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Special Associated Curves in Galilean 4-Space G

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Abstract In this study, we define generalized Mannheim curves which one of the special associated curves in the four dimensional Galilean Space G_4 and give some characterization for this curves.

Key Words Galilean 4-Space, Associated curve, Mannheim curve

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1 Introduction and Preliminaries

In the theory of space curves in differential geometry, the associated curve, the curves for which at the corresponding points of them one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve have an important role for the characterizations of space curves. The well-known example of such curves is Bertrand curve. Another kind of associated curve have been called Mannheim curve and Mannheim partner curve. The notion of Mannheim curves was discovered by A.Mannheim in 1878. These curves in Euclidean 3-space are characterized in terms of the curvature and torsion as follows:

A space curve is a Mannheim curve if end only if its curvature κ and torsion τ satisfy the relation $\kappa = c(\kappa^2 + \tau^2)$ for some constant.

On the other hand, Galilean 3-space G_3 is simply defined as a Klein geometry of the product space $\mathbb{R}X\mathbb{E}^2$ whose symmetry group is Galilean transformation group which has an important place in classical and modern physics. A curve in Galilean 3-space G_3 is a graph of a plane motion. Note that such a curve is called a worldline in 3-dimensional Galilean space. It is well known that, the idea of worldlines originates in physics and was pioneered by Einstein. The term is now most often in relativity theories, i.e., general relativity and special relativity.

From the differential geometric point of view, the study of curves in G_3 has its own interest. In recent years, many interesting results on curves in G_3 have been obtained by many authors in literature [1,3,4,10,11].

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In [6], firstly, the author constructed Frenet-Serret frame of a curve in the Galilean 4-space and obtained the mentioned curve's Frenet-Serret equations.

In this paper, using the method in [7], we study the generalized Mannheim Curves in 4-dimensional Galilean Space. We obtain the caracterizations of the generalized Mannheim curves in this space.

In the following, some fundamental properties of curves in 4D Galilean space is given for the purpose of the requirements [2].

In Affine coordinates the Galilean scalar product between two points

$$P_i = (p_{i1}, p_{i2}, p_{i3}, p_{i4}), \ i = 1, 2$$

is defined by

$$g(P_1, P_2) = \begin{cases} |p_{21} - p_{11}|, & if \quad p_{21} \neq p_{11}, \\ \sqrt{|(p_{22} - p_{12})^2 + (p_{23} - p_{13})^2 + (p_{24} - p_{14})^2|}, & if \quad p_{21} = p_{11}. \end{cases}$$

The Galilean cross product in G_4 for the vectors $\overrightarrow{u} = (u_1, u_2, u_3, u_4)$, $\overrightarrow{v} = (v_1, v_2, v_3, v_4)$ and $\overrightarrow{w} = (w_1, w_2, w_3, w_4)$ is defined by:

$$\overrightarrow{u} \wedge \overrightarrow{v} \wedge \overrightarrow{w} = - \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix},$$

where e_i , $1 \leqslant i \leqslant 4$, are the standart basis vectors.

The scalar product of two vectors $\overrightarrow{U} = (u_1, u_2, u_3, u_4)$ and $\overrightarrow{V} = (v_1, v_2, v_3, v_4)$ in G_4 is defined by

$$\left\langle \overrightarrow{U}, \overrightarrow{V} \right\rangle_{G_4} = \begin{cases} u_1 v_1, & \text{if } u_1 \neq 0 \text{ or } v_1 \neq 0, \\ u_2 v_2 + u_3 v_3 + u_4 v_4 & \text{if } u_1 = 0 \text{ and } v_1 = 0. \end{cases}$$

The norm of vector $\overrightarrow{U} = (u_1, u_2, u_3, u_4)$ is defined by

$$\left\| \overrightarrow{U} \right\|_{G_4} = \sqrt{\left| \left\langle \overrightarrow{U}, \overrightarrow{U} \right\rangle_{G_4} \right|},$$

[2].

Let $\alpha : I \subset R \longrightarrow G_4$, $\alpha(s) = (s, y(s), z(s), w(s))$ be a curve parametrized by arclength s in G_4. The first vector of the Frenet-Serret frame, that is the tangent vector of α is defined by

$$t = \alpha'(s) = (1, y'(s), z'(s), w'(s)).$$

Since t is a unit vector, so we can express

$$\langle t, t \rangle_{G_4} = 1. \tag{1.1}$$

Differentiating the equation (1.1) with respect to s, we have

$$\langle t', t \rangle_{G_4} = 0.$$

The vector function t' gives us the rotation measurement of the curve α. The real valued function

$$\kappa(s) = \|t'(s)\| = \sqrt{(y''(s))^2 + (z''(s))^2 + (w''(s))^2}$$

is called the first curvature of the curve α. We assume that, $\kappa(s) \neq 0$, for all $s \in I$. Similar to space G_3, the principal vector is defined by

$$n(s) = \frac{t'(s)}{\kappa(s)}$$

in another words

$$n(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s), w''(s)), \tag{1.2}$$

[2].

By the aid of the differentiation of the principal normal vector given in (1.2), define the second curvature function is defined by

$$\tau(s) = \|n'(s)\|_{G_4}. \tag{1.3}$$

This real valued function is called torsion of the curve α. The third vector field, namely binormal vector field of the curve α is defined by

$$b(s) = \frac{1}{\tau(s)} \left(0, \left(\frac{y''(s)}{\kappa(s)} \right)', \left(\frac{z''(s)}{\kappa(s)} \right)', \left(\frac{w''(s)}{\kappa(s)} \right)' \right). \tag{1.4}$$

Thus the vector $b(s)$ is both perpendicular to t and n. The fourth unit vector is defined by

$$e(s) = \mu t(s) \Lambda n(s) \Lambda b(s). \tag{1.5}$$

Here the coefficient μ is taken ± 1 to make $+1$ determinant of the matrix $[t, n, b, e]$.

The third curvature of the curve α by the Galilean inner product is defined by

$$\sigma = \langle b', e \rangle_{G_4}. \tag{1.6}$$

Here, as well known, the set $\{t, n, b, e, \kappa, \tau, \sigma\}$ is called the Frenet-Serret apparatus of the curve α. We know that the vectors $\{t, n, b, e\}$ are mutually orthogonal vectors satisfying

$$\langle t, t \rangle_{G_4} = \langle n, n \rangle_{G_4} = \langle b, b \rangle_{G_4} = \langle e, e \rangle_{G_4} = 1, \tag{1.7}$$

$$\langle t, n \rangle_{G_4} = \langle t, b \rangle_{G_4} = \langle t, e \rangle_{G_4} = \langle n, b \rangle_{G_4} = \langle n, e \rangle_{G_4} = \langle b, e \rangle_{G_4} = 0.$$

For the curve α in G_4, we have following the Frenet-Serret equations

$$\begin{aligned} t' &= \kappa(s) n(s) \\ n' &= \tau(s) b(s), 1.8 \\ b' &= -\tau(s) n(s) + \sigma(s) e(s), \\ e' &= -\sigma(s) b(s), \end{aligned} \tag{1}$$

[2].

During this paper, we will denote the Frenet-Serret vectors with $\{e_1, e_2, e_3, e_4\}$.

2 On Generalized Mannheim Curves in 4D Galilean Space

First we give some definitions.

Definition 2.1. *Let C be a curve parametrized by arclength s in G_4. The curve C is a generalized Mannheim curve if there exists a $\widetilde{C}$ in G_4 such that the first normal line at each point of C is included in the plane generated by the second normal lşne and the third normal line of $\widetilde{C}$ at corresponding point under ψ. Here ψ is a bijection from C to $\widetilde{C}$. The curve $\widetilde{C}$ is called the generalized Mannheim mate curve of C,* [4].

Then, by the definition, a generalized Mannheim mate curve $\widetilde{C}$ is given by the map $\widetilde{x} : \widetilde{I} \longrightarrow G_4$ such that

$$\widetilde{x}(f(s)) = x(s) + \alpha(s)e_2(s), \ \ s \in \widetilde{I}, \tag{2.1}$$

where α is a smooth function on I. We remark that the parameter s generally is not an arc-length parameter of $\widetilde{C}$. Let $\widetilde{s}$ be the arc-length of $\widetilde{C}$ defined by

$$\widetilde{s} = \int_0^s \left\| \frac{d\widetilde{x}(s)}{ds} \right\| ds.$$

We can consider a smooth function $f : I \to \widetilde{I}$, given by $f(s) = \widetilde{s}$. Then we have

$$f'(s) = \sqrt{1 + (\alpha(s)\tau(s))^2}, \quad s \in I, \tag{2.2}$$

Theorem 2.1.([4, p.5]) *Let C be a special Frenet-Serret curve in G_4. If the curve α is a generalized Mannheim curve, then the first curvature functions κ and second curvature functions τ of C satisfy the equation*

$$\frac{\kappa(s)}{(\tau(s))^2} = const. = \alpha, \ \ s \in I, \tag{2.3}$$

where α is positive constant number.

Proof. Let C be a generalized Mannheim curve in G_4 and $\widetilde{C}$ be the generalized Mannheim mate curve of C. Then we have from (2.1)

$$\widetilde{x}(f(s)) = x(s) + \alpha(s)e_2(s).$$

Also,we can set

$$e_2(s) = g(s)\widetilde{e}_3(f(s)) + h(s)\widetilde{e}_4(f(s)), \tag{2.4}$$

for some smooth functions g and h on I.

If we differentiate both sides of this equation with respect to s, we get

$$f'(s)\widetilde{e}_1(f(s)) = e_1(s) + \alpha'(s)e_2(s) + \alpha(s)\tau(s)e_3(s). \tag{2.5}$$

If we take the Galilean inner product both sides of (2.5) with $e_2(s)$ and considering (2.4), we obtain that α is constant. If we consider this fact in (2.5), we get

$$\widetilde{e}_1(f(s)) = \frac{1}{f'(s)}e_1(s) + \frac{\alpha(s)\tau(s)}{f'(s)}e_3(s), \tag{2.6}$$

where $f'(s)=\sqrt{1+(\alpha(s)\tau(s))^2}$ for $s\in I$. By differentiation of (2.6) with respect to s, we have

$$f'(s)\widetilde{\kappa}(f(s))\widetilde{e}_2(f(s)) = \left(\frac{1}{f'(s)}\right)' e_1(s)+\frac{\kappa(s)-\alpha(\tau(s))^2}{f'(s)}e_2(s) + \left(\frac{\alpha(s)\tau(s)}{f'(s)}\right)' e_3(s)+\frac{\alpha(s)\tau(s)\sigma(s)}{f'(s)}e_4(s). \tag{2.7}$$

If we take the Galilean inner product both sides of (2.7) with $e_2(s)$ and considering (2.4), we obtain

$$\alpha=\frac{\kappa(s)}{(\tau(s))^2}$$

from α is constant, we have $\frac{\kappa(s)}{(\tau(s))^2}=const.$Thus the proof is completed.

Theorem 2.2. *Let C be a special Frenet-Serret curve in G_4 whose curvature functions κ and τ are non-constant functions and satisfy the equation $\kappa(s)=\alpha(\tau(s))^2$, where $s\in I$ and α is a positive number. If the curve $\widetilde{C}$ given by $\widetilde{x}(s)=x(s)+\alpha.e_2(s)$ is a special Frenet-Serret curve, then C is a generalized Mannheim curve and $\widetilde{C}$ is the generalized Mannheim mate curve of C.*

Proof. Let $\widetilde{s}$ be the arc-length of $\widetilde{C}$. That is, $\widetilde{s}$ is defined by

$$\widetilde{s}=\int_0^s\left\|\frac{d\widetilde{x}(s)}{ds}\right\|ds.$$

We can consider a smooth function function $f:I\to\widetilde{I}$, given by $f(s)=\widetilde{s}$. Then we have

$$f'(s)=\sqrt{1+(\alpha(s)\tau(s))^2},$$

for $s\in I$. Also , from Theorem 2.1., we have

$$f'(s)=\sqrt{1+\alpha(s)\kappa(s)}. \tag{2.8}$$

The representation of $\widetilde{C}$ by arc-length parameter $\widetilde{s}$ is denoted by $\widetilde{x}(\widetilde{s})$, then we can write

$$\widetilde{x}(\hat{s})=\widetilde{x}(f(s))=x(s)+\alpha(s)e_2(s),$$

for curve $\widetilde{C}$. Thus we obtain

$$\widetilde{e}_1(f(s))=\frac{1}{\sqrt{1+\alpha(s)\kappa(s)}}e_1(s)+\frac{\alpha(s)\tau(s)}{\sqrt{1+\alpha(s)\kappa(s)}}e_3(s). \tag{2.9}$$

Differentiating both side of the equation (2.9) with respect to s, then we have

$$f'(s)\widetilde{\kappa}(f(s))\widetilde{e}_2(f(s)) = \left(\frac{1}{\sqrt{1+\alpha(s)\kappa(s)}}\right)' e_1(s) + \left(\frac{\kappa(s)-\alpha(s)(\tau(s))^2}{\sqrt{1+\alpha(s)\kappa(s)}}\right)e_2(s) + \left(\frac{\alpha(s)\tau(s)}{\sqrt{1+\alpha(s)\kappa(s)}}\right)' e_3(s) \tag{2.10}$$

$$+\left(\frac{\alpha(s)\tau(s)\sigma(s)}{\sqrt{1+\alpha(s)\kappa(s)}}\right)e_4(s).$$

Under our assumption, that is, $\kappa(s)-\alpha(\tau(s))^2=0$,we have that the coefficient of $e_2(s)$ in the equation (2.10) vanishes. Thus , for each $s\in I$, the vector $\widetilde{e}_2(f(s))$ is given by linear combination of $e_1(s), e_3(s)$ and $e_4(s)$. Also, from (2.9), the vector $\widetilde{e}_1(f(s))$ is given by linear combination of $e_1(s)$ and $e_3(s)$. Since the curve $\widetilde{C}$ is a special Frenet-Serret curve in G_4, the vector $e_2(s)$ is given by linear combination of $\widetilde{e}_3(f(s))$ and $\widetilde{e}_4(f(s))$.

Therefore, the first normal line at each point of C is concluded in the plane generated the second normal line and the third normal line of $\widetilde{C}$ at corresponding point under the bijection $\psi: C\to\widetilde{C}$. This completes the proof.

Theorem 2.3. *Let C be a special Frenet-Serret curve in G_4 such that its third curvature function σ doesn't vanish. The curvature functions κ and τ of C are constant functions if and only if there exists a special Frenet-Serret curve $\widetilde{C}$ in G_4 such that the first normal line at each point of C is the third normal line of $\widetilde{C}$ at corresponding each point under a bijection $\psi: C\to\widetilde{C}$.*

Proof. Let C be a special Frenet-Serret curve in G_4 with the Frenet-Serret frame field $\{e_1, e_2, e_3, e_4\}$ and curvature functons κ,τ,σ. The first curvature function κ and the second curvature function τ of C are positive constants. Thus $\alpha=\frac{\kappa(s)}{(\tau(s))^2}$ is a positive constant number. We define a regular smooth curve $\widetilde{C}$ by

$$\begin{aligned}\widetilde{x} &: I\to G_4\\ \widetilde{x}(s) &= x(s)+\alpha.e_2(s).\end{aligned}$$

Let $\widetilde{s}$ denote the arc-length parameter of $\widetilde{C}$, and let $f: I\to\widetilde{I}$ be a function defined by

$$\hat{s}=f(s)=\sqrt{1+(\alpha\tau)^2}s.$$

Then we have

$$f'(s)=\sqrt{1+(\alpha\tau)^2}$$

and

$$\widetilde{e}_1(f(s))=\frac{1}{\sqrt{1+(\alpha\tau)^2}}e_1(s)+\frac{\alpha\tau}{\sqrt{1+(\alpha\tau)^2}}e_3(s).$$

Differentiating both sides of the last equation with respect to s we obtain

$$f'(s)\widetilde{\kappa}(f(s))\widetilde{e}_2(f(s))=\frac{\alpha\tau\sigma}{\sqrt{1+(\alpha\tau)^2}}e_4.$$

Thus, we have

$$\widetilde{\kappa}(f(s))=\left\|\frac{d\widetilde{e}_1(\hat{s})}{d\hat{s}}\right\|=sign\sigma\frac{\alpha\tau\sigma}{1+(\alpha\tau)^2}>0$$

and

$$\widetilde{e}_2(\hat{s})=sign(\sigma)e_4(s).$$

After some calculations, we obtain following equations

$$\begin{aligned}\widetilde{\tau}(f(s)) &= \frac{sign(\sigma)}{\sqrt{1+(\alpha\tau)^2}}\sigma(s) > 0\\ \widetilde{e}_3(f(s)) &= -e_3(s)\\ \widetilde{e}_4(f(s)) &= = \frac{sign(\sigma)}{\sqrt{1+(\alpha\tau)^2}}e_2,\end{aligned}$$

that means that, $\widetilde{C}$ is a special Frenet-Serret curve in G_4 and the first normal line at each point of C is the third normal line of $\widetilde{C}$ at corresponding each point under the bijection $\psi : C \to \widetilde{C}$.

Now, let C is be a special Frenet-Serret curve in G_4 with the Frenet frame field and curvature functions κ, τ and σ. Let $\widetilde{C}$ is a special Frenet-Serret curve in $G_4$with the Frenet frame field $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ and curvature function $\widetilde{\kappa}, \widetilde{\tau}$ and $\widetilde{\sigma}$. We suppose that the first normal line at each point of C is the third normal line of $\widetilde{C}$ at corresponding each point under the bijection $\psi : C \to \widetilde{C}$. then the curve $\widetilde{C}$ is parametrized by

$$\widetilde{x}(s) = x(s) + \alpha.e_2(s), s \in I. \tag{2.11}$$

Let $\widetilde{s}$ be the arc-length parameter of $\widetilde{C}$ given by

$$\widetilde{s} = f(s) = \int_0^s \sqrt{1 + (\alpha'(s))^2 + (\alpha(s)\tau(s))^2}ds.$$

Thus our assumption imlies that

$$\widetilde{e}_4 = \pm e_2(s).$$

Differentiating the equation (2.11)with respect to s, we have

$$f'(s)\widetilde{e}_1(f(s)) = e_1(s) + \alpha'(s)e_2(s) + \alpha(s)\tau(s)e_3(s).$$

Taking the inner product of $f'(s)\widetilde{e}_1(f(s))$ and $\widetilde{e}_4(f(s))$, we obtain

$$f'(s) = \sqrt{1 + (\alpha(s)\tau(s))^2} > 0$$

and

$$\widetilde{e}_1(f(s)) = \frac{1}{f'(s)}e_1(s) + \frac{\alpha(s)\tau(s)}{f'(s)}e_3(s). \tag{2.12}$$

We differentiate the equation (2.12) with respect to s, we have

$$\begin{aligned}f'(s)\widetilde{\kappa}(f(s))\widetilde{e}_2(f(s)) &= \left(\frac{1}{f'(s)}\right)' e_1(s) + \frac{\kappa(s) - \alpha(\tau(s))^2}{f'(s)}e_2(s)\\ &+ \left(\frac{\alpha(s)\tau(s)}{f'(s)}\right)' e_3(s) + \frac{\alpha(s)\tau(s)\sigma(s)}{f'(s)}e_4(s).\end{aligned} \tag{2.13}$$

We know that

$$\langle f'(s)\widetilde{\kappa}(f(s))\widetilde{e}_2(f(s)), \widetilde{e}_4(s)\rangle = 0,$$

It holds that

$$\kappa(s) - \alpha(\tau(s))^2 = 0,$$

so that α is a positive constant number. By differentiation of the equation (2.13) with respect to s,we have

$$(\kappa(s)-\alpha(s)\tau^2(s))f''(s)+\alpha(s)\tau(s)\tau'(s)f'(s)=0.$$

And we know that, $\kappa(s)-\alpha(\tau(s))^2=0$. Hence we obtain

$$\alpha(s)\tau(s)\tau'(s)=0,$$

that means that , $\tau'=0$, that is $\tau=const$. From the relation $\alpha(s)=\frac{\kappa(s)}{\tau^2(s)}$, the curvature function κ is positive constant too. Thus the proof is completed.

Theorem 2.4. *Let C be a curve defined by*

$$x(s)=\left(s,\alpha\int(\int h(s)\sin s ds)ds,\alpha\int(\int h(s)\cos s ds)ds,\alpha\int(\int h(s)g(s)ds)ds\right),\ s\in I,$$

where α is a positive constant number, g and h are any smooth functions: $I\to R$, and f defined by

$$h(s)=\frac{(gg')^2+g^2(g')^3}{(1+g(s))^{\frac{9}{2}}},$$

for $s\in I$. Then the curvature functions κ and τ of the curve C satisfy

$$\kappa(s)-\alpha(\tau(s))^2=0,$$

for each $s\in I$.

Proof. Considering the formula of curvature calculations in G_4, the proof is can be done easily.

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