly β-Kenmotsu, nearly Kenmotsu and nearly cosympletic manifolds are given by

$$(\bar{\bar{\nabla}}_X \phi)Y + (\bar{\bar{\nabla}}_Y \phi)X = \alpha(2g(X, Y)\xi - \eta(Y)X - \eta(X)Y), \tag{2.7}$$

$$(\bar{\bar{\nabla}}_X \phi)Y + (\bar{\bar{\nabla}}_Y \phi)X = (2g(X, Y)\xi - \eta(Y)X - \eta(X)Y), \tag{2.8}$$

$$(\bar{\bar{\nabla}}_X \phi)Y + (\bar{\bar{\nabla}}_Y \phi)X = -\beta(\eta(Y)\phi X + \eta(X)\phi Y), \tag{2.9}$$

$$(\bar{\bar{\nabla}}_X \phi)Y + (\bar{\bar{\nabla}}_Y \phi)X = -(\eta(Y)\phi X + \eta(X)\phi Y), \tag{2.10}$$

$$(\bar{\bar{\nabla}}_X \phi)Y + (\bar{\bar{\nabla}}_Y \phi)X = 0 \tag{2.11}$$

respectively.

We can Owing to the existence of 1-form η, we can define a semi-symmetric non-metric connection in contact manifold by

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y + \eta(Y)X. \tag{2.12}$$

for all $X, Y \in TM$. From (2.6) and (2.12), we have

$$\begin{aligned} (\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X &= \alpha(2g(X, Y)\xi) - \eta(Y)X - \eta(X)Y) \\ &\quad -(\beta + 1)(\eta(Y)\phi X + \eta(X)\phi Y) \end{aligned} \tag{2.13}$$

A trans-Sasakian structure is always a nearly trnas-Sasakian structure. Moreover, a nearly trans-Sasakian structure with semi-symmetric non-metric connection of type (α, β) is nearly Sasakian [15] or nearly Kenmotsu [1] or nearly Cosympletic [3] according as $\beta = 0$, $\alpha = 1$; or $\alpha = 0$ $\beta = 1$; or $\alpha = 0, \beta = 0$ recpectively.

A nearly trans-Sasakian structure of type (α, β) will be called nearly α-Sasakian (resp. nearly β-Kenmotsu) if β=0 (resp. $\alpha = 0$).

Thus the structural equations for nearly $\alpha-$Sasakian, nearly Sasakian, nearly $\beta-$Kenmotsu, nearly Kenmotsu and nearly cosympletic manifolds are given by

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = \alpha(2g(X,Y)\xi) - \eta(Y)X - \eta(X)Y) \tag{2.14}$$

$$-(\eta(Y)\phi X + \eta(X)\phi Y)$$

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = (2g(X,Y)\xi) - \eta(X)Y - \eta(X)Y) \tag{2.15}$$

$$-(\eta(Y)\phi X + \eta(X)\phi Y)$$

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = -(\beta+1)(\eta(Y)\phi X + \eta(X)\phi Y) \tag{2.16}$$

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = -2(\eta(Y)\phi X + \eta(X)\phi Y) \tag{2.17}$$

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = -(\eta(Y)\phi X + \eta(X)\phi Y). \tag{2.18}$$

3 Semi-invariant submanifolds.

Let M be a submanifold of a Riemannian manifold $\bar{M}$ with Riemannian metric g. Then Gauss and Weingarten formulae are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \qquad (X, Y \in TM), \tag{3.1}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N + \eta(N)X \qquad (N \in T^{\perp}M), \tag{3.2}$$

where $\bar{\nabla}, \nabla$ and $\nabla^{\perp}$ respectively the semi-symmetric non-metric, induced connection and induced normal connections in $\bar{M}$, M and the normal bundel $T^{\perp}M$ of M respectively and h is the second fundamental form related to A by

$$g(h(X,Y), N) = g(A_N X, Y). \tag{3.3}$$

Moreover, if ϕ is a $(1,1)$ tensor field on $\bar{M}$, $X \in TM$ and $N \in T^{\perp}M$, we have

$$(\bar{\nabla}_X\phi)Y = ((\nabla_X P)Y - A_{FY}X - th(X,Y)) \tag{3.4}$$

$$+((\nabla_X F)Y + h(X,PY) - fh(X,Y)) + \eta(FY)X,$$

$$(\bar{\nabla}_X\phi)N = ((\nabla_X t)N - A_{fN}X - PA_N X - \eta(N)PX) \tag{3.5}$$

$$+((\nabla_X f)N + h(X,tN) - FA_N X - \eta(N)FX) + \eta(fN)X,$$

where

$$\phi X = PX + FX, \qquad (PX \in TM, FX \in T^{\perp}M), \tag{3.6}$$

$$\phi N = tN + fN, \qquad (tN \in TM, fN \in T^{\perp}M), \tag{3.7}$$

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y, \qquad (\nabla_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y,$$

$$(\nabla_X t)N = {}_X tN - t\nabla^{\perp}_X N, \qquad (\nabla_X f)N = \nabla^{\perp}_X fN - f\nabla^{\perp}_X N.$$

The submanifold M is known to be totally geodesic in $\bar{M}$ if $h = 0$, minimal in $\bar{M}$ if $H = trace(h)/dim(M) = 0$ and totally umbilical in $\bar{M}$ if $h(X,Y) = g(X,Y)H$.
For a distribution D on M, M is said to be *D-totally geodesic* if for all $X, Y \in D$, we have $h(X,Y) = 0$. If for all $X, Y \in D$, we have $h(X,Y) = g(X,Y)K$, for some normal vector K, then M is called *D-totally umbilical*. For two distributions D and $\mathcal{E}$ defined on M, M is said to be $(D,\ \mathcal{E})$-*mixed totally geodesic* $if\ for\ all$ $X \in D$ and $Y \in \mathcal{E}$ we have $h(X,Y) = 0$. Let D and $\mathcal{E}$ be two distributions defined on a manifold M. We say that D is $\mathcal{E}$-parallel if for all $X \in \mathcal{E}$ and $Y \in D$, we have $\nabla_X Y \in D$. If D is $D-$parallel then it is called *autoparallel*. D is called $X-$*parallel* for some $X \in TM$, if for all $Y \in D$ we have $\nabla_X Y \in D$. D is said to be if for all $X \in TM$ and $Y \in D$, $\nabla_X Y \in D$. If a distribution D on M is autoparallel, then it is clearly integrable and by Gauss formula D is totally geodesic in M. If D is parallel then orthogonal complementary distribution $D^{\perp}$ is also parallel which implies that D is parallel if and only if $D^{\perp}$is parallel.

In this case M is locally the product of the leaves of D and $D^{\perp}$. Let M be a submanifold of an almost contact metric manifold. If $\xi \in TM$ then we write $TM = \{\xi\} \oplus \{\xi\}^{\perp}$, where $\{\xi\}$ is the distribution spanned by ξ and $\{\xi\}^{\perp}$ is the complementary orthogonal distribution of $\{\xi\}$ in M. Then one gets

$$P\xi = 0 = F\xi, \qquad \eta o P = 0 = \eta o F, \tag{3.8}$$

$$P^2 + tF = -I + \eta \otimes \xi, \qquad FP + fF = 0, \tag{3.9}$$

$$f^2 + Ft = -I, \qquad tf + Pt = 0. \tag{3.10}$$

A submanifold M of an almost contact metric manifold $\bar{M}$ with $\xi \in TM$ is called semi-invariant submanifold (Bejancu[2]) of $\bar{M}$ if there exists two differentiable distributions D^1 and D^0 on M such that

(i) $TM = D^1 \oplus D^0 \oplus \{\xi\}$,

(ii) *the distribution* D^1 *is invariant by* ϕ, *that is* $\phi(D^1) = D^1$, and

(iii) *the distribution* D^0 *is anti* $-$ *invariant by* ϕ, *that is* $\phi(D^0) \subseteq T^{\perp}M$.

For $X \in TM$, we can write

$$X = U^1 X + U^0 X + \eta(X)\xi, \tag{3.11}$$

where U^1 and U^0 are the projection operators of TM on D^1 and D^0 respectively. A semi-invariant submanifold of an almost contact metric manifold becomes an invariant submanifold [2, 3] (resp. anti-invariant submanifold [2, 3]) if $D^0 = \{0\}$ (resp. $D^1 = \{0\}$). Moreover, we have

4 Nijenhuis tensor

An almost contact metric manifold is said to be normal ([3]) if the torsion tensor $N^{(1)}$ vanishes:

$$N^{(1)} \equiv [\phi, \phi] + 2d\eta \otimes \xi = 0, \tag{4.1}$$

where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ and d denotes the exterior derivative operator.

In this section we obtain expression for Nijenhuis tensor $[\phi,\phi]$ of the structure tensor field ϕ given by

$$[\phi,\phi](X,Y) = ((\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X) - \phi((\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X) \tag{4.2}$$

in a nearly trans-Sasakian manifold. In particular, we derive the expressions for the Nijenhuis tensor $[\phi,\phi]$ in nearly Sasakian manifold and nearly Kenmotsu manifolds.

First, we need the following Lemma.

Lemma 4.1 *In an almost contact metric manifold we have*

$$(\bar{\nabla}_Y\phi)\phi X = -(\phi(\bar{\nabla}_Y\phi)X + (\bar{\nabla}_Y\eta)X)\xi + \eta(X)\bar{\nabla}_Y\xi \tag{4.3}$$

Proof. For X,$Y \in TM$, we have

$$\begin{aligned}(\bar{\nabla}_Y\phi)\phi X &= \bar{\nabla}_Y(\phi^2X) - \phi(\bar{\nabla}_Y\phi X) + \phi(\phi\bar{\nabla}_YX) - \phi^2\bar{\nabla}_YX\\ &= \bar{\nabla}_Y(-X+\eta(X)\xi) - \phi(\bar{\nabla}_Y\phi X)\\ &\quad +\phi(\phi\bar{\nabla}_YX) - (-\bar{\nabla}_YX + \eta(\bar{\nabla}_YX)\xi),\end{aligned}$$

which gives the equation (4.3).

Now, we prove the following theorem.

Theorem 4.2 *In a nearly trans-Sasakian manifold with semi-symmetric non-metric connection, the Nijenhuis tensor* $[\phi,\phi]$ *of* ϕ *is given by*

$$\begin{aligned}[\phi,\phi](X,Y) &= 4\phi(\bar{\nabla}_Y\phi)X + 2d\eta(X,Y)\xi - \eta(X)\bar{\nabla}_Y\xi + \eta(Y)\bar{\nabla}_X\xi \\ &\quad +\alpha(\eta(Y)\phi X + 3\eta(X)\phi Y) + 4g(\phi X,Y)\xi\\ &\quad +4(\beta+1)(\eta(X)\eta(Y)\xi) + (\beta+1)(-\eta(Y)X - 3\eta(X)Y)\end{aligned} \tag{4.4}$$

Proof. Using lemma(4.1) and $\eta o\phi = 0$ in (2.6) we get

$$\begin{aligned}(\bar{\nabla}_{\phi X}\phi)Y &= \phi(\bar{\nabla}_Y\phi)X - ((\bar{\nabla}_Y\eta)X)\xi - \eta(X)\bar{\nabla}_Y\xi\\ &\quad +\alpha(2g(\phi X,Y)\xi - \eta(Y)\phi X)\\ &\quad -(\beta+1)(-\eta(Y)X + \eta(Y)\eta(X)\xi)\end{aligned} \tag{4.5}$$

Thus

$$\begin{aligned}[\phi,\phi](X,Y) &= (((\bar{\nabla}_{\phi X}\phi)Y + \phi((\bar{\nabla}_Y\phi)X))) - ((\bar{\nabla}_{\phi Y}\phi)X + \phi(\bar{\nabla}_X\phi)Y))\\ &= 2\phi(\bar{\nabla}_Y\phi)X - ((\bar{\nabla}_Y\eta)X)\xi - \eta(X)\bar{\nabla}_Y\xi\\ &\quad +((\bar{\nabla}_X\eta)Y)\xi + \eta(Y)\bar{\nabla}_X\xi - \alpha(2g(\phi Y,X)\xi - \eta(Y)\phi X)\\ &\quad -(\beta+1)(-\eta(Y)X + \eta(Y)\eta(X)\xi) - 2\phi(\bar{\nabla}_X\phi)Y\end{aligned}$$

$$-(\beta+1)(-\eta(X)Y+\eta(X)\eta(Y)\xi)$$

$$+\alpha(2g(\phi X,Y)\xi-\eta(Y)\phi X))$$

$$=2\phi((\bar{\nabla}_Y\phi)X-(\bar{\nabla}_X\phi)Y)+2d\eta(X,Y)\xi-\eta(X)\bar{\nabla}_Y\xi+\eta(Y)\bar{\nabla}_X\xi$$

$$-(\beta+1)(-\eta(X)Y-\eta(Y)X)+2(\beta+1)(\eta(X)\eta(Y)\xi)$$

$$+\alpha(4g(\phi X,Y)\xi-\eta(Y)\phi X+\eta(X)\phi Y)$$

$$=2\phi((\bar{\nabla}_Y\phi)X+(\bar{\nabla}_Y\phi)X)-2\alpha\phi(2g(X,Y)\xi-\eta(Y)X-\eta(X)Y)\xi$$

$$+\alpha(4g(\phi X,Y)\xi+\eta(Y)\phi X+\eta(X)\phi Y)-(\beta+1)(-\eta(Y)\phi X-\eta(X)\phi Y)$$

$$-\eta(X)\bar{\nabla}_Y\xi+\eta(Y)\bar{\nabla}_X\xi-(\beta+1)(-\eta(X)Y-\eta(Y)X)$$

$$+2(\beta+1)(\eta(X)\eta(Y)\xi)+2d\eta(X,Y)$$

$$=4\phi(\bar{\nabla}_Y\phi)X-\eta(X)\bar{\nabla}_Y\xi+\eta(Y)\bar{\nabla}_X\xi$$

$$+\alpha(\eta(Y)\phi X+3\eta(X)\phi Y+4g(\phi X,Y)\xi)$$

$$+2d\eta(X,Y)\xi-(\beta+1)(\eta(Y)\phi^2X+3\eta(X)\phi^2Y)$$

which implies the equation(4.4).
From equation (4.4), we get

$$\eta(N^1(X,Y))=4d\eta(X,Y)-4\alpha g(X,\phi Y). \tag{4.6}$$

In particular, if X and Y are perpendicular to ξ, then (4.4) gives

$$[\phi,\phi](X,Y)=4\phi(\bar{\nabla}_Y\phi)X-2\eta([X,Y])\xi. \tag{4.7}$$

Corollary 4.3. In a nearly Sasakian manifold with semi-symmetric non-metric connnection the Nijenhuis tensor $[\phi,\phi]$ of ϕ is given by

$$[\phi,\phi](X,Y)=4\phi(\bar{\nabla}_Y\phi)X+2d\eta(X,Y)\xi-\eta(X)\bar{\nabla}_Y\xi+\eta(Y)\bar{\nabla}_X\xi \tag{4.8}$$

$$-\eta(X)Y-3\eta(Y)X+\eta(Y)\phi X+3\eta(X)\phi Y-4g(X,\phi Y)\xi)$$

$$-4\eta(X)\eta(Y)\xi).$$

Consequently,

$$\eta(N^1(X,Y))=4d\eta(X,Y)-4g(X,\phi Y)-4\eta(X)\eta(Y). \tag{4.9}$$

In particular, if X and Y are perpendicular to ξ, then

$$[\phi,\phi](X,Y)=4\phi(\bar{\nabla}_Y\phi)X-2\eta([X,Y])\xi-4g(X,\phi Y)\xi \tag{4.10}$$

$$-4\eta(X)\eta(Y)\xi.$$

Corollary 4.4. In a nearly Kenmotsu manifold with semi-symmetric non-metric connection the Nijenhuis tensor $[\phi, \phi]$ of ϕ is given by

$$[\phi,\phi](X,Y) = 4\phi(\bar{\nabla}_Y\phi)X + 2d\eta(X,Y)\xi - \eta(X)\bar{\nabla}_Y\xi + \eta(Y)\bar{\nabla}_X\xi \tag{4.11}$$

$$+8\eta(X)\eta(Y)\xi - 2(\eta(Y)\phi X + 3\eta(X)\phi Y).$$

Consequently,

$$\eta(N^1(X,Y)) = 4d\eta(X,Y). \tag{4.12}$$

In particular, if X and Y are perpendicular to ξ, then

$$[\phi,\phi](X,Y) = 4\phi(\bar{\nabla}_Y\phi)X - 2\eta([X,Y])\xi. \tag{4.13}$$

5 Some basic results

Let M be a submanifold of a nearly trans-Sasakian manifold. Using (3.4), (3.6) in (2.17) for $X, Y \in TM$, we get

$$\alpha(2g(X,Y)\xi - \eta(X)Y - \eta(Y)X) - (\beta+1)(\eta(Y)PX + \eta(Y)FX + \eta(X)PY + \eta(X)FY)$$

$$+2\eta(X)\eta(Y)\xi$$

$$= (\nabla_X P)Y + (\nabla_Y P)X - A_{FY}X - A_{FX}Y - 2th(X,Y) - 2fh(X,Y)$$

$$+(\nabla_X F)Y + (\nabla_Y F)X + h(X,PY) + h(PX,Y)$$

$$+\eta(FY)X + \eta(FX)Y. \tag{5.1}$$

Thus, we have following.

Proposition 5.1 Let M be a submanifold of a nearly trans-Sasakian manifold with semi-symmetric non-metric connetion then for all $X, Y \in TM$, we have

$$(\nabla_X P)Y + (\nabla_Y P)X - A_{FY}X - A_{FX}Y - 2th(X,Y) \tag{5.2}$$

$$= \alpha(2g(X,Y)\xi - \eta(X)Y - \eta(Y)X)$$

$$-(\beta+1)(\eta(Y)PX + +\eta(X)PY)$$

and

$$(\nabla_X F)Y + (\nabla_Y F)X + h(X,PY) + h(PX,Y) - 2fh(X,Y) \tag{5.3}$$

$$= -(\beta+1)(\eta(Y)FX + \eta(X)FY) - \eta(FX)Y - \eta(FY)X$$

Now, we state the following proposition

Proposition 5.2 Let M be a submanifold of a nearly trans-Sasakian manifold with semi-symmetric non-metric connetion. Then

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X - \phi[X,Y] = 2((\nabla_X P)Y - A_{FY}X - th(X,Y) + \eta(FY)X) \tag{5.4}$$

$$+2((\nabla_X F)Y + h(X,PY) - fh(X,Y) + \eta(FX)Y)$$
$$+\alpha(\eta(Y)X + \eta(X)Y - (2g(X,Y)\xi)$$
$$+(\beta+1)(\eta(Y)PX + \eta(X)PY)$$
$$+(\beta+1)(\eta(Y)FX + \eta(X)FY).$$

Consequently,

$$P[X,Y] = -\nabla_X PY - \nabla_Y PX + A_{FX}Y + A_{FY}X + 2P\nabla_X Y + 2th(X,Y) \tag{5.5}$$
$$+(\alpha(\eta(Y)X + \eta(X)Y - 2g(X,Y)\xi)$$
$$-(\beta+1)(\eta(Y)PX + \eta(X)PY),$$
$$F[X,Y] = -\nabla^{\perp}_X FY - \nabla^{\perp}_Y FX - h(X,PY) - h(PX,Y) + 2F\nabla_X Y \tag{5.6}$$
$$+2fh(X,Y) - (\beta+1)(\eta(Y)FX + \eta(X)FY) - \eta(FY)X + \eta(FX)Y$$

for all $X, Y \in\ TM$.

The proof is straightforward.

Proposition 5.3 Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold with semi-symmetric non-metric connection. Then (P,ξ,η,g) is a nearly trans-Sasakian structure on the distribution $D^1 \oplus \{\xi\}$ $\ if \ \ th(X,Y) = 0 \ \ for\ all$ X,Y∈ $D^1 \oplus \{\xi\}$.

Proof. From $D^1 \oplus \{\xi\} = Ker(F)$ and (3.9) we have $P^2 = -I + \eta \oplus \xi$ on $D^1 \oplus \{\xi\}$. We also get $P\xi = 0$, $\eta(\xi) = 1$, $\eta o P = 0$. Using $D^1 \oplus \{\xi\} = Ker(F)$ and $th(X,Y) = 0$ in (5.2) we get

$$(\nabla_X P)Y + (\nabla_Y P)X = -(\beta+1)(\eta(Y)PX + \eta(X)PY) \tag{5.7}$$
$$+\alpha(2g(X,Y)\xi - \eta(X)Y - \eta(Y)X),$$
$$X, Y \in D^1 \oplus \{\xi\}$$

This completes the proof.

Theorem 5.4 Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold with semi-symmetric semi-metric connection,we have

(i) if $D^0 \oplus \{\xi\}$ is autoparallel then

$$A_{FX}Y + A_{FY}X + 2th(X,Y) = 0, \qquad X, Y \in D^0 \oplus \{\xi\}, \tag{5.8}$$

(ii) if $D^1 \oplus \{\xi\}$ is autoparallel then

$$h(X,PY) + h(PX,Y) = 2fh(X,Y), \qquad X, Y \in D^1 \oplus \{\xi\}. \tag{5.9}$$

Proof. In view of (5.2) and autoparallelness of $D^0 \oplus \{\xi\}$ we get (1), while in view of (5.3) and appropriateness of $D^1 \oplus \{\xi\}$, we get (2).

In view of proposition(5.3) and (*ii*) (Theorem 5.4), we get

Theorem 5.5. Let M be a submanifold of nearly trans-Sasakian manifold with semi-symmetric non-metric connection with $\xi \in TM$. If M is invariant then M is nearly trans-Sasakian. Moreover,

$$h(X, PY) + h(PX, Y) - 2fh(X, Y) = 0,$$

X,Y∈ TM.

6 Integrability of the distribution $D^1 \oplus \{\xi\}$

We begin with a lemma

Lemma 6.1. Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold with semi-symmetric non-metric connection for $X, Y \in \mathrm{D}^1 \oplus \{\xi\}$, we get

$$F[X, Y] = -h(X, PY) - h(PX, Y) + 2F\nabla_X Y + 2fh(X, Y) \tag{6.1}$$

or equivalently

$$-h(X, PX) + F\nabla_X X + fh(X, X) = 0 \tag{6.2}$$

and

$$\eta(FX)Y = \eta(FY)X$$

for all $X, Y \in D^1 \oplus \{\xi\}$.

Proof. Equation (6.1) follows from $D^1 \oplus \{\xi\} = Ker(F)$ and (5.6) equivalence of (6.1) and $D^1 \oplus \{\xi\} = Ker(F)$. We can state following theorem.

Theorem 6.2. The distribution $D^1 \oplus \{\xi\}$ on semi-invariant submanifold of a nearly trans-Sasakian manifold with semi symmetric non- metric connection is integrable if and only if

$$h(X, PY) + h(PX, Y) = 2(F\nabla_X Y + fh(X, Y)). \tag{6.3}$$

Now, we need the following

Definition 6.3. Let M be a Riemannian manifold with the Riemannian connection ∇. A distribution D on M is said to be *nearly autoparallel* if for all $X, Y \in D$ we have $(\nabla_X Y + \nabla_Y X) \in D$ or equivalently $\nabla_X X \in D$.

Thus we have the following flow chart:

Parallel⇒ $Autoparallel \Rightarrow Nearly\ autoparallel$,

$Parallel \Rightarrow Integrable$,

$Autoparallel \Rightarrow Integrable$, and

$Nearly\ autoparallel + Integrable \Rightarrow Autoparallel.$

Theorem 6.4. Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold with semi-symmetric non-metric connection then following four statements

(a) the distribution $D^1 \oplus \{\xi\}$ is autoparallel,

(b) $h(X, PY) + h(PX, Y) = 2fh(X, Y) \qquad for X, Y \in D^1 \oplus \{\xi\}$,

(c) $h(X, PX) = fh(X, X) \qquad for X \in D^1 \oplus \{\xi\}$,

(d) *the distribution* $D^1 \oplus \{\xi\}$ *is nearly autoparallel,*

are related by (a)$\Rightarrow$ $(b) \Leftrightarrow (c) \Rightarrow (d)$. *In particular,* if $D^1 \oplus \{\xi\}$ is integrable then the above four statements are equivalent.

Let $X, Y \in D^1 \oplus \{\xi\}$. Using (2.1) and (3.6) in (4.1) and we get

$$N^1(X, Y) = 2d\eta(X, Y)\xi + [\phi X, \phi Y] - [X, Y] + \eta([X, Y])\xi \tag{6.4}$$

$$-P([X, \phi X] + [\phi X, Y]) - F([X, \phi Y] + [\phi X, Y]).$$

On the other hand from equation (4.5), we have

$$(\bar{\nabla}_{\phi X}\phi)Y = \phi(\bar{\nabla}_Y \phi)X - ((\bar{\nabla}_Y \eta)X)\xi - \eta(X)\bar{\nabla}_Y \xi + \alpha(2g(\phi X, Y)\xi - \eta(Y)\phi X)$$

$$-(\beta + 1)(-\eta(Y)X + \eta(X)\eta(Y)\xi)$$

which implies that

$$(\bar{\nabla}_{\phi X}\phi)Y - (\bar{\nabla}_{\phi Y}\phi)X = \phi(\bar{\nabla}_Y \phi)X - (\bar{\nabla}_X \phi)Y + 2d\eta(X, Y)\xi$$

$$-\eta(X)U^1\nabla_Y \xi - \eta(X)U^0\nabla_Y \xi + \eta(Y)U^1\nabla_X \xi(6.5)$$

$$+\eta(Y)U^0\nabla_X \xi - \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi)$$

$$+\alpha(\eta(X)\phi X - \eta(Y)\phi X) - (\beta + 1)(-\eta(X)Y - \eta(Y)X)$$

$$-(\beta + 1)(2\eta(X)\eta(Y)\xi.$$

Next, we can easily get

$$\phi(\bar{\nabla}_Y \phi)X = \phi\bar{\nabla}_Y \phi X - \phi^2\bar{\nabla}_Y X \tag{6.6}$$

$$= \phi(\nabla_Y \phi X + h(Y, \phi X)) + \bar{\nabla}_Y X - \eta(\bar{\nabla}_Y X)\xi$$

so that

$$\phi((\bar{\nabla}_Y \phi)X - (\bar{\nabla}_X \phi)Y) = -[X, Y] + \eta([X, Y])\xi + P(\nabla_Y \phi X - \nabla_X \phi Y) \tag{6.7}$$

$$+F(\nabla_Y \phi X - \nabla_X \phi Y) + \phi(h(Y, \phi X) - h(X, \phi Y))$$

In view of (6.5) and (6.7), we get

$$N^1(X,Y) = -2[X,Y] + 2P(\nabla_Y \phi X - \nabla_X \phi Y) + 2F(\nabla_Y \phi X - \nabla_X \phi Y) \tag{6.8}$$
$$+2\phi(h(Y,\phi X) - h(X,\phi Y)) - \eta(X)U^1\nabla_Y\xi - \eta(X)U^0\nabla_Y\xi + \eta(Y)U^1\nabla_X\xi$$
$$+\eta(Y)U^0\nabla_X\xi - \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi) + 4d\eta(X,Y)\xi$$
$$+\alpha(-\eta(X)\phi X - \eta(Y)\phi X) - (\beta+1)(-\eta(X)Y - \eta(Y)X)$$
$$+2\eta([X,Y])\xi - (\beta+1)(2\eta(X)\eta(Y)\xi).$$

Theorem 6.5. The distribution $D^1 \oplus \{\xi\}$ is integrable on a semi-invariant submanifold M of nearly trans-Sasakian manifold with semi symmetric non-metric connection if and only if

$$N^1(X,Y) \in D^1 \oplus \{\xi\}, \tag{6.9}$$

$$2(h(Y,\phi X) - h(X,\phi Y)) = \eta(X)(\phi U^0\nabla_Y\xi + \alpha U^0 Y + (\beta+1)\phi U^0 Y + fh(Y,\xi)) \tag{6.10}$$
$$-\eta(Y)(\phi U^0\nabla_X\xi + \alpha U^0 X + (\beta+1)\phi U^0 X + fh(X,\xi))$$

for all $X, Y \in D^1 \oplus \{\xi\}$.

Proof. Let $X, Y \in D^1 \oplus \{\xi\}$, if $D^1 \oplus \{\xi\}$ is integrable, then (6.9) is true and from (6.8), we get

$$0 = 2F(\nabla_Y\phi X - \nabla_X\phi Y) + 2\phi(h(Y,\phi X) - h(X,\phi Y) + \eta(Y)U^0\nabla_X\xi - \eta(X)U^0\nabla_Y\xi$$
$$+\eta(Y)h(X,\xi) - \eta(X)h(Y,\xi) - \alpha(-\eta(X)FY + \eta(Y)FX) - (\beta+1)(\eta(Y)X - \eta(X)Y).$$

Applying ϕ to the above equation,

$$0 = -2U^0(\nabla_Y\phi X - \nabla_X\phi Y) + 2(h(Y,\phi X) - h(X,\phi Y) + \eta(Y)\phi U^0\nabla_X\xi$$
$$+\eta(Y)th(X,\xi) + \eta(Y)fh(X,\xi) - \eta(X)th(Y,\xi) - \eta(X)fh(Y,\xi)$$
$$+\alpha(\eta(X)U^0Y + \eta(Y)U^0X) - (\beta+1)\phi(\eta(X)U^0Y + \eta(Y)U^0X).$$
$$-\eta(X)\phi U^0\nabla_Y\xi$$

Hence taking the normal part, we get (6.10).
Conversely, let (6.9) and (6.10) be true. Using (6.10) in (6.8), we get

$$0 = -2U^0[X,Y] + 2F(\nabla_Y\phi X - \nabla_X\phi Y) + 2\phi(h(Y,\phi X) - h(X,\phi Y) + \eta(Y)U^0\nabla_X\xi$$
$$-\eta(X)U^0\nabla_Y\xi + \eta(Y)h(X,\xi) - \eta(X)h(Y,\xi)$$

.

$$+\alpha(-\eta(X)FY - \eta(Y)FX) - (\beta+1)(\eta(Y)X - \eta(X)Y)$$

Applying ϕ to the above equation and using (6.10), we get $\phi U^0[X,Y] = 0$, from which we get $U^0[X,Y] = 0$. Hence $D^1 \oplus \{\xi\}$ is integrable.

If $\bar{M}$ is a trans-Sasakian manifold then for all $X \in D^1 \oplus \{\xi\}$ it is know that $h(X, \xi) = 0$ and $U^0 \nabla_X \xi = 0$. Hence in view of previous theorem, we have

Corollary 6.6. If M is a semi-invariant submanifold of a trans-Sasakian manifold with semi-symmetric non-metric connection then the distribution $D^1 \oplus \{\xi\}$ is integrable if and only if for all $X, Y \in D^1 \oplus \{\xi\}$

$$h(X, \phi Y) = h(Y, \phi X).$$

7 Integrability of the distribution $D^0 \oplus \{\xi\}$

Lemma 7.1. Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold with semi-symmetric non-metric connection. Then

$$3(A_{FX}Y - A_{FY}X) = P[X, Y] - (\beta + 1)(\eta(Y)PX + \eta(X)PY) \tag{7.1}$$

for all X, Y$\in$D$^0 \oplus \{\xi\}$.

Proof. For $X, Y \in D^0 \oplus \{\xi\}$ and $Z \in TM$, we have

$$-A_{\phi X}Z + \nabla^{\perp}_Z \phi X = \bar{\nabla}_Z \phi X = (\bar{\nabla}_Z \phi)X + \phi(\bar{\nabla}_Z)X.$$

Using Equation (2.13)

$$\begin{aligned} -A_{\phi X}Z + \nabla^{\perp}_Z \phi X = & -(\bar{\nabla}_X \phi)Z + \alpha(2g(X, Z)\xi - \eta(X)Z - \eta(Z)X) \\ & +(\beta + 1)(\eta(X)\phi Z + \eta(Z)\phi X) \\ & +\phi\bar{\nabla}_Z X + \phi h(Z, X) \end{aligned}$$

so that

$$\begin{aligned} \phi h(Z, X) = & -A_{\phi X}Z + \nabla^{\perp}_Z \phi X + (\bar{\nabla}_X \phi)Z - \alpha(2g(X, Z)\xi - \eta(X)Z - \eta(Z)X) \\ & +(\beta + 1)(-\eta(X)\phi Z - \eta(Z)\phi X) - \phi\bar{\nabla}_Z X \end{aligned}$$

and hence we have

$$g(\phi h(Z, X), Y) = -g(A_{\phi X}Y, Z) - g((\bar{\nabla}_X \phi)Y, Z).$$

On the other hand

$$g(\phi h(Z, X), Y) = -g(h(Z, X), \phi Y) = -g(A_{\phi Y}X, Z).$$

Thus from above two relation, we get

$$g(A_{\phi Y}X, Z) = g(A_{\phi X}Y, Z) + g((\bar{\nabla}_X \phi)Y, Z). \tag{7.2}$$

For $X, Y \in D^0 \oplus \{\xi\}$, we calculate $(\bar{\nabla}_X \phi)Y$ as follows. In view of

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X$$

and

$$\bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X = (\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X + \phi[X,Y],$$

we have

$$(\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X = A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X - \phi[X,Y]$$

which in view of (2.13) gives

$$(\bar{\nabla}_X \phi)Y = \frac{1}{2}(A_{\phi X}Y - A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X - \phi[X,Y] \tag{7.3}$$

$$+\frac{\alpha}{2}(2g(X,Y)\xi) - \eta(X)Y - \eta(Y)X)$$

$$-\frac{(\beta+1)}{2}(\eta(X)\phi Y + \eta(Y)\phi X).$$

Now using (7.3) in (7.2) we get (7.1).

In view of $Ker(P) = D^0 \oplus \{\xi\}$, this leads to the following

Theorem 7.2. Let M be semi-invariant submanifold of a nearly trans-Sasakian manifold with semi-symmetric non-metric connection. Then the distribution $D^0 \oplus \{\xi\}$ is integrable if and only if

$$A_{FX}Y = A_{FY}X,$$

$X, Y \in D^0 \oplus \{\xi\}$.
Using (2.4) in (7.2) for $X, Y \in D^0 \oplus \{\xi\}$ we get $A_{FX}Y = A_{FY}X$. Hence in view of the above theorem, we get the following result.

Corollary 7.3. Let M be a semi-invariant submanifold of a trans-Sasakian manifold with semi-symmetric non-metric connection. Then the distribution $D^0 \oplus \{\xi\}$ is integrable.

8 Integrability of the distribution D^0 and D$'$

We calculated the torsion tensor $N^1(Y,X)\ for\ Y, X \in D^0$, it can be verified that

$$\phi((\bar{\nabla}_X \phi)Y - (\bar{\nabla}_Y \phi)X) = \phi(A_{\phi X}Y - A_{\phi Y}X) + \phi(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X) \tag{8.1}$$

$$+[X,Y] - \eta([X,Y])\xi,$$

$$(\bar{\nabla}_{\phi Y} \phi)X - (\bar{\nabla}_{\phi X} \phi)Y = [X,Y] + \phi(A_{\phi X}Y - A_{\phi Y}X) \tag{8.2}$$

$$+\phi(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X)$$

Using (8.1), (8.2) and (7.1), we get

$$N^1(Y,X) = \frac{8}{3}[X,Y] + \frac{2}{3}\phi(\nabla^{\perp}_X \phi Y - \nabla^{\perp}_Y \phi X) + \frac{8}{3}d\eta(X,Y)\xi \tag{8.3}$$

for $Y, X \in D^0$.

Theorem 8.1. The distribution D^0 is integrable on a semi-nvariant submanifold M of a nearly trans-Sasakian manifold with semi -symmetric non-metric connection if and only if

$$N^1(Y, X) \in D^0 \oplus \bar{D}^1, \tag{8.4}$$

$$A_{FX}Y = A_{FY}X, \tag{8.5}$$

$X, Y \in D^0$.

Proof. If D^0 is integrable, then in view of (8.2) and (8.3) the relation (8.4) and (8.5) follow easily. Conversely, Let $X, Y \in D^0$ and let the relation (8.4) and (8.5) be true. Then in view (8.2) we get $P[X, Y] = 0$. In view of (8.3) we get

$$0 = g(\xi, N^1(Y, X)) = g(\xi, 2[Y, X]).$$

Thus $[X, Y] \in D^0$.

Theorem 8.2. Let M be a semi-invariant submanifold of a nearly trans-Sasakian manifold with semi-symmetric non-metric connection with $\alpha \neq 0$. Then the non-zero invariant distribution D^1 is not integrable.

Proof. If D^1 is integrable then for $X, Y \in D^1$ it follows that $d\eta(X, Y) = 0$ and $[\phi, \phi](X, Y) \in D^1$. Therefore, for $X \in D^1$ in view of (4.6) we get

$$0 = \eta([\phi, \phi](X, PX) + 2d\eta(X, PX)\xi)$$

$$\eta(N^1(X, PX)) = 4\alpha g(\phi X, PX) = 4\alpha g(PX, PX)$$

which is a contradiction.

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