Fuzzy soft ideal topological spaces


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Abstract The purpose of this paper is to introduce the notion of fuzzy soft ideal in fuzzy soft set theory. The concept of fuzzy soft local function is also introduced by using quasi-coincident relation and Q-fuzzy soft neighborhood system. These concepts are discussed with a view to find new fuzzy soft topologies from the original one. The basic structure, especially a basis for such generated fuzzy soft topologies also studied here. Finally, the notion of compatibility of fuzzy soft ideals with fuzzy soft topologies is introduced and some equivalent conditions concerning this topic are established here.

Key Words fuzzy soft set, fuzzy soft topological space, fuzzy soft ideal, fuzzy soft local function

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1 Introduction

We are not able to use classical methods to solve some kinds of problems given in sociology, economics, environment, engineering etc., since, these kinds of problems have their own uncertainties. Fuzzy set theory, which was firstly proposed by Zadeh [22] in 1965, has become a very important tool to solve these kinds of problems and provides an appropriate framework for representing vague concepts by allowing partial membership. Fuzzy set theory has been studied by both mathematicians and computer scientists and many applications of fuzzy set theory have arisen over the years, such as fuzzy control systems, fuzzy automata, fuzzy logic, fuzzy topology etc. Beside this theory, there are also theory of probability, rough set theory which deal with to solve these problems. Each of these theories has its inherent difficulties as pointed out in 1999 by Molodtsov [12] who introduced the concept of soft set theory which is a completely new approach for modeling uncertainty. In this paper, Molodtsov established the fundamental results of this new theory and successfully applied the soft set theory into several directions, such as smoothness of functions, operations research, Riemann integration, game theory, theory of probability and so on.

Maji et al. [10] initiated the study involving both fuzzy sets and soft sets. In this paper, the notion of fuzzy soft sets was introduced as a fuzzy generalizations of soft sets and some basic properties of fuzzy soft sets are discussed in detail. Maji et al. combined fuzzy sets and soft sets and introduced the concept of fuzzy soft sets. To continue the investigation on fuzzy soft sets, Ahmad and Kharal [1] presented some more properties of fuzzy soft sets and introduced the notion of a mapping on fuzzy soft sets. In 2011, Tanay et al. [19] gave the topological structure of fuzzy soft sets.

The local properties of a space which may also be in certain cases the properties of the whole space, are important field for study in general, fuzzy topology, and soft topology. In general topology, by introducing the notion of ideal, Kuratowski [8], Vaidyanathaswamy [20, 21] and several other authors carried out such analyses. Recently, there has been an extensive study on the importance of ideal in general topology in the paper of Jankovic and Hamlett [4], in fuzzy topology: by D. Sakar [16] and Nasef et al. [13], in soft set theory: by Kandil et. al. [6] in 2014.

Our aim in this paper is to extend those ideas of general topology, fuzzy topology, and soft topology in fuzzy soft topological space. In Section 3, we define fuzzy soft ideal and introduce the notion of fuzzy soft local function corresponding to a fuzzy soft topological space. We have deduced some characterization theorems for such concepts exactly analogous to general topology, fuzzy topology, and soft topology and succeeded in finding out the generated new fuzzy soft topologies for any fuzzy topological space. In Section 4, we discuss the basic structure of new fuzzy soft topology and it is established that the new fuzzy soft topology cannot be further generated with the same fuzzy soft ideal. Finally, in Section 5, we define quasi-cover of a fuzzy soft set and introduce the notion of compatibility of fuzzy soft ideal with a fuzzy soft topological space and obtain some results concerning this concept.

2 Preliminaries

Throughout this paper $X$ denotes initial universe, $E$ denotes the set of all possible parameters which are attributes, characteristic or properties of the objects in $X$, and the set of all subsets of $X$ will be denoted by $P(X)$. In this section, we present the basic definitions and results of soft set theory which will be needed in the sequel.

**Definition 2.1** [3] A fuzzy set $A$ of a non-empty set $X$ is characterized by a membership function $\mu_A : X \rightarrow [0, 1] = I$ whose value $\mu_A(x)$ represents the "degree of membership" of $x$ in $A$ for $x \in X$. Let $I^X$ denotes the family of all fuzzy sets on $X$.

**Definition 2.2** [12] Let $A$ be a non-empty subset of $E$. A pair $(F, A)$ denoted by $F_A$ is called a soft set over $X$, where $F$ is a mapping given by $F : A \rightarrow P(X)$. In other words, a soft set over $X$ is a parametrized family of subsets of the universe $X$. For a particular $e \in A$, $F(e)$ may be considered the set of $e$-approximate elements of the soft set $(F, A)$ and if $e \notin A$, then $F(e) = \phi$ i.e $F = \{F(e) : e \in A \subseteq E, F : A \rightarrow P(X)\}$. The family of all these soft sets over $X$ denoted by $SS(X)_E$. 

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Aktaş and Çağman [2] showed that every fuzzy set may be considered as a soft set. That is, fuzzy sets are a special class of soft sets.

**Definition 2.3** [9, 13, 14, 17, 19] Let $A \subseteq E$. A pair $(f, A)$, denoted by $f_A$, is called fuzzy soft set over $X$, where $f$ is a mapping given by $f : A \rightarrow f^X$ defined by $f_A(e) = \mu^f_A$; where $\mu^f_A = \overline{0}$ if $e \notin A$, and $\mu^f_A \neq \overline{0}$ if $e \in A$, where $\overline{0}(x) = 0 \forall x \in X$. The family of all these fuzzy soft sets over $X$ denoted by $FSS(X)_E$.

**Definition 2.4** [9, 13, 14, 17, 19] The complement of a fuzzy soft set $(f, A)$, denoted by $(f, A)^c$, is defined by $(f, A)^c = (f^c, A)$. For each $e \in E$, $(f, A)^c$ is a mapping given by $\mu^c_A = 1 - \mu^f_A \forall e \in A$, where $\overline{1}(x) = 1 \forall x \in X$. Clearly, $(f_A)^c = f_A$.

**Definition 2.5** [10, 14, 15, 18, 19] A fuzzy soft set $f_A$ over $X$ is said to be a null fuzzy soft set, denoted by $\overline{0}_E$, if for all $e \in E$, $f_A(e) = \overline{0}$.

**Definition 2.6** [10, 14, 15, 18, 19] A fuzzy soft set $f_A$ over $X$ is said to be an absolute fuzzy soft set, denoted by $\overline{1}_E$, if $\overline{1}_E(e) = \overline{1} \forall e \in E$. Clearly, we have $(\overline{0}_E)^c = \overline{1}_E$ and $(\overline{1}_E)^c = \overline{0}_E$.

**Definition 2.7** [10, 14, 15, 18, 19] Let $f_A, g_B \in FSS(X)_E$. Then $f_A$ is fuzzy soft subset of $g_B$, denoted by $f_A \subseteq g_B$, if $A \subseteq B$ and $\mu^c_A(x) \leq \mu^c_B(x) \forall x \in X, \forall e \in E$. Also, $g_B$ is called fuzzy soft superset of $f_A$ denoted by $g_B \supseteq f_A$.

**Definition 2.8** [10, 14, 15, 18, 19] Two fuzzy soft sets $f_A$ and $g_B$ on $X$ are called equal if $f_A \subseteq g_B$ and $g_B \subseteq f_A$.

**Definition 2.9** [10, 14, 15, 18, 19] The union of two fuzzy soft sets $f_A$ and $g_B$ over the common universe $X$, denoted by $f_A \cup g_B$, is also a fuzzy soft set $h_C$, where $C = A \cup B$ and for all, $e \in C$, $h_C(e) = \mu^c_C = \mu^c_A \lor \mu^c_B \forall e \in C$.

**Definition 2.10** [10, 14, 15, 18, 19] The intersection of two fuzzy soft sets $f_A$ and $g_B$ over the common universe $X$, denoted by $f_A \cap g_B$, is also a fuzzy soft set $h_C$, where $C = A \cap B$ and for all, $e \in C$, $h_C(e) = \mu^c_C = \mu^c_A \land \mu^c_B \forall e \in C$.

**Definition 2.11** [5] The difference of two fuzzy soft sets $f_A$ and $g_B$ over the common universe $X$, denoted by $f_A - g_B$, is the fuzzy soft set $h_C$, where $C = A \cap B \neq \emptyset$ and $C \subseteq X$, $\mu^c_C(x) = \max\{\mu^c_A(x) - \mu^c_B(x), 0\}$. Clearly, we have $f_A - g_B = f_A \cap g_B^c$.

**Definition 2.12** [19] Let $FSS(X)_E$ be a collection of fuzzy soft sets over a universe $X$ with a fixed set of parameters $E$. Then $\tau \subseteq FSS(X)_E$ is called fuzzy soft topology on $X$ if

1. $0_E, 1_E \in \tau$, where $0_E(e) = \overline{0}$ and $1_E(e) = \overline{1} \forall e \in E$.
2. The union of any members of $\tau$ belongs to $\tau$.
3. The intersection of any two members of $\tau$ belongs to $\tau$.

The triplet $(X, \tau, E)$ is called a fuzzy soft topological space over $X$. Also, each member of $\tau$ is a called fuzzy soft open set in $(X, \tau, E)$.

**Examples 2.1** The following are fuzzy soft topology on $X$:
(1) \( \tau_0 = \{ \bar{0}_E, \bar{1}_E \} \) is called fuzzy soft indiscrete topology on \( X \).

(2) \( \tau_D = FSS(X)_E \) is called fuzzy soft discrete topology on \( X \).

Note that, the intersection of any family of fuzzy soft topologies on \( X \) is also a fuzzy soft topology on \( X \).

**Definition 2.13** [14] A fuzzy soft topology \( \tau_1 \) is called weaker (or coarser) than a fuzzy soft topology \( \tau_2 \) if and only if \( \tau_1 \subseteq \tau_2 \). In that case \( \tau_2 \) is said to be stronger (or finer) than \( \tau_1 \).

**Definition 2.14** [19] Let \((X, \tau, E)\) be a fuzzy soft topological space. A fuzzy soft set \( f \) over \( X \) is said to be fuzzy soft closed set in \( X \), if its relative complement \( f^c \) is fuzzy soft open set. The collection of all fuzzy soft sets will be denoted by \( \tau^c \).

**Definition 2.15** [14, 15] Let \((X, \tau, E)\) be a fuzzy soft topological space and \( f_A \in FSS(X)_E \). The fuzzy soft closure of \( f_A \), denoted by \( Fcl(f_A) \) is the intersection of all fuzzy soft closed supersets of \( f_A \). i.e., \( Fcl(f_A) = \cap \{ h_C; h_C \in \tau^c \text{ and } f_A \subseteq h_C \} \). Clearly, \( Fcl(f_A) \) is the smallest fuzzy soft closed set over \( X \) which contains \( f_A \), and \( Fcl(f_A) \) is fuzzy soft closed set.

**Theorem 2.1** [14] Let \( c : FSS(X)_E \to FSS(X)_E \) be an operator satisfying the following:

\[(c1) \ c(\bar{0}_E) = \bar{0}_E, \]

\[(c2) \ f_A \subseteq c(f_A), \forall f_A \in FSS(X)_E, \]

\[(c3) \ c(f_A \cup g_B) = c(f_A) \cup c(g_B), \forall f_A, g_B \in FSS(X)_E, \]

\[(c4) \ c(c(f_A)) = c(f_A), \forall f_A \in FSS(X)_E. \]

The operator \( c \) is called the fuzzy soft closure operator. Then we can associate a fuzzy soft topology in the following way: \( \tau = \{ f^c_A \in F(X, E) | c(f_A) = f_A \} \). Moreover with this fuzzy soft topology \( \tau \), \( Fcl(f_A) = c(f_A) \) for every \( f_A \in FSS(X)_E \).

**Definition 2.16** [15, 18] The fuzzy soft set \( f_A \in FSS(X)_E \) is called fuzzy soft point if there exist \( x \in X \) and \( e \in E \) such that \( \mu^e_{f_A}(x) = \alpha; (0 < \alpha \leq 1) \) and \( \mu^e_{f_A}(y) = 0 \ \forall y \in X - \{x\} \) and this fuzzy soft point is denoted by \( x^e_\alpha \) or \( f_e \). The class of all fuzzy soft points of \( X \), denoted by \( FSP(X)_E \).

**Definition 2.17** [9] The fuzzy soft point \( x^e_\alpha \) is said to be belonging to the fuzzy soft set \( f_A \), denoted by \( x^e_\alpha \in f_A \), if for the element \( e \in A \), \( \alpha \leq \mu^e_{f_A}(x) \).

**Definition 2.18** [15, 18] A fuzzy soft point \( x^e_\alpha \) is said to be a quasi-coincident with a fuzzy soft set \( f_A \), denoted by \( x^e_\alpha \q f_A \), if \( \alpha + \mu^e_{f_A}(x) > 1 \). The negation of this statement is written as \( x^e_\alpha \not\q f_A \).

**Definition 2.19** [15, 18] A fuzzy soft set \( f_A \) is said to be quasi-coincident with \( g_B \), denoted by \( f_A \q g_B \), if there exists \( x \in X \) such that \( \mu^e_{f_A}(x) + \mu^e_{g_B}(x) > 1 \), for some \( e \in A \cap B \). If this is true we can say that \( f_A \) and \( g_B \) are quasi-coincident at \( x \).

**Proposition 2.1** [15, 18] Let \( f_A \) and \( g_B \) be two fuzzy soft sets, \( f_A \subseteq g_B \) if and only if \( f_A \not\q g_B^c \). In particular, \( x^e_\alpha \in f_A \) if and only if \( x^e_\alpha \not\q (f_A)^c \).

**Definition 2.20** [15, 18] Let \((X, \tau, E)\) be a fuzzy soft topological space and \( x^e_\alpha \) be a fuzzy soft point in \( X \). A fuzzy soft set \( f_A \) is called Q-fuzzy soft neighborhood of \( x^e_\alpha \) (Q-fuzzy soft nbd, for short), if ther
exists \( g_B \in \tau \) such that \( x_\alpha^e \sqsubseteq g_B \) and \( g_B \subseteq f_A \). Let \( N(x_\alpha^e) \) be the family of all Q-fuzzy soft nbd of \( x_\alpha^e \) in a fuzzy soft topological space \((X, \tau, E)\).

**Theorem 2.2** [15, 18] Let \((X, \tau, E)\) be a fuzzy soft topological space and \( x_\alpha^e \) be a fuzzy soft point in \( X \). Then \( x_\alpha^e \in \text{Fcl}(f_A) \) if and only if for each \( g_B \in N(x_\alpha^e) \), \( g_B \sqsubseteq f_A \).

**Theorem 2.3** [15] A subfamily \( \beta \) of a fuzzy soft topology \( \tau \) is a basis for \( \tau \) if and only if for each fuzzy soft point \( x_\alpha^e \) and for each Q-fuzzy soft nbd \( f_A \) of \( x_\alpha^e \), there exists a member \( g_B \in \beta \) such that \( x_\alpha^e \sqsubseteq g_B \) and \( g_B \subseteq f_A \).

**Definition 2.21** [15] A fuzzy soft point \( x_\alpha^e \) is called an adherence point of a fuzzy soft set \( f_A \) if every Q-fuzzy soft nbd of \( x_\alpha^e \) is a quasi-coincident with \( f_A \).

**Proposition 2.2** [15] Every fuzzy soft point of \( f_A \) is an adherence point of \( f_A \).

**Definition 2.22** [15] A fuzzy soft point \( x_\alpha^e \) is called an accumulation point of a fuzzy soft set \( f_A \) if:
(a) \( x_\alpha^e \) is an adherence point of \( f_A \),
(b) whenever \( x_\alpha^e \notin f_A \), every Q-fuzzy soft nbd \( g_B \) of \( x_\alpha^e \), is quasi-coincident with \( f_A \) at some fuzzy soft point different from \( x_\alpha^e \).

The union of all accumulation points of \( f_A \) is called the fuzzy soft driven set of \( f_A \), denoted by \( f_A^d \).

**Theorem 2.4** [15] \( \text{Fcl}(f_A) = f_A \sqcup f_A^d \).

**Corollary 2.1** [15] A fuzzy soft set \( f_A \) is closed in a fuzzy soft topological space \((X, \tau, E)\) if and only if \( f_A \) contains all its accumulation points.

**Theorem 2.5** [7] Let \((X, \tau, E)\) be a fuzzy soft topological space and \( f_A, g_B \) be fuzzy soft subsets over \( X \), the following properties are satisfied:
1. \( \overline{g_B} = \overline{E} \).
2. \( (f_A \sqcup g_B)^d = f_A^d \sqcup g_B^d \).
3. \( (f_A^d)^d \subseteq f_A^d \).

3 Fuzzy soft ideal topological spaces

In this section, we generate a new fuzzy soft topological space from the old one \((X, \tau, E)\) on \( X \) by using the fuzzy soft ideal notion. We denote \((X, \tau, \widetilde{I}, E)\) as a fuzzy soft ideal topological space on \( X \).

**Definition 3.1** A non-empty collection \( \widetilde{I} \) of fuzzy soft sets over a universe \( X \) with a fixed set of parameters \( E \) is said to be a fuzzy soft ideal on \( X \) if it satisfies the following conditions:
(i) If \( f_A \in \widetilde{I}, g_B \subseteq f_A \implies g_B \in \widetilde{I} \), (heredity)
(ii) If \( f_A, g_B \in \widetilde{I} \implies f_A \sqcup g_B \in \widetilde{I} \), (finite additivity)

**Definition 3.2** Let \((X, \tau, E)\) be a fuzzy soft topological space and \( \widetilde{I} \) be a fuzzy soft ideal on \( X \). Let \( f_A \) be any fuzzy soft set of \( X \). The fuzzy soft local function \( f_A^*(\widetilde{I}, \tau) \) of \( f_A \) is the union of all fuzzy soft points \( x_\alpha^e \) such that if \( g_B \in N(x_\alpha^e) \) and \( h_C \in \widetilde{I} \) there is at least one \( y \in X \) for which \( \mu_{f_A}^*(y) + \mu_{g_B}^*(y) - 1 > \mu_{h_C}^*(y) \) for some \( \epsilon \in E \).
Therefore, any \( x_\alpha^e \not\in f_A^*(\bar{I}, \tau) \) implies there is at least \( g_B \in N(x_\alpha^e) \) such that for every \( y \in Y \) and \( \epsilon \in E \), 
\( \mu^e_{f_A}(y) + \mu^e_{gb}(y) - 1 \leq \mu^e_{hC}(y) \) for some \( hC \in \bar{I} \). We will occasionally write \( f^*_A \) or \( f^*_A(\bar{I}) \) for \( f^*_A(\bar{I}, \tau) \) and it will cause no ambiguity.

**Example 3.1** The simplest fuzzy soft ideals on \( X \) are \( \{\bar{0}_E\} \) and \( FSS(X)_E \), the set of all fuzzy soft sets of \( X \). Obviously, \( \bar{I} = \{\bar{0}_E\} \iff f^*_A = Fcl(f_A) \) for any fuzzy soft set \( f_A \) of \( X \) and \( \bar{I} = FSS(X)_E \iff f^*_A = \bar{0}_E \).

The following theorem gives basic results of fuzzy soft local function which are known in general topology [4], fuzzy topology [16], and soft topology [6]. Also, we show the difference between these structures.

**Theorem 3.1** Let \((X, \tau, E)\) be a fuzzy soft topological space and \( \bar{I}, \bar{J} \) be two fuzzy soft ideals on \( X \). Then for any fuzzy soft sets \( f_A, g_B \) of \( X \),

(i) \( \bar{0}_E = \bar{0}_E \),
(ii) \( f_A \subseteq g_B \implies f_A^*(\bar{I}, \tau) \subseteq g_A^*(\bar{I}, \tau) \),
(iii) \( \bar{I} \subseteq \bar{J} \implies f_A^*(\bar{I}, \tau) \subseteq f_A^*(\bar{J}, \tau) \),
(iv) \( f_A^* = Fcl(f_A) \subseteq Fcl(f_A) \),
(v) \( (f_A)^* \subseteq f_A^* \),
(vi) \( (f_A \cup g_B)^* = f_A^* \cup g_B^* \),
(vii) \( (f_A \cap g_B)^* = f_A^* \cup g_B^* \),
(viii) \( h_C \in \bar{I} \implies (f_A \cup h_C)^* = f_A^* \).

**Proof.** (i) Obvious from Definition 3.2.

(ii) Since \( f_A \subseteq g_B \) implies \( \mu^e_{f_A}(x) \leq \mu^e_{gb}(x) \) for every \( x \in X \) and \( e \in E \). Let \( x_\alpha^e \not\in f_A^* \). Then for every \( s_D \in N(x_\alpha^e) \) and \( h_C \in \bar{I} \), there exists \( y \in X \) such that \( \mu^e_{f_A}(y) + \mu^e_{s_D}(y) - 1 \leq \mu^e_{hC}(y) \) for some \( e \in E \). So \( \mu^e_{s_D}(y) - 1 \leq \mu^e_{hC}(y) \) for some \( e \in E \) and \( x_\alpha^e \notin g_B^* \). Hence \( f_A \subseteq g_B^* \).

(iii) Clearly, \( \bar{I} \subseteq \bar{J} \) implies \( f_A^*(\bar{I}, \tau) \subseteq f_A^*(\bar{J}, \tau) \), as there may be other fuzzy soft sets which belong to \( \bar{J} \) so that for a fuzzy soft point \( x_\alpha^e ; x_\alpha^e \in f_A^*(\bar{J}, \tau) \) but \( x_\alpha^e \notin f_A^*(\bar{I}, \tau) \).

(iv) Since \( \{\bar{0}_E\} \subseteq \bar{I} \) for any fuzzy soft ideal \( \bar{I} \) on \( X \), therefore, by (iii), \( f_A^*(\bar{I}) \subseteq f_A^*(\{\bar{0}_E\}) = Fcl(f_A) \) for any fuzzy soft set \( f_A \) of \( X \). Suppose \( x_\alpha^e \in Fcl(f_A) \). Then every Q-fuzzy soft nbd \( g_B \) of \( x_\alpha^e \) is a quasi-coincident of \( f_A^* \), therefore there exists \( y \in X \) such that \( \mu^e_{f_A}(y) + \mu^e_{g_B}(y) > 1 \) for some \( e \in E \cap B \). Hence \( \mu^e_{f_A}(y) \neq 0 \). Let \( \mu^e_{f_A}(y) = \beta \). Clearly, \( y_\beta^e \in f_A^* \) and \( \beta + \mu^e_{g_B}(y) > 1 \) so that \( g_B \) is also a Q-fuzzy soft nbd of \( y_\beta^e \). Now \( y_\beta^e \in f_A^* \) implies that there is at least one \( z \in X \) such that \( \mu^e_{f_A}(z) + \mu^e_{g_B}(z) - 1 > \mu^e_{hC}(z) \) for each \( s_D \in N(y_\beta^e) \) and \( h_C \in \bar{I} \) and for some \( e \in E \). This is also true for \( g_B \). So, \( \mu^e_{f_A}(z) + \mu^e_{g_B}(z) - 1 \geq \mu^e_{hC}(z) \) for each \( h_C \in \bar{I} \) and for some \( e \in E \). Since \( g_B \) is an arbitrary Q-fuzzy soft nbd of \( x_\alpha^e \), therefore \( x_\alpha^e \not\in f_A^* \).

Hence \( f_A^* = Fcl(f_A^*) \subseteq Fcl(f_A) \).

(v) By (iv), we have \( (f_A)^* = Fcl((f_A)^*) \subseteq Fcl(f_A) = f_A^* \).

(vi) Suppose \( x_\alpha^e \notin f_A^* \cup g_B^* \), i.e., \( \alpha > \mu^e_{f_A \cup g_B}(x) = \max\{\mu^e_{f_A}(x), \mu^e_{g_B}(x)\} \); \( e \in A \cup B \). So \( x_\alpha^e \notin f_A^* \) and \( x_\alpha^e \notin g_B^* \). This implies there is at least one Q-fuzzy soft nbd \( s_D \) of \( x_\alpha^e \) such that for every \( y \in X \), \( \mu^e_{f_A}(y) + \mu^e_{s_D}(y) - 1 \leq \mu^e_{hC}(y) \), for some \( h \in \bar{I} \) and for every \( e \in E \). Similarly, there is at least one Q-fuzzy soft nbd \( v_L \) of \( x_\alpha^e \) such that for every \( y \in X \), \( \mu^e_{g_B}(y) + \mu^e_{v_L}(y) - 1 \leq \mu^e_{hC}(y) \), for some \( h \in \bar{I} \) and for every \( e \in E \). Let \( j_N = s_D \cap v_L \). So \( j_N \) is also a Q-fuzzy soft nbd of \( x_\alpha^e \), \( h_C \cup u_M \in \bar{I} \), and \( \mu^e_{f_A \cup g_B}(y) + \mu^e_{s_D}(y) - 1 \leq \mu^e_{hC}(y) \), for some \( h \in \bar{I} \) and for every \( e \in E \).
\[\mu'_{\alpha}(y) - 1 \leq \mu_{\beta_{c\cup\mu_{y}}}(y)\] for every \(y \in X\). Therefore \(x_{\alpha}^{\sim} \notin (f_{A} \cup gb)^{\sim}\). Hence \((f_{A} \cup gb)^{\sim} \subseteq f_{A}^{\sim} \cup g_{B}^{\sim}\). On the other hand, both \(f_{A}\) and \(g_{B} \subseteq f_{A} \cup gb\) implies \(f_{A}^{\sim} \cup g_{B}^{\sim} \subseteq (f_{A} \cup gb)^{\sim}\) and this complete the proof of (vi).

(vii) Since \(f_{A} \cap gb \nsubseteq f_{A}\) and \(g_{B}\), then \((f_{A} \cap gb)^{\sim} \nsubseteq f_{A}^{\sim} \cap g_{B}^{\sim}\).

(viii) It is clear that \(h_{C} \subseteq \hat{I}\) implies \(h_{C}^{b} = \hat{0}_{E}\), so that \((f_{A} \cup h_{C})^{\sim} = f_{A}^{\sim}\)\(\cup h_{C}^{b} = f_{A}^{\sim} \cup \hat{0}_{E} = f_{A}^{\sim}\).

\[\textbf{Theorem 3.2}\] Let \((X, \tau, E)\) be a fuzzy soft topological space and \(\hat{I}\) be a fuzzy soft ideal on \(X\). Then \(Fcl^{\sim}(f_{A}) = f_{A} \cup f_{A}^{\sim}\) is a fuzzy soft closure operator which induces a fuzzy soft topology \(\tau^{\sim}(\hat{I})\) given by \(\tau^{\sim}(\hat{I}) = \{f_{A} \in FSS(X)_{E};\ Fcl^{\sim}(f_{A}^{\sim}) = f_{A}^{\sim}\}\).

\[\textbf{Proof.}\] Since \(\hat{0}_{E}^{\sim} = \overline{0}_{E}\) by Theorem 3.1 (i), then \(Fcl^{\sim}(\hat{0}_{E}) = \overline{0}_{E}\). It is clear that, \(f_{A} \subseteq Fcl^{\sim}(f_{A})\).

So \((f_{A} \cup gb)^{\sim} = f_{A}^{\sim} \cup g_{B}^{\sim}\) by Theorem 3.1 (vi), then \(Fcl^{\sim}(f_{A} \cup gb) = Fcl^{\sim}(f_{A}) \cup Fcl^{\sim}(gb)\). Since \((f_{A}^{\sim})^{\sim} \subseteq f_{A}\) by Theorem 3.1 (v), then \(Fcl^{\sim}[Fcl^{\sim}(f_{A})] \subseteq Fcl^{\sim}(f_{A})\). Therefore by Theorem 2.5, \(Fcl^{\sim}(f_{A}) = f_{A} \cup f_{A}^{\sim}\) is a fuzzy soft closure operator induces a fuzzy soft topology \(\tau^{\sim}(\hat{I})\) given by \(\tau^{\sim}(\hat{I}) = \{f_{A} \in FSS(X)_{E};\ Fcl^{\sim}(f_{A}^{\sim}) = f_{A}^{\sim}\}\).

\[\textbf{Remark 3.1}\] (i) If \(\hat{I} = \{\hat{0}_{E}\}\), then \(f_{A}^{\sim} = Fcl(f_{A})\) and so, \(Fcl^{\sim}(f_{A}) = f_{A} \cup f_{A}^{\sim} = f_{A} \cup Fcl(f_{A}) = Fcl(f_{A})\), for every \(f_{A} \in FSS(X)_{E}\). So, \(\tau^{\sim}(\{\hat{0}_{E}\}) = \tau\).

(ii) If \(\hat{I} = FSS(X)_{E}\), then \(f_{A}^{\sim} = \hat{0}_{E}\) and so, \(Fcl^{\sim}(f_{A}) = f_{A} \cup f_{A}^{\sim} = f_{A} \cup \hat{0}_{E} = f_{A}\) for every \(f_{A} \in FSS(X)_{E}\).

So, \(\tau^{\sim}(FSS(X)_{E})\) is the fuzzy soft discrete topology on \(X\).

(iii) Since \(\{\hat{0}_{E}\}\) and \(FSS(X)_{E}\) are the two extreme fuzzy soft ideals on \(X\) we have \(\{\hat{0}_{E}\} \subseteq \hat{I} \subseteq FSS(X)_{E}\). So we can conclude by Theorem 3.1 (iii), \(\tau^{\sim}(\{\hat{0}_{E}\}) \subseteq \tau^{\sim}(FSS(X)_{E})\), i.e., \(\tau \subseteq \tau^{\sim}(\hat{I}) \subseteq \tau^{\sim}(\hat{J})\), i.e., \(\tau \subseteq \tau^{\sim}(\hat{I})\), \(\tau \subseteq \tau^{\sim}(\hat{J})\), i.e., \(\tau \subseteq \tau^{\sim}(\hat{J})\).

\[\textbf{Theorem 3.3}\] Let \(\tau_{1}, \tau_{2}\) be two fuzzy soft topologies on \(X\). Then for any fuzzy soft ideal \(\hat{I}\) on \(X\), \(\tau_{1} \subseteq \tau_{2}\) implies

(i) \(f_{A}^{\sim}(\hat{I}, \tau_{2}) \subseteq f_{A}^{\sim}(\hat{I}, \tau_{1})\) for every \(f_{A} \in FSS(X)_{E}\),

(ii) \(\tau_{1}^{\sim}(\hat{I}) \subseteq \tau_{2}^{\sim}(\hat{I})\).

\[\textbf{Proof.}\] Since every \(\tau_{1}\) Q-fuzzy soft nbd of any fuzzy soft point \(x_{\alpha}^{\sim}\) is also a \(\tau_{2}\) Q-fuzzy soft nbd of \(x_{\alpha}^{\sim}\). Therefore, \(f_{A}^{\sim}(\hat{I}, \tau_{2}) \subseteq f_{A}^{\sim}(\hat{I}, \tau_{1})\) as there may be other \(\tau_{2}\) Q-fuzzy soft nbd of \(x_{\alpha}^{\sim}\) where the condition for \(x_{\alpha}^{\sim} \notin f_{A}^{\sim}(\hat{I}, \tau_{2})\) may not hold true, although \(x_{\alpha}^{\sim} \notin f_{A}^{\sim}(\hat{I}, \tau_{1})\). Clearly, \(\tau_{1}^{\sim}(\hat{I}) \subseteq \tau_{2}^{\sim}(\hat{I})\) as \(f_{A}^{\sim}(\hat{I}, \tau_{2}) \subseteq f_{A}^{\sim}(\hat{I}, \tau_{1})\).

\[\textbf{Theorem 3.4}\] \(f_{A}^{d} \subseteq f_{A}^{\sim}\) and \(f_{A}^{d^*} \subseteq f_{A}^{\sim}\) for all fuzzy soft set \(f_{A}\) of \(X\), where \(f_{A}^{d^*}\) denotes the fuzzy soft driven set of \(f_{A}\) in \(\tau^{\sim}\)-fuzzy soft topology.

\[\textbf{Proof.}\] Since \(\tau \subseteq \tau^{\sim}\). Therefore, \(x_{\alpha}^{\sim} \notin f_{A}^{d^*}\) implies every Q-fuzzy soft nbd of \(x_{\alpha}^{\sim}\) in \(\tau^{\sim}\)-fuzzy soft topology is quasi-coincident with \(f_{A} \Longrightarrow \forall Q\)-fuzzy soft nbd of \(x_{\alpha}^{\sim}\) in \(\tau\)-fuzzy soft topology is quasi-coincident with \(f_{A} \Longrightarrow x_{\alpha}^{\sim} \notin f_{A}\), so that \(f_{A}^{d^*} \subseteq f_{A}^{d}\).

\[\text{Again, for any fuzzy soft point} x_{\alpha}^{\sim} \notin f_{A}^{d}\text{ implies} x_{\alpha}^{\sim} \in Fcl^{\sim}(f_{A}) = f_{A} \cup f_{A}^{\sim}\text{, i.e.,} x_{\alpha}^{\sim} \in f_{A}\text{ or} x_{\alpha}^{\sim} \in f_{A}^{\sim}\text{ or both. Now, if} x_{\alpha}^{\sim} \notin f_{A}\text{, then the result. If} x_{\alpha}^{\sim} \in f_{A}\text{, then for any Q-fuzzy soft nbd} g_{A}\text{ of} x_{\alpha}^{\sim}\text{ in} \tau^{\sim}\text{-fuzzy soft topology, there exists} y \in X\text{ such that} x \neq y\text{ and} \mu_{f_{A}(y)}^{\sim} + \mu_{g_{A}(y)} > 1\). This implies \(x_{\alpha}^{\sim}\) is a fuzzy soft accumulation point of the fuzzy soft set \(f_{A}\) such that
\[ \mu_{f_A}^\epsilon(p) = \begin{cases} \mu_{f_A}^\epsilon(p) & \text{if } p \neq x \\ \alpha^{\prime} & \text{if } p = x \text{ and } \alpha^{\prime} < \alpha \end{cases} \]

Obviously, \( f_A \subseteq f_A \), so that \( (f_A)^* \subseteq f_A^* \) and also \( x_\alpha^\epsilon \notin f_A \). Hence, \( x_\alpha^\epsilon \in (f_A^*)^* \), because \( x_\alpha^\epsilon \in \text{Fcl}^*(f_A^*) = f_A \cup (f_A)^* \). So, \( x_\alpha^\epsilon \notin f_A^* \). Therefore, \( x_\alpha^\epsilon \in f_A^* \Rightarrow x_\alpha^\epsilon \in f_A^* \Rightarrow f_A \subseteq f_A^* \). \( \square \)

**Definition 3.3** A fuzzy soft set \( f_A \) of \( X \) is called fuzzy soft closed and discrete if and only if \( f_A^* = \emptyset \).

**Theorem 3.5** Let \( (X, \tau, E) \) be a fuzzy soft topological space and \( \tilde{I} \) be a fuzzy soft ideal on \( X \). Then

(i) \( h_C \in \tilde{I} \Rightarrow h_C \) is fuzzy soft closed and discrete in \( (X, \tau^*, E) \),

(ii) \( f_A^* = \text{Fcl}(f_A \tilde{\setminus} h_C) \) for every \( h_C \in \tilde{I} \) and for any fuzzy soft set \( f_A \) of \( X \).

**Proof.** (i) Let \( h_C \in \tilde{I} \Rightarrow h_C = \emptyset \). Therefore, by Theorem 3.4, \( h_C^* = \emptyset \) and \( h_C \) is fuzzy soft closed and discrete in \( (X, \tau^*, E) \).

(ii) Clear from the definition of fuzzy soft local function and closure of fuzzy soft set. \( \square \)

**Remark 3.2.** The following example shows some cases where the two fuzzy soft topologies \( \tau \) and \( \tau^* \) are equal.

**Example 3.2.** (i) If \( \tilde{I} \) is a fuzzy soft ideal on \( X \) such that \( f_A^* \subseteq \text{Fcl}(f_A \tilde{\setminus} h_C) \) for every \( h_C \in \tilde{I} \) and for any fuzzy soft set \( f_A \) of \( X \), then it is clear that \( f_A^* \subseteq f_A^* \) so \( \text{Fcl}(f_A) \subseteq \text{Fcl}^*(f_A) \). Therefore, \( \tau = \tau^* \).

(ii) If \( \tilde{I} \) is any fuzzy soft ideal on \( X \) such that \( f_A^* = (f_A \tilde{\setminus} h_C)^\epsilon \) for every \( h_C \in \tilde{I} \), then \( f_A^* \subseteq \text{Fcl}(f_A \tilde{\setminus} h_C) = f_A^* \). This implies, \( \text{Fcl}(f_A) \subseteq \text{Fcl}^*(f_A) \). Therefore, \( \tau = \tau^* \). (iii) If \( f_A^* = f_A^* \) for any fuzzy soft set \( f_A \) of \( X \) implies \( \tau = \tau^* \).

4 Basic structure of generated fuzzy soft topology

In this section, we discuss the basic structure of new fuzzy soft topology and it is established that the new fuzzy soft topology cannot be further generated with the same fuzzy soft ideal.

**Theorem 4.1** Let \( (X, \tau, E) \) be a fuzzy soft topological space and \( \tilde{I} \) be a fuzzy soft ideal on \( X \). Then \( \beta(\tilde{I}, \tau) = \{ f_A \tilde{\setminus} h_C; f_A \in \tau, h_C \in \tilde{I} \} \) forms a basis for the generated fuzzy soft topology \( \tau^* (\tilde{I}) \).

**Proof.** Let \( (X, \tau, E) \) be a fuzzy soft topological space, \( \tilde{I} \) be a fuzzy soft ideal on \( X \) and \( f_A \) be a Q-fuzzy soft nbd of a fuzzy soft point \( x_\alpha^\epsilon \) in \( \tau^*-\text{topology} \). Therefore, there exists \( g_B \in \tau^* \) such that \( x_\alpha^\epsilon \in \text{Fcl}^*(g_B) \) and \( g_B \subseteq f_A \), i.e., \( \alpha + \mu_{g_B}^\epsilon(x) > 1 \).

Now, \( g_B \in \tau \iff g_B^* \) is fuzzy soft \( \tau^*-\text{closed} \iff \text{Fcl}^*(g_B^*) = g_B^* \iff g_B^* \subseteq [g_B^*]^\epsilon \). Therefore, \( \alpha + \mu_{g_B}^\epsilon(x) > 1 \iff \alpha + \mu_{[g_B^*]^\epsilon}(x) > 1 \iff \alpha > \mu_{[g_B^*]^\epsilon}(x) \iff x_\alpha^\epsilon \notin [g_B^*]^\epsilon \).

This implies there exists at least one Q-fuzzy soft nbd \( s_D \) of \( x_\alpha^\epsilon \) (in \( \tau \)) such that for every \( y \in X \), \( \mu_{s_D}^\epsilon(y) + \mu_{s_D}^\epsilon(y) > 1 \leq \mu_{h_C}^\epsilon(y) \), for some \( h_C \in \tilde{I} \) and for every \( \epsilon \in E \), i.e., \( \mu_{s_D}^\epsilon(y) > \mu_{h_C}^\epsilon(y) \) for every \( y \in X \) and \( \epsilon \in E \). Therefore, as \( s_D \) is a Q-fuzzy soft nbd of \( x_\alpha^\epsilon \) (in \( \tau \)), there is \( u_M \in \tau \) such that \( x_\alpha^\epsilon \in u_M \subseteq s_D \) and by heredity property of fuzzy soft ideal, we have \( u_L \in \tilde{I} \) for which \( x_\alpha^\epsilon \in u_M = u_L \).
where \( \mu_{\alpha}^\epsilon \sim_{\epsilon} (y) = \max\{\mu_{\alpha}^\epsilon (y) - \mu_{\alpha}^\epsilon (y), 0\} \) for every \( y \in X \) and \( \epsilon \in E \). Hence, for \( g_B \in \tau^* \), we have \( u_M \in \tau \) and \( v_L \in \tilde{I} \) such that \( u_M \sim v_L \subseteq g_B \subseteq f_A \). Thus \( \beta \) is a basis for \( \tau^*(\tilde{I}) \). \hfill \Box

We site an example which is very important for the further results that justifies the above construction.

**Example 4.1** Let \( \tau_0 \) be the fuzzy soft indiscrete topology on \( X \), i.e., \( \tau_0 = \{ \tilde{I}_E, \tilde{0}_E \} \). So \( \tilde{I}_E \) is the only Q-fuzzy soft nbd of every fuzzy soft point \( x^\alpha_n \). Now, \( x^\alpha_n \in f_A^\alpha \iff \) for each \( h_C \in \tilde{I} \), there is at least one \( y \in X \), \( \epsilon \in E \) such that \( \mu_{f_A}^\epsilon (y) > \mu_{h_C}^\epsilon (y) \). Therefore, \( f_A^\alpha = \tilde{I}_E \iff f_A \notin \tilde{I} \) and \( f_A^\alpha = \tilde{0}_E \iff f_A \in \tilde{I} \).

This implies that we have, \( \text{Ff} \tau^*(f_A) = f_A \cup f_A^\alpha = \tilde{I}_E \) if \( f_A \notin \tilde{I} \) and \( \text{Ff} \tau^*(f_A) = f_A \) if \( f_A \in \tilde{I} \) for any fuzzy soft set \( f_A \) of \( X \). Hence \( \tau_0^\alpha = \{ \tilde{0}_E \} \cup \{ g_B; g_B \in \tilde{I} \} \).

Let \( \tau \cup \tau_0^\alpha (\tilde{I}) \) be the supremum fuzzy soft topology of \( \tau \) and \( \tau_0^\alpha (\tilde{I}) \), i.e., the smallest fuzzy soft topology generated by \( \tau \cup \tau_0^\alpha (\tilde{I}) \). Then we have the following theorem.

**Theorem 4.2** \( \tau^*(\tilde{I}) = \tau \cup \tau_0^\alpha (\tilde{I}) \).

**Proof.** Since \( \tau \cup \tau_0^\alpha (\tilde{I}) \) forms a subbasis for a fuzzy soft topology on \( X \). Let \( f_A, g_B \in \tau \cup \tau_0^\alpha (\tilde{I}) \) such that \( f_A \in \tau \) and \( g_B \in \tau_0^\alpha (\tilde{I}) \). Then \( \beta = \{ f_A \cap g_B; f_A \in \tau, g_B \in \tau_0^\alpha (\tilde{I}) \} = \{ f_A \cap g_B; f_A \in \tau, g_B \in \tau_0^\alpha (\tilde{I}) \} \) forms a basis for \( \tau^*(\tilde{I}) \). Then \( \tau^*(\tilde{I}) = \tau \cup \tau_0^\alpha (\tilde{I}) \). \hfill \Box

**Lemma 4.1** For any two fuzzy soft ideals \( \tilde{I} \) and \( \tilde{J} \) on \( X \), \( \tilde{I} \cap \tilde{J} = \{ h_C \cap s_D; h_C \in \tilde{I}, s_D \in \tilde{J} \} \) and \( \tilde{I} \cup \tilde{J} = \{ h_C \cap s_D; h_C \in \tilde{I}, s_D \in \tilde{J} \} \) are fuzzy soft ideals on \( X \).

**Proof.** Let \( f_A \in \tilde{I} \cap \tilde{J} \) and \( g_B \subseteq f_A \). Then \( f_A = h_C \cup s_D \) for some \( h_C \in \tilde{I} \) and \( s_D \in \tilde{J} \). Since \( g_B \subseteq f_A \), then \( g_B \subseteq h_C \) or \( g_B \subseteq s_D \) or both and so \( g_B \in \tilde{I} \) or \( g_B \in \tilde{J} \) or both. Therefore, \( g_B \in \tilde{I} \cap \tilde{J} \).

**Theorem 4.3** Let \( (X, \tau, E) \) be a fuzzy soft topological space and \( \tilde{I}, \tilde{J} \) be two fuzzy soft ideal on \( X \). Then, for any fuzzy soft set \( f_A \) of \( X \),

(i) \( f_A^\alpha (\tilde{I} \cap \tilde{J}) = f_A^\alpha (\tilde{I}) \cup f_A^\alpha (\tilde{J}) \),

(ii) \( f_A^\alpha (\tilde{I} \cap \tilde{J}) = f_A^\alpha (\tilde{I}, \tau^*(\tilde{J})) \cup f_A^\alpha (\tilde{J}, \tau^*(\tilde{I})) \).

**Proof.** (i) Let \( x^\alpha_n \notin f_A^\alpha (\tilde{I}) \cup f_A^\alpha (\tilde{J}) \). Then \( x^\alpha_n \notin \) both \( f_A^\alpha (\tilde{I}) \) and \( f_A^\alpha (\tilde{J}) \). Now, \( x^\alpha_n \notin f_A^\alpha (\tilde{I}) \) implies there is at least one Q-fuzzy soft nbd \( g_B \) of \( x^\alpha_n \) (in \( \tau \)) such that for every \( y \in X \), \( \mu_{f_A}^\epsilon (y) + \mu_{h_C}^\epsilon (y) - 1 \leq \mu_{h_C}^\epsilon (y) \), for some \( h_C \in \tilde{I} \) and for every \( \epsilon \in E \). Again, \( x^\alpha_n \notin f_A^\alpha (\tilde{J}) \) implies there is at least one Q-fuzzy soft nbd \( s_D \) of \( x^\alpha_n \) (in \( \tau \)) such that for every \( y \in X \), \( \mu_{f_A}^\epsilon (y) + \mu_{g_B}^\epsilon (y) - 1 \leq \mu_{g_B}^\epsilon (y) \), for every \( \epsilon \in E \). Therefore, we have \( \mu_{f_A}^\epsilon (y) + \mu_{g_B}^\epsilon (y) - 1 \leq \mu_{g_B}^\epsilon (y) \), for every \( \epsilon \in E \). Since, \( g_B \cap s_D \) is also a Q-fuzzy soft nbd of \( x^\alpha_n \) (in \( \tau \)) and \( h_C \cap u_M \in \tilde{I} \cap \tilde{J} \), therefore \( x^\alpha_n \notin f_A^\alpha (\tilde{I} \cap \tilde{J}) \), so that \( f_A^\alpha (\tilde{I} \cap \tilde{J}) \subseteq f_A^\alpha (\tilde{I}) \cup f_A^\alpha (\tilde{J}) \). Also, \( \tilde{I} \subseteq \tilde{I} \cap \tilde{J} \), so by Theorem 3.1 (iii), \( f_A^\alpha (\tilde{I}) \cup f_A^\alpha (\tilde{J}) \subseteq f_A^\alpha (\tilde{I} \cap \tilde{J}) \), which complete the proof of (i).

(ii) Since \( x^\alpha_n \notin f_A^\alpha (\tilde{I} \cap \tilde{J}, \tau) \), there is at least one Q-fuzzy soft nbd \( g_B \) of \( x^\alpha_n \) (in \( \tau \)) such that, for every \( y \in X \), \( \epsilon \in E \), \( \mu_{f_A}^\epsilon (y) + \mu_{g_B}^\epsilon (y) - 1 \leq \mu_{u_M}^\epsilon (y) \) for some \( u_M \in \tilde{I} \). Then \( u_M = h_C \cup s_D \).
for every $\tau \in \tau^*(\tilde{I})$ and $\tau^*(\tilde{J})$, then $g_B$ is a $Q$-fuzzy soft nbd of $x^\epsilon_A$ in $\tau^*(\tilde{I})$ and $\tau^*(\tilde{J})$. Therefore, for every $y \in X$, $\epsilon \in E$, $\mu_{f_A}(y)+\mu_{g_B}(y)-1 \leq \mu_{h_C}(y)$ or $\mu_{f_A}(y)+\mu_{g_B}(y)-1 \leq \mu_{h_D}(y)$. This implies $x^\epsilon_A \notin f_A(\tilde{I}, \tau^*(\tilde{J}))$ or $x^\epsilon_A \notin f_A(\tilde{J}, \tau^*(\tilde{I}))$. Thus we have, $f_A(\tilde{I}, \tau^*(\tilde{J})) \cap f_A(\tilde{J}, \tau^*(\tilde{I})) \subseteq f_A(\tilde{I} \lor \tilde{J}, \tau)$.

Conversely, let $x^\epsilon_A \notin f_A(\tilde{I}, \tau^*(\tilde{J}))$. This implies there is at least one $Q$-fuzzy soft nbd $g_B$ in $\tau^*(\tilde{J})$ of $x^\epsilon_A$ such that for every $y \in X$, $\mu_{f_A}(y)+\mu_{g_B}(y)-1 \leq \mu_{h_C}(y)$ for some $h_C \in \tilde{I}$ and for every $\epsilon \in E$. Since $g_B$ is a $\tau^*(\tilde{J})$ Q-fuzzy soft nbd of $x^\epsilon_A$, by heredity of fuzzy soft ideals we have a $Q$-fuzzy soft nbd $u_M$ of $x^\epsilon_A$ (in $\tau$) such that for every $y \in X$, $\mu_{f_A}(y)+\mu_{g_B}(y)-1 \leq \mu_{h_C}(y)$ for some $h_C \in \tilde{I}$, $s_D \in \tilde{J}$ and for every $\epsilon \in E$, i.e., $x^\epsilon_A \notin f_A(\tilde{I} \lor \tilde{J}, \tau)$. Thus, $f_A(\tilde{I} \lor \tilde{J}, \tau) \subseteq f_A(\tilde{I}, \tau^*(\tilde{J}))$. Similarly, $f_A(\tilde{I} \lor \tilde{J}, \tau) \subseteq f_A(\tilde{J}, \tau^*(\tilde{I}))$ and this complete the proof. \[\Box\]

An important results follows from the above theorem that $\tau^*(\tilde{I})$ and $[\tau^*(\tilde{I})]^*(\tilde{I})$ (in short $\tau^{**}$) are equal for any fuzzy soft ideal on $X$.

**Corollary 4.1** Let $(X, \tau, E)$ be a fuzzy soft topological space and $\tilde{I}$ be a fuzzy soft ideal on $X$. Then $f_A^*(\tilde{I}, \tau) = f_A^*(\tilde{I}, \tau^*)$ and $\tau^*(\tilde{I}) = [\tau^*(\tilde{I})]^*(\tilde{I})$.

**Proof.** Putting $\tilde{I} = \tilde{J}$ in Theorem 4.3 (ii) we have $f_A^*(\tilde{I}, \tau) = f_A^*(\tilde{I}, \tau^*)$ and so that $\tau^*(\tilde{I}) = [\tau^*(\tilde{I})]^*(\tilde{I})$. \[\Box\]

**Corollary 4.2** Let $(X, \tau, E)$ be a fuzzy soft topological space and $\tilde{I}$, $\tilde{J}$ be two fuzzy soft ideal on $X$. Then,

(i) $\tau^*(\tilde{I} \lor \tilde{J}) = [\tau^*(\tilde{J})]^*(\tilde{I}) = [\tau^*(\tilde{I})]^*(\tilde{J})$,

(ii) $\tau^*(\tilde{I} \lor \tilde{J}) = \tau^*(\tilde{I}) \lor \tau^*(\tilde{J})$,

(iii) $\tau^*(\tilde{I} \land \tilde{J}) = \tau^*(\tilde{I}) \land \tau^*(\tilde{J})$.

**Proof.** (i) By Theorem 4.3 (ii), the result follow.

(ii) By (i), we have $\tau^*(\tilde{I} \lor \tilde{J}) = [\tau^*(\tilde{J})]^*(\tilde{I}) = \tau^*(\tilde{J}) \lor \tau_0(\tilde{I})$ (by Theorem 4.2). Since $\tau \subseteq \tau^*$ for any fuzzy soft ideal on $X$. Therefore, $\tau^*(\tilde{I} \lor \tilde{J}) = \tau \lor \tau^*(\tilde{J}) \lor \tau_0(\tilde{I}) = \tau^*(\tilde{I}) \lor \tau^*(\tilde{J})$.

(iii) Since $\tilde{I} \land \tilde{J}$ included in both $\tilde{I}$ and $\tilde{J}$, then $\tau^*(\tilde{I} \land \tilde{J})$ included in both $\tau^*(\tilde{I})$ and $\tau^*(\tilde{J})$. Now $g_B$ is a fuzzy soft open set in $\tau^*(\tilde{I}) \lor \tau^*(\tilde{J})$, implies $g_B^*$ is a fuzzy soft closed set in both $\tau^*(\tilde{I}) \cap \tau^*(\tilde{J})$. That means $(g_B^*)^*(\tilde{I}) \subseteq g_B^*$ and $(g_B^*)^*(\tilde{J}) \subseteq g_B^*$. So, $(g_B^*)^*(\tilde{I} \lor (g_B^*)^*(\tilde{J}) \subseteq g_B^*$. Therefore, by Theorem 4.3 (i), $(g_B^*)^*(\tilde{I} \land \tilde{J}) \subseteq g_B$. Hence, $g_B \in \tau^*(\tilde{I} \land \tilde{J})$. This complete the proof. \[\Box\]

5 **Compatibility of fuzzy soft ideals with fuzzy soft topology**

In this section, we define quasi-cover of a fuzzy soft set and introduce the notion of compatibility of fuzzy soft ideal with a fuzzy soft topological space and obtain some results concerning this concept.

**Definition 5.1** Let $(X, \tau, E)$ be a fuzzy soft topological space and $\tilde{I}$ be a fuzzy soft ideal on $X$. $\tau$ is said to be compatible with $\tilde{I}$, denoted by $\tau \sim \tilde{I}$, if for every fuzzy soft set $f_A$ of $X$, if for all fuzzy soft point $x^\epsilon_A \in f_A$, there exists a $Q$-fuzzy soft nbd $g_B$ of $x^\epsilon_A$ (in $\tau$) such that $\mu_{f_A}(y)+\mu_{g_B}(y)-1 \leq \mu_{h_C}(y)$ holds for every $y \in X$, $\epsilon \in E$, and for some $h_C \in \tilde{I}$. Then $f_A \in \tilde{I}$. 

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Definition 5.2 Let \( \{(g_B)_\alpha; \alpha \in \Lambda \} \) be an indexed family of fuzzy soft sets of \( X \) such that \((g_B)_\alpha \) is fuzzy soft for each \( \alpha \in \Lambda \) where \( f_A \) is a fuzzy soft set of \( X \). Then \( \{(g_B)_\alpha; \alpha \in \Lambda \} \) is said to be a fuzzy soft Q-cover of a fuzzy soft set \( f_A \) if and only if \( \mu^e_{f_A}(y) + \mu^e_{(g_B)_\alpha}(y) \geq 1 \) for every \( y \in X \) and \( e \in A \cap (\cup_{\alpha \in \Lambda} B_\alpha) \). Further, if each \((g_B)_\alpha \) is fuzzy soft open, then this fuzzy soft Q-cover will be called a fuzzy soft Q-open cover of the fuzzy soft set \( f_A \) of \( X \).

Theorem 5.1 For any fuzzy soft topological space \((X, \tau, E)\) with fuzzy soft ideal \( \tilde{I} \). The following are equivalent:

(i) \( \tau \sim \tilde{I} \),

(ii) If for every fuzzy soft set \( f_A \) of \( X \) has a fuzzy soft Q-open cover \( \{(g_B)_\alpha; \alpha \in \Lambda \} \) such that for each \( \alpha \in \Lambda \), \( \mu^e_{f_A}(y) + \mu^e_{(g_B)_\alpha}(y) - 1 \leq \mu^e_{h_c}(y) \), for some \( h_c \in \tilde{I} \) and for every \( y \in X \), \( e \in E \). Then \( f_A \in \tilde{I} \),

(iii) For every fuzzy soft set \( f_A \) of \( X \), \( f_A \cap f_A^* = 0_E \) implies \( f_A \in \tilde{I} \),

(iv) For every fuzzy soft set \( f_A \) of \( X \), \( f_A \in \tilde{I} \) where \( f_A = \bigcup x^e_\alpha \) such that \( x^e_\alpha \in f_A \) but \( x^e_\alpha \notin f_A^* \\

Proof. We prove most of the equivalent conditions which ultimately prove all the equivalence.

(i) \( \Rightarrow \) (ii): Let \( \{(g_B)_\alpha; \alpha \in \Lambda \} \) be a fuzzy soft Q-open cover of a fuzzy soft set \( f_A \) of \( X \) such that for each \( \alpha \in \Lambda \), \( \mu^e_{f_A}(y) + \mu^e_{(g_B)_\alpha}(y) - 1 \leq \mu^e_{h_c}(y) \), for some \( h_c \in \tilde{I} \) and for every \( y \in X \), \( e \in E \). Therefore, as \( \{(g_B)_\alpha; \alpha \in \Lambda \} \) is a fuzzy soft Q-open cover of \( f_A \), for each \( x^e_\alpha \in f_A \), there exists at least one \((g_B)_\alpha \) such that \( x^e_\alpha \) \( q \) \((g_B)_\alpha \) and for every \( y \in X \), \( \mu^e_{f_A}(y) + \mu^e_{(g_B)_\alpha}(y) - 1 \leq \mu^e_{h_c}(y) \), for some \( h_c \in \tilde{I} \) and for every \( e \in E \). Obviously, \((g_B)_\alpha \) is a Q-fuzzy soft nbd of \( x^e_\alpha \) \( \in \tilde{I} \). Therefore, as \( \tau \sim \tilde{I} \), \( f_A \in \tilde{I} \).

(ii) \( \Rightarrow \) (i): Clear from the fact that a collection of fuzzy soft open sets which contains at least one open Q-fuzzy soft nbd of each fuzzy soft point of \( f_A \), constitutes a fuzzy soft Q-open cover of \( f_A \).

(iii) \( \Rightarrow \) (ii): For every fuzzy soft point \( x^e_\alpha \in f_A \), there is a Q-fuzzy soft nbd \( g_B \) of \( x^e_\alpha \) \( \in \tilde{I} \) such that for every \( y \in X \), \( \mu^e_{f_A}(y) + \mu^e_{g_B}(y) - 1 \leq \mu^e_{h_c}(y) \), for every \( e \in E \), if \( x^e_\alpha \in f_A \). Since \( g_B \) is a Q-fuzzy soft nbd of \( x^e_\alpha \), therefore, there is a fuzzy soft open set \( s_D \) \( \in \tilde{I} \) such that \( x^e_\alpha \) \( q \) \( s_D \) and \( s_D \subseteq g_B \). So, the collection of such \( s_D \)'s for each \( x^e_\alpha \in f_A \), constitutes a fuzzy soft Q-open cover of \( f_A \).

(iii) \( \Rightarrow \) (iii): Let for every fuzzy point \( x^e_\alpha \in f_A \), there is a Q-fuzzy soft nbd \( g_B \) of \( x^e_\alpha \) \( \in \tilde{I} \) such that for every \( y \in X \), \( \mu^e_{f_A}(y) + \mu^e_{g_B}(y) - 1 \leq \mu^e_{h_c}(y) \), for some \( e \in E \). Since \( g_B \in \tilde{I} \), this contradicts the assumption for every fuzzy soft point of \( f_A \). So, \( \mu^e_{f_A}(x) = 0 \). That means, \( x^e_\alpha \notin f_A \) implies \( x^e_\alpha \notin f_A^* \). Now this is true for every fuzzy soft set \( f_A \) of \( X \), \( f_A \cap f_A^* = 0_E \). Hence, by condition (iii), we have \( f_A \in \tilde{I} \), which implies \( \tau \sim \tilde{I} \).

(iii) \( \Rightarrow \) (iv): Let the fuzzy soft point \( x^e_\alpha \in f_A \). That means \( x^e_\alpha \in f_A \) but \( x^e_\alpha \notin f_A^* \). So, there is a Q-fuzzy soft nbd \( g_B \) of \( x^e_\alpha \) such that for every \( y \in X \), \( \mu^e_{f_A}(y) + \mu^e_{g_B}(y) - 1 \leq \mu^e_{h_c}(y) \), for some \( h_c \in \tilde{I} \) and for every \( e \in E \). Since, \( f_A \subseteq f_A \), so for every \( y \in X \), \( \mu^e_{f_A}(y) + \mu^e_{g_B}(y) - 1 \leq \mu^e_{h_c}(y) \), for some \( h_c \in \tilde{I} \) and for
every $\epsilon \in E$. Therefore, $x^\alpha_\beta \not\in (\tilde{f}_A)^*$, so that either $\mu^e_{(f_A)}(x) = 0$ or $\mu^e_{(f_A)}(x) \neq 0$ but $\mu^e_{(f_A)}(x) \neq 0$. Let $x^\alpha_\beta$ be a fuzzy point such that $t \leq \mu^e_{(f_A)}(x)< \alpha$, i.e., $x^\alpha_\beta \not\in (\tilde{f}_A)^*$. So for each Q-fuzzy soft nbd $s^\beta_D$ of $x^\alpha_\beta$ and for each $h_C \in \tilde{I}$, there is at least one $y \in X$ such that $\mu^e_{(f_A)}(y) + \mu^e_{s^\beta_D}(y) - 1 > \mu^e_{h_C}(y)$ for some $\epsilon \in E$.

Since $\tilde{f}_A \subseteq f_A$, for each Q-fuzzy soft nbd $s^\beta_D$ of $x^\alpha_\beta$ and for each $h_C \in \tilde{I}$, there is at least one $y \in X$ such that $\mu^e_{s^\beta_D}(y) + \mu^e_{s^\beta_D}(y) - 1 > \mu^e_{h_C}(y)$ for some $\epsilon \in E$. This implies $x^\alpha_\beta \not\in (f_A \cap (f_A \cup f_A^*) = (\tilde{f}_A)^* \cup (f_A \cap f_A^*)$. This contradicts the hypothesis about every fuzzy soft set $f_A$ of $X$ that contains no non-null fuzzy soft subset $g_B$ with $g_B \subseteq g^*_B$. Therefore, $f_A \cap f_A^* = 0_E$ so that $\tilde{f}_A = f_A$ and hence by condition (iv) $f_A = \tilde{f}_A \in \tilde{I}$.

(v) $\implies$ (i) Let $f_A$ be any fuzzy soft set of $X$. Let for every fuzzy soft point $x^\alpha_\beta \in f_A$, there is Q-fuzzy soft nbd $g_B$ of $x^\alpha_\beta$ (in $\tau$) such that for every $y \in X$, $\mu^e_{f_A}(y) + \mu^e_{g_B}(y) - 1 \leq \mu^e_{h_C}(y)$, for some $h_C \in \tilde{I}$ and for every $\epsilon \in E$. This implies $x^\alpha_\beta \not\in (f_A \cup f_A^*) = f_A \cup (f_A \cup f_A^*) \subseteq f_A^*$. Therefore, $\text{Fcl}(s^\beta_D) = s^\beta_D \cup s^\beta_D = s^\beta_D$. That means, $s^\beta_D$ is a fuzzy soft $\tau^*$-closed set. Therefore, by condition (v), $s^\beta_D \subseteq \tilde{I}$. Again, since $(f_A \cup f_A^*) = (f_A \cup f_A^*) \subseteq (f_A \cup f_A^*) = f_A$, then $s^\beta_D \subseteq f_A$. Let $y^\alpha_\beta \not\in (f_A \cup f_A^*)$ and so $y^\alpha_\beta \not\in f_A$. Hence $f_A \in \tilde{I}$, i.e., $\tau \sim \tilde{I}$. $\square$

**Theorem 5.2** Let $(X, \tau, E)$ be a fuzzy soft topological space with $\tilde{I}$ be any fuzzy soft ideal on $X$. The following are implied by $\tau \sim \tilde{I}$:

(i) For any fuzzy soft set $f_A$, $f_A \cap f_A^* = 0_E$ implies $f_A^* = 0_E$.

(ii) For any fuzzy soft set $f_A$ of $X$, $(\tilde{f}_A)^* = 0_E$.

(iii) For any fuzzy soft set $f_A$ of $X$, $(f_A \cap f_A^*)^* = f_A^*$.

**Proof.** Clear from Theorem 5.1. $\square$

**Theorem 5.3** Let $(X, \tau, E)$ be a fuzzy soft topological space with $\tilde{I}$ be any fuzzy soft ideal on $X$. Let $\tau \sim \tilde{I}$. Then a fuzzy soft set $f_A$ of $X$ is $\tau^*$-closed if and only if it is the union of a fuzzy soft $\tau^*$-closed set and a fuzzy soft set in $\tilde{I}$.

**Proof.** Let $f_A$ be a fuzzy soft $\tau^*$-closed set of $X$. That means $f_A^* \subseteq f_A$ and we have, $f_A = \tilde{f}_A \cup f_A^*$. Since $\tau \sim \tilde{I}$, therefore $\tilde{f}_A \in \tilde{I}$. Also, $f_A^*$ is always $\tau$-closed (by Theorem 3.1 (iv)). This complete the necessary part of the proof.

Conversely, let $f_A$ be any fuzzy soft set of $X$ such that $f_A = g_B \cup h_C$ where $g_B$ is a fuzzy soft $\tau$-closed set and $h_C \in \tilde{I}$. Therefore, by Theorem 3.1, $f_A^* = g_B^* \cup h_C^* = g_B^* = \text{Fcl}(g_B) = g_B \subseteq f_A$. That means $f_A^* \subseteq f_A$. So, we have $\text{Fcl}(f_A) = f_A$ and this implies $f_A$ is fuzzy soft $\tau^*$-closed set of $X$. $\square$

An important consequence of Theorem 5.3 is the following corollary.
Corollary 5.1 The fuzzy soft topology $\tau$ is compatible with the fuzzy soft ideal $\tilde{I}$ on $X$ implies $\beta(\tilde{I}, \tau)$, a basis for $\tau^\ast$ is itself a fuzzy soft topology and also $\beta = \tau^\ast$.

Proof. Clear by the previous theorem. \square

References