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Generalized closed sets in bitopological spaces

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Abstract In this paper, we introduce the notions of generalized pairwise closed sets and I- generalized pairwise sets via ideal on bitopological spaces and study some of their properties. The properties of the space (X, τ_1, τ_2) are studied through the study of the space $(X, , \tau_{12})$ which is a supra topology associated to the bitopological space (X, τ_1, τ_2). The relation between these approaches has studied. We note that the finite union (intersection) of gp- (Igp)- closed sets may be not a gp-(Igp)-closed set. The study of the properties $(X, , \tau_1, \tau_2)$ via properties of (X, τ_{12}) is very important, since we are dealing with one family instead of two families and the class of supra topological spaces is wider than the class of bitopological spaces.

Key Words Bitopological space, Supra topology, P-closed, P-open, Generalized pairwise closed

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1 Introduction

(Levine, 1970)[12] introduced the fundamental concept of generalized closed sets. He defined a set A to be generalized closed if its closure contained in every open superset of A. A bitopological space (X, τ_1, τ_2) was introduced by Kelly [11] in 1963, as a method of generalizes topological space (X, τ). Every bitopological space (X, τ_1, τ_2) can be regarded as a topological space (X, τ) if $\tau_1 = \tau_2 = \tau$. Furthermore, he extended some of the standerd results of separation axioms and mappings in a topological space to a bitopological space. The notion of connectedness in bitopological space has been studied by Pervin[14], Reilly [15] and Swart [17]. Fukutake[7] introduce and investigate the concept of $(\tau_i\tau_j) - g-$closed in a bitopological space. In 1983 Mashhor et al. [13] introduced supra topological spaces by dropping only the intersection condition. Kandil,[10] generated a supra topological space (X, τ_{12}) from the bitopological space (X, τ_1, τ_2) and they studied some properties of the space (X, τ_1, τ_2) via properties of the associated

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space (X, τ_{12}). Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting [2, 3, 4, 5, 10, 16].

In this paper we introduce the notion of generalized P-closed (P-open)sets and study some of their properties. Also, we introduce the notion of generalize pairwise closed (open) sets via ideals, study some properties and investigate the relation between two approaches.

2 Preliminaries

This section contains the basic concept proprieties of generalized closed sets, generalized open set, supra topological spaces and bitopological spaces.

Definition 2.1. *[11] A bitopological spaces (bts, for short) is a triple (X, τ_1, τ_2) where τ_1 and τ_2 are arbitrary topologies on X.*

Definition 2.2. *[3]. Let (X, τ_1, τ_2) be a bitopological space. Then, $A \subseteq X$ is said to be pairwise open (P-open, for short) if $A = U_1 \cup U_2$, $U_i \in \tau_i$, $(i = 1, 2)$. A set A is a P-closed if its complement A' is P-open.*
Note that the notion of P.open sets as well as P-closed set has studied in [6, 10] under the name of P^*-open and P^*-closed.

Definition 2.3. *[1]. A family $\eta \subseteq P(X)$ is said to be a supra topology η contains X, ϕ and closed under arbitrary union. The element of η are supraopen sets and their complements are said to be supraclosed sets.*

Proposition 2.1. *[6] Let (X, τ_1, τ_2) be a bts. The family of all P-open subsets of X, denoted by*

$$\tau_{12} = \{U_1 \cup U_2 : U_i \in \tau_i, i = 1, 2\}$$

is a supratopology on X and (X, τ_{12}) is the supra topological space associated to the bts (X, τ_1, τ_2).

Definition 2.4. *[10]. An operator $C : P(X) \longrightarrow P(X)$ is a supra closure closure operator if it satisfies the following conditions for all $A, B \subseteq X$.*

1. *$C(\phi) = \phi$.*
2. *$A \subseteq C(A)$.*
3. *$C(A \cup B) \supseteq C(A) \cup C(B)$.*
4. *$C(C(A)) = C(A)$.*

Proposition 2.2. *[6]. Let (X, τ_1, τ_2) be a bts. Then, the operator $cl_{12} : P(X) \longrightarrow P(X)$ defined by $cl_{12}(A) = \overline{A}^1 \cap \overline{A}^2$, is a supra closure operator such that, $\tau_{12} = \{A \subseteq X : cl_{12}(A') = A'\}$ and $\overline{A}^i$is the closure of A with respect to τ_i and $i = 1, 2$.*

Proposition 2.3. *[6]. Let (X, τ_1, τ_2) be a bts. Then the operator, $int_{12} : P(X) \longrightarrow P(X)$ defined by, $int_{12}(A) = A^{o1} \cup A^{o2}$ is a supra interior operator such that $\tau_{12} = \{A \subseteq X : int_{12}(A) = A\}$, where $A^{oi}, (i = 1, 2)$ is the τ_i- interior with respect to τ_i.*

It is easy to verify the following proposition.

Proposition 2.4. *[6]. Let (X, τ_1, τ_2) be a bts and $A \subseteq X$. Then*

1. $\tau_1, \tau_2 \subseteq \tau_{12}$.
2. $cl_{12}(A) = X \backslash int_{12}(A)$.
3. $int_{12}(A) = X \backslash cl_{12}(A)$.
4. *A is P-open* $\Leftrightarrow A = int_{12}(A)$.
5. *A is P-closed* $\Leftrightarrow A = cl_{12}(A)$.

Definition 2.5. *[12]. A subset A of a topological space (X, τ) is said to be generalized closed (briefly g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).*

Proposition 2.5. *[12]. If A and B are g-closed sets in space (X, τ), then $A \cup B$ is a g-closed set in X.*

Definition 2.6. *[10]. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \gamma_1, \gamma_2)$ is said to be P^* continuous function (P^*-cts, for short) if and only if $f^{-1}(B) \in \tau_{12}$ for all $B \in \gamma_{12}$.*

Definition 2.7. *[10]. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \gamma_1, \gamma_2)$ is said to be P^* closed (P^*-open) if and only if $f(A)$is γ_{12}closed (open) for all τ_{12}-closed (open) set A in X.*

Definition 2.8. *[9]. An ideal I on a set X is a nonempty collection of subset of X (i.e $I \subseteq P(X)$) such that*

1. *if $A \in I$ and $B \subseteq A$, then $B \in I$,*
2. *if $A, B \in I$, then $A \cup B \in I$.*

3 Generalized pairwise closed and generalized pairwise open sets

In this section we defined the concepts of generalized pairwise closed sets and generalized pairwise open sets and study some of their properties.

Definition 3.1. *Let (X, τ_1, τ_2) be a bts and (X, τ_{12}) be its associated supra topological space. Then, $A \subseteq X$ is called a generalized pairwise closed set (gp-closed, for short) if, $cl_{12}(A) \subseteq O$ whenever $A \subseteq O$, O is a P-open.*

Theorem 3.1. *A set A is a gp-closed set if and only if $cl_{12}(A) \backslash A$ contains no non empty P-closed sets.*

Proof. Let $\phi \neq F$ be a P-closed set such that $F \subseteq cl_{12}(A) \backslash A$. It is follows that $A \subseteq F^{'}$ but A is a *gp*-closed set, $F^{'}$ is P-open, then $cl_{12}(A) \subseteq F^{'}$. This implies that $F \subseteq (cl_{12}(A))^{'}$, this means $F \subseteq (cl_{12}(A))^{'} \cap cl_{12}(A) \backslash A = \phi$. Then, $F = \phi$ which is a contradiction.

Conversely, let $A \subseteq X$ such that $cl_{12}(A) \backslash A$ be contains no non empty P-closed set, $O \in \tau_{12}$ such that $A \subseteq O$. Suppose $cl_{12}(A) \nsubseteq O$. Then $cl_{12}(A) \cap O^{'}$ is P-closed set. Thus, $cl_{12}(A) \cap O^{'} \subseteq cl_{12}(A) \backslash A$ which is a contradiction. Therefore, $cl_{12}(A) \subseteq O$. Hence, A is a *gp*-closed set. □

Corollary 3.1. *If A is a gp-closed set, then A is P-closed if and only if $cl_{12}(A) \backslash A$ is a P-closed set.*

Proof. Necessity. Let A be P-closed.Then, $cl_{12}(A) = A$. Since, $cl_{12}(A)\backslash A = \phi$ which is a P-closed set. Sufficiency. Let $cl_{12}(A)\backslash A$ be P-closed. Since, A is gp-closed set. Then, $cl_{12}(A)\backslash A = \phi$ gives $cl_{12}(A) \subseteq A$, but we have $A \subseteq cl_{12}(A)$. Therefore, $cl_{12}(A) = A$. Hence, A is P-closed set. □

Remark 3.1. *1. If A is P-closed set, then A is gp-closed set.*

2. If $A$and B are gp-closed sets, then $A \cup B$ and $A \cap B$ are not necessary gp-closed sets.

Example 3.1. *Let $X = \{a, b, c, d, e\}$, $\tau_1 = \{X, \phi, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{c, b\}\}$. Then, $\tau_{12} = \{X, \phi, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Take $A = \{a, b, d\}$ and $B = \{b, e, c\}$. Then A and B are gp-closed sets but $A \cap B = \{b\}$ is not gp-closed set, since $\{b\} \subseteq \{b, c\}$ but $cl_{12}\{b\} = X \nsubseteq \{b, c\}$.*

Example 3.2. *Let $X = \{1, 2, 3, 4, 5\}$, $\tau_1 = \{X, \phi, \{1, 2, 3\}, \{3, 4, 5\}, \{3\}\}$, $\tau_2 = \{X, \phi, \{1, 2, 4\}\}$. Then, $\tau_{12} = \{X, \phi, \{1, 2, 3\}, \{3, 4, 5\}, \{3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$. Now, $\{4, 5\}$ and $\{3, 5\}$ are gp-closed sets but $\{4, 5\} \cup \{3, 5\} = \{3, 4, 5\}$ is not a gp-closed set.*

Definition 3.2. *Let (X, τ_1, τ_2) be a bts, $A \subseteq X$ and $(A, \tau_1|_A, \tau_2|_A)$ be a bitopological subspace of (X, τ_1, τ_2). Then, $\tau_{12}(A) = \tau_1|_A \sqcup \tau_2|_A = \{U_1 \cup U_2 : U_1 \in \tau_1|_A, U_2 \in \tau_2|_A\}$ such that $\tau_i|_A = \{A \cap U_i : U_i \in \tau_i, i = 1, 2\}$.*

Theorem 3.2. *Let (X, τ_1, τ_2) be a bts, $A \subseteq X$. Then, $\tau_{12}|_A = \tau_{12}(A)$, where $\tau_{12}|_A = \{A \cap U : U \in \tau_{12}\}$.*

Proof. Let $V \in \tau_{12}|_A$. Then, $\exists V_1 \in \tau_{12}$ such that $V = V_1 \cap A$. Also $\exists U_i \in \tau_i$; $(i = 1, 2)$ such that $V_1 = U_1 \cup U_2$. Thus, $V = (U_1 \cup U_2) \cap A = (U_1 \cap A) \cup (U_2 \cap A)$. But $U_i \cap A \in \tau_i|_A \forall i = 1, 2$, then we get $V \in \tau_{12}(A)$. Hence, $\tau_{12}|_A \subseteq \tau_{12}(A)$. On the other hand, Let $F \in \tau_{12}(A)$. Then $\exists F_1, F_2$ such that $F = F_1 \cup F_2, F_1 \in \tau_1|_A, F_2 \in \tau_2|_A$. Also, $F_1 = U_1 \cap A, F_2 = U_2 \cap A$, such that $U_i \in \tau_i, i = 1, 2$. Then we get, $F = F_1 \cup F_2 = (U_1 \cap A) \cup (U_2 \cap A) = (U_1 \cup U_2) \cap A$, but $U_1 \cup U_2 \in \tau_{12}$. Then $F \in \tau_{12}|_A$. Hence, $\tau_{12}(A) \subseteq \tau_{12}|_A$, and consequently $\tau_{12}(A) = \tau_{12}|_A$. □

Theorem 3.3. *Let (X, τ_1, τ_2) be a bts. Suppose that $B \subseteq A \subseteq X$, B is a gp-closed set relative to A and A is a gp-closed set of X. Then, B is a gp-closed set relative to X.*

Proof. Let $B \subseteq O$,O be P-open in X. Then, $B \cap A \subseteq O \cap A$. Hence, $B \subseteq O \cap A$, but B is gp-closed relative to A, then $cl^A_{12}(B) \subseteq A \cap O$. ($cl^A_{12}$, the closure is taken w.r.t A). Then $cl_{1A}(B) \cap cl_{2A}(B) \subseteq A \cap O$. $\Rightarrow$ $[A \cap cl_1(B)] \cap [A \cap cl_2(B)] \subseteq A \cap O$. $\Rightarrow$ $cl_1(B) \cap cl_2(B) \cap A \subseteq A \cap O$. $\Rightarrow$ $cl_{12}(B) \cap A \subseteq A \cap O$. $\Rightarrow A \subseteq O \cup [cl_{12}(B)]^{'}$. $O \cup [cl_{12}(B)]^{'}$ is P-open in X. Then $cl_{12}(A) \subseteq O \cup [cl_{12}(B)]^{'}$. Since $B \subseteq A$, then $cl_{12}(B) \subseteq cl_{12}(A) \subseteq O \cup [cl_{12}(B)]^{'}$. Hence, $cl_{12}(B) \cap [O \cup (cl_{12}(B))^{'}]^{'} = \phi$, $\Longrightarrow cl_{12}(B) \cap O^{'} \cap cl_{12}(B) = \phi$. $\Longrightarrow cl_{12}(B) \subseteq O$. It is follows that B is a gp-closed set in X. □

Corollary 3.2. *If A is a gp-closed set and F is P-closed set, then $A \cap F$ is a gp-closed set.*

Theorem 3.4. *Let (X, τ_1, τ_2) be a bts and $A \subseteq X$. If A is a gp closed set and $A \subseteq B \subseteq cl_{12}(A)$. Then, $B$is a gp-closed set.*

Proof. Let A be a gp-closed set, $A \subseteq B \subseteq cl_{12}(A)$ and let $B \subseteq O$, O be P-open set. Then, $A \subseteq O$. But, A is gp-closed, this implies that $cl_{12}(A) \subseteq O$, but $cl_{12}(B) \subseteq cl_{12}(cl_{12}(A)) = cl_{12}(A)$, thus $cl_{12}(B) \subseteq O$. Hence, B is a gp-closed set. □

Theorem 3.5. *Let (X, τ_1, τ_2) be a bts and $A \subseteq Y \subseteq X$. Suppose that A is gp-closed in X. Then, A is gp-closed relative to Y.*

Proof. Let $A \subseteq Y \subseteq X$, A be a *gp*-closed set in X. Let $A \subseteq O^*, O^*$ be P-open in Y. Then, $\exists O \in \tau_{12}$ such that $O \cap Y = O^*$. Now, $A \subseteq O \cap Y \subseteq O$, but A is *gp*-closed relative to X. So, $cl_{12}(A) \subseteq O \Rightarrow cl_{12}(A) \cap Y \subseteq O \cap Y$, then $cl_{12}^A(A) \subseteq O^*$. Hence, A is *gp*-closed relative to Y. □

Theorem 3.6. *Let (X, τ_1, τ_2) be a bts, τ_{12} be supra topology on X induced by τ_1, τ_2. Then, $\tau_{12} = \tau_{12}^{'}$ iff every subset of X is a gp- closed set.*

Proof. " $\Longrightarrow$ " : Let $\tau_{12} = \tau_{12}^{'}$, $A \subseteq X$ and $A \subseteq O$, O be a P-open set in X. Then, $cl_{12}(A) = cl_{12}(O) = O$, since $\tau_{12} = \tau_{12}^{'}$. Hence, A is a *gp*-closed set.

" $\Longleftarrow$ " : Let $A \subseteq X; A \in \tau_{12}$ and let every subset of X be a *gp*-closed set. Then, $cl_{12}(A) \subseteq A$, but we have $A \subseteq cl_{12}(A)$, so $cl_{12}(A) = A$. Hence, $\tau_{12} \subseteq \tau_{12}^{'}$. On the other hand, let $F \in \tau_{12}^{'}$. Then, $F^{'} \in \tau_{12}, F^{'} \subseteq F^{'}$(by hypothesis), $cl_{12}(F^{'}) \subseteq F^{'}$ but, $F^{'} \subseteq cl_{12}(F^{'})$, then $F^{'} = cl_{12}(F^{'})$. Thus, $F \in \tau_{12}$ and consequently $\tau_{12}^{'} \subseteq \tau_{12}$. Hence, $\tau_{12} = \tau_{12}^{'}$. □

Definition 3.3. *A set A is called a generalized pairwise open (for short, gp-open) set if and only if $A^{'}$ is a gp-closed set.*

Theorem 3.7. *Let (X, τ_1, τ_2) be a bts. Then, A set A is a gp-open set if and only if $F \subseteq int_{12}(A)$ whenever F is a P-closed set and $F \subseteq A$.*

Proof. " $\Longrightarrow$ " Let A be a *gp*-open set and $F \subseteq A$ such that F is P-closed set. Then, $A^{'}$ is *gp*- closed, $A^{'} \subseteq F^{'}$ and $F^{'}$ is P-open thus, $cl_{12}(A^{'}) \subseteq F^{'}$, implies $F \subseteq [cl_{12}(A^{'})]^{'} = int_{12}(A)$. Hence, $F \subseteq int_{12}(A)$.

" $\Longleftarrow$ "Let $A \subseteq X, F$ be a P-closed set and $F \subseteq int_{12}(A)$ whenever $F \subseteq A$. We prove that A is a *gp*-open or $A^{'}$ is a *gp*-closed. So, let $A^{'} \subseteq O$; O is P-open. Then $O^{'} \subseteq A$; $O^{'}$ is P-closed. By hypothesis, $O^{'} \subseteq int_{12}(A)$, thus, $[int_{12}(A)]^{'} \subseteq O$ and $cl_{12}(A^{'}) \subseteq O$. Hence, $A^{'}$ is *gp*-closed and consequently A is a *gp*-open. □

Definition 3.4. *[4] Let (X, τ_1, τ_2) be a bts and $A \subseteq B \subseteq X$. Then A and B are P^*- separated in X if $A \cap cl_{12}(B) = \phi$ and $cl_{12}(A) \cap B = \phi$.*

Not that if A and B are P-separated and $C \subseteq A, D \subseteq B$, then $CandD$ are P^*- separated.

Theorem 3.8. *Let (X, τ_1, τ_2) be a bts. If A and B are P^*-separated and gp-open sets, then $A \cup B$ is a gp-open set.*

Proof. Let F be a P-closed set such that $F \subseteq A \cup B$. Also, let A, B be *gp*-open and P^*-separated. Then $F \cap cl_{12}(A) \subseteq (A \cup B) \cap cl_{12}(A) = (A \cap cl_{12}(A) \cup \phi)$, since A, B are P-separated, so, $F \cap cl_{12}(A) \subseteq (A)$. Similarly, $F \cap cl_{12}(B) \subseteq B$. Now, $F \cap cl_{12}(A)$ and $F \cap cl_{12}(B)$ are P-closed and A, B are *gp*-open, then, $F \cap cl_{12}(A) \subseteq int_{12}(A)$ and $F \cap cl_{12}(B) \subseteq int_{12}(B)$. It is follows $F \cap cl_{12}(A) \cup F \cap cl_{12}(B) \subseteq int_{12}(A) \cup int_{12}(B) \subseteq int_{12}(A \cup B)$, but $F = F \cap (A \cup B) \subseteq F \cap (cl_{12}(A) \cup cl_{12}(B))$, this implies that $F \subseteq int_{12}(A \cup B)$, then $A \cup B$ is *gp*-open set. □

Corollary 3.3. *Let (X, τ_1, τ_2) be a bts and let A and B be two gp-closed sets and suppose that $A^{'}, B^{'}$ are $P^{*}-$ separated. Then, $A \cap B$ is a gp-closed set.*

Proof. Let A and B be a gp-closed subsets. Then, $A^{'}, B^{'}$ are gp-open, by Theorem 3.8 we get $A^{'} \cup B^{'}$ is gp-open. Hence, $(A^{'} \cup B^{'})^{'} = A \cap B$ is a gp- closed set. □

Theorem 3.9. *Let (X, τ_1, τ_2) be a bts. Then, a set A is gp- open if and only if $O = X$ whenever O is P-open and $int_{12}(A) \cup A^{'} \subseteq O$.*

Proof. Necessity. Let A be a gp-open set and O be a P-open set such that $int_{12}(A) \cup A^{'} \subseteq O$. Then $O^{'} \subseteq (int_{12}(A) \cup A^{'})^{'} = (int_{12}(A))^{'} \cap A = cl_{12}(A^{'}) \setminus A^{'}$, but $O^{'}$ is P-closed set and $A^{'}$ is gp-closed set, then by Theorem 3.1 we get $O^{'} = \phi$. Hence, $O = X$.

Sufficiency. Let $F \subseteq A$, F be a P-closed set. Then, $A^{'} \subseteq F^{'}$, thus $int_{12}(A) \cup A^{'} \subseteq int_{12}(A) \cup F^{'} = X$ (by hypothesis), then $int_{12}(A) \supseteq F$. Hence, A is gp-open. □

Theorem 3.10. *Let (X, τ_1, τ_2) be a bts and $A, B \in P(X)$. If $int_{12}(A) \subseteq B \subseteq A$ and A is gp-open set, then B is gp-open set.*

Proof. Let $A, B \in P(X)$ and A be gp-open set such that $int_{12}(A) \subseteq B \subseteq A$. Then, $A^{'} \subseteq B^{'} \subseteq [int_{12}(A)]^{'} = cl_{12}(A^{'})$, by Theorem 3.4 we get $A^{'}$ is gp-closed, then $B^{'}$ is gp-closed set. Hence, B is gp-open set. □

Theorem 3.11. *Let (X, τ_1, τ_2) be a bts. Then a set A is a gp-closed if and only if $cl_{12}(A) \setminus A$ is gp-open.*

Proof. Necessity. Let A be a gp- closed set, $F \subseteq cl_{12}(A) \setminus A$ such that F be a P-closed set. By Theorem 3.1 we get, $cl_{12}(A) \setminus A$ has no non empty subset, then $F = \phi$, thus $\phi \subseteq int_{12}[cl_{12}(A) \setminus A]$. Hence, $cl_{12}(A) \setminus A$ is gp- open set.

Sufficiency. Let $cl_{12}(A) \setminus A$ be a gp-open set and $A \subseteq O$ such that O be a P-open set. Then, $O^{'} \subseteq A^{'}$, so we get $cl_{12}(A) \cap O^{'} \subseteq cl_{12}(A) \cap A^{'} = cl_{12}(A) \setminus A$, but $cl_{12}(A) \setminus A$ is gp-open set and $cl_{12}(A) \cap O^{'}$ is P-closed set. Thus, $cl_{12}(A) \cap O^{'} \subseteq int_{12}(cl_{12}(A) \setminus A) = \phi$ so, $cl_{12}(A) \cap O^{'} = \phi$ then, $cl_{12}(A) \subseteq O$. Hence A is gp-closed set. □

Definition 3.5. *Let (X, τ_1, τ_2) be a bts. A space (X, τ_1, τ_2) is said to be a pairwise symmetric (P^{*}-symmetric for short) if for all x and y in X, then $x \in cl_{12}\{y\} \Longrightarrow y \in cl_{12}\{x\}$.*

Theorem 3.12. *A space (X, τ_1, τ_2) is a P^{*}-symmetric if and only if $\{x\}$ is a gp- closed $\forall x \in X$.*

Proof. Necessity. Let (X, τ_1, τ_2) be a P^{*}-symmetric space, $\{x\} \subseteq O$ such that $O \in \tau_{12}$, we want to prove $cl_{12}\{x\} \subseteq O$. Suppose that $cl_{12}\{x\} \nsubseteq O$, thus $cl_{12}\{x\} \cap O^{'} \neq \phi$. Let $y \in cl_{12}\{x\} \cap O^{'}$. Then a X is P-symmetric and $x \neq y$, then $x \in cl_{12}\{y\}$, therefore $cl_{12}\{y\} \subseteq cl_{12}\{x\} \cap O^{'}$, so $x \in O^{'}$ but $x \in O$ which is contradiction, then $cl_{12}\{x\} \subseteq O$. Hence, $\{x\}$ is gp-closed set.

Sufficiency.Let $x, y \in X, x \neq y$ and $x \in cl_{12}\{y\}$. Suppose that $y \notin cl_{12}\{x\}$, then $\{y\} \subseteq (cl_{12}\{x\})^{'}$, as $\{y\}$ is gp-closed set, implies $cl_{12}\{y\} \subseteq (cl_{12}\{x\})^{'}$, then $x \in (cl_{12}\{x\})^{'}$ which is contradiction. Hence, $y \in cl_{12}\{x\}$. □

Theorem 3.13. *If A is gp-closed set in X and $f : X \longrightarrow Y$ is P^*-cts and is P^*-closed function, then $f(A)$ is a gp-closed set.*

Proof. Let A be a gp-closed set in $X, f(A) \subseteq O$ such that O is P-open subset of Y. So, $A \subseteq f^{-1}(O)$, but f is P^*-cts, then $f^{-1}(O)$ is P-open set in X. As A is a gp-closed set, then $cl_{12}(A) \subseteq f^{-1}(O)$. Since f isP^*-closed, then $cl_{12}(f(A)) \subseteq f(cl_{12}(A))$. It follows that $cl_{12}[f(A)] \subseteq f(cl_{12}(A)) \subseteq ff^{-1}(O) \subseteq O$. Hence, $f(A)$ is gp-closed set in Y. □

Remark 3.2. *If f is P^*- continuous and P^*-closed function and A is gp-open set, then $f(A)$ is not necessary to be a gp-open set, as shown in the follwing example.*

Example 3.3. *Let $X = \{a\}, Y = \{b, c\}, \tau_1 = \{X, \phi\}, \tau_1 = \tau_2, \nu_1 = \{Y, \phi\}, \nu_2 = \{Y, \phi, \{b\}\}, \nu_{12} = \{Y, \phi, \{b\}\}, f\{a\} = c, f$ is P^*-closed mapping and P^*- cts but $\{a\}$ is gp-open and $f\{a\} = \{c\}$ is not a gp-open set.*

Theorem 3.14. *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \upsilon_1, \upsilon_2)$ is P^*-cts and P^*-closed function, B is a gp-closed(or gp-open) subset of Y, then $f^{-1}(B)$ is gp-closed (or gp-open) set in X.*

Proof. Suppose that B is gp-closed subset of Y and $f^{-1}(B) \subseteq O, O$ is P-open in X. We will show that $cl_{12}[f^{-1}(B)] \subseteq O$ or $cl_{12}[f^{-1}(B)] \cap O' = \phi$, therefore $f[cl_{12}[f^{-1}(B)] \cap O'] \subseteq f(cl_{12}[f^{-1}(B)]) \cap f[O'] \subseteq cl_{12}(B) \cap f[O'] \subseteq cl_{12}(B) \backslash B = \phi$, that is by Theorem 3.1, then $f[cl_{12} f^{-1}(B) \cap O'] = \phi$, so $cl_{12}[f^{-1}(B)] \subseteq O$. Hence, $f^{-1}(B)$ is gp-closed. By taking complements we can show that if B is gp-open in Y, then $f^{-1}[B]$ is gp-open in X. □

4 Generalized pairwise open and pairwise closed sets via Ideals

In this section we induced and study the concept of gp- closed sets and gp-open sets with respect to an ideal, which is the extension of the concept of gp-closed sets and gpopen sets.

Definition 4.1. *Let (X, τ_1, τ_2) be a bts, (X, τ_{12}) its associated supra topological space and I be an ideal on X. Then $A \subseteq X$ is called a generalized pairwise closed set with respect to an ideal (Igp-closed, for short) iff $cl_{12}(A) \backslash B \in I$ whenever $A \subseteq B$, B is P-open.*

Remark 4.1. *Every gp-closed set is an Igp-closed set, but the converse need not be true as shown in the following example.*

Example 4.1. *Let $X = \{a, b, c, d, e\}$, $\tau_1 = \{X, \phi, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{b, c\}\}$. Then $\tau_{12} = \{X, \phi, \{a, b\}, \{b, c\}\{a, b, c\}, I = \{\phi, \{a\}, \{c\}, \{e\}, \{d\}, \{a, c\}, \{a, e\}, \{a, d\}, \{c, e\}, \{c, d\}\{e, d\}, \{a, c, e\}, \{a, c, d\}, \{c, e, d\}\}$. We have $\{b\}$ is not gp-closed set but it is an Igp-closed set.*

Example 4.2. *Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{b, c\}\}$, $\tau_2 = \{X, \phi, \{c, d\}\}$. Then $\tau_{12} = \{X, \phi, \{b, c\}, \{c, d\} \{b, c, d\}$, $I = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. We have here $\{b\}$ is an Igp-closed sets, but not a gp-closed set.*

Theorem 4.1. *Let (X, τ_1, τ_2) be a bts. A set A is an Igp-closed set if and only if $F \subseteq cl_{12}(A) \backslash A$ and F is a P-closed set $\Longrightarrow F \in I$.*

Proof. Assume that A is a Igp- closed. Let $F \subseteq cl_{12}(A)\backslash A$. Suppose F is a P-closed set. Then, $A \subseteq F^{'}$, but A is Igp-closed. By our assumption, $cl(A)\backslash F^{'} \in I$. But $F \subseteq cl_{12}(A)\backslash F^{'} \Longrightarrow F \in I$.

Conversely, assume that $F \subseteq cl_{12}(A)\backslash A$ and F is P-closed in X implies that $F \in I$. Suppose $A \subseteq U$ and U is P-open. Then, $cl_{12}(A)\backslash U = cl_{12}(A) \cap U^{'}$ which is a P-closed set and contained in $cl_{12}(A)\backslash A$, since, $A \subseteq U$. By assumption, $cl_{12}(A)\backslash U \in I$. Hence, A is Igp-closed. □

Remark 4.2. *The intersection and union of two an Igp-closed sets need not be an Igp-closed set as shown by next examples.*

Example 4.3. *Let $X = \{1,2,3,4,5\}$, $\tau_1 = \{X, \phi, \{1,2,3\}\{3,4,5\}, \{3\}\}$, $\tau_2 = \{X, \phi, \{1,2,4\}\}$. Then $\tau_{12} = \{X, \phi, \{3\}, \{1,2,3\}, \{3,4,5\}, \{1,2,4\}, \{1,2,3,4\}\}$, $I = \{\phi\}$, we have here $\{4,5\}$ and $\{3,5\}$ are Igp-closed sets, but $\{4,5\} \cup \{3,5\} = \{3,4,5\}$ is not an Igp-closed set.*

Example 4.4. *Let $X = \{a,b,c,d,e\}$, $\tau_1 = \{X, \phi, \{a,b\}\}$, $\tau_2 = \{X, \phi, \{b,c\}\}$. Then $\tau_{12} = \{X, \phi, \{a,b\}, \{b,c\}\{a,b,c\}$, $I = \{\phi, \{a\}\}$ we have here $\{a,b,d\}$ and $\{d,e,c\}$ are Igp-closed sets, but $\{a,b,d\} \cap \{d,e,c\} = \{d\}$ is not Igp-closed set.*

Example 4.5. *In example 4.2 we can see that $\{b\}$ and $\{d\}$ are Igp-closed sets, but $\{b\} \cup \{d\} = \{b,d\}$ is not a Igp-closed set.*

Theorem 4.2. *Let (X, τ_1, τ_2) be a bts. Let $A \subseteq Y \subseteq X$ and suppose that A is an Igp-closed. Then A is an Igp-closed relative to the subspace Y of X, with respect to the ideal $I_Y = \{F \subseteq Y : F \in I\}$.*

Proof. Suppose $A \subseteq U \cap Y$and U is P-open in (X, τ_{12}). Then $A \subseteq U$. Since, A is Igp-closed relative to X, we have $cl_{12}(A)\backslash U \in I$. Now, $cl^{Y}_{12}(A)\backslash(U \cap Y) = Y \cap (cl_{12}(A) \cap (Y \cap U)^{'}$ =$[Y \cap cl_{12}(A) \cap Y^{'}] \cup Y \cap cl_{12}(A) \cap U^{'}$ =$\phi \cup [cl_{12}(A)\backslash U] \cap Y = [cl_{12}(A)\backslash U] \cap Y \in I$. Then,$cl^{Y}_{12}(A)\backslash(U \cap Y) \in I_Y$. Hence, A is Igp-closed in Y. □

Theorem 4.3. *Let (X, τ_1, τ_2) be a bts, A be an Igp-closed set and F be a P-closed set. Then, $A \cap F$ is an Igp-closed set.*

Proof. Let A be an Igp-closed and F be a P-closed set. Suppose $A \cap F \subseteq U$ and U is a P-open. Then, $A \subseteq U \cup F^{'}$. Since, A is Igp-closed and $U \cup F^{'}$ is a P-open, then $cl_{12}(A)\backslash U \cup F^{'} \in I$. Now, $cl_{12}(A \cap F) \subseteq cl_{12}(A) \cap F = [cl_{12}(A) \cap F]\backslash F^{'}$. Therefore, $cl_{12}(A \cap F)\backslash U \subseteq [cl_{12}(A \cap F)\backslash F^{'}]\backslash U = [cl_{12}(A) \cap F]\backslash[U \cup F^{'}] \in I$ since, $cl_{12}(A) \cap F \backslash(U \cup F^{'}) \subseteq cl_{12}(A)\backslash U \cup F^{'} \in I$. Hence, $A \cap F$ is an Igp-closed. □

Corollary 4.1. *Let (X, τ_1, τ_2) be a bts. If A is an Igp-closed and $A \subseteq B \subseteq cl_{12}(A)$, then B is an Ig-closed set.*

Proof. Suppose A is an Igp-closed and $A \subseteq B \subseteq cl_{12}(A)$. Suppose that $B \subseteq G$ and G is P-open. Then, $A \subseteq G$. Since, A is Igp-closed. Then $cl_{12}(A)\backslash G \in I$, we have, $B \subseteq cl_{12}(B)$. This implies that $cl_{12}(B) \subseteq cl_{12}(A)$. Now, $cl_{12}(B)\backslash G \subseteq cl_{12}(A)\backslash\ G$, but $cl_{12}(A)\backslash\ G \in I \Longrightarrow cl_{12}(B)\backslash\ G \in I$. Hence, B is an Igp-closed. □

Definition 4.2. *Let (X, τ_1, τ_2) be a bts, (X, τ_{12}) its associated supra topological space and I be an ideal on X. Then, $A \subseteq X$ is called a generalized pairwise open set with respect to an ideal (Igp-open for short) if and only if $A^{'}$ is Igp-closed.*

Theorem 4.4. *Let (X, τ_1, τ_2) be a bts. Then, a set A is Igp-open if and only if $F \backslash U \subseteq int_{12}(A)$, for some $U \in I$ whenever $F \subseteq A$ and F is a P-closed.*

Proof. Suppose A is an Igp-open. Suppose $F \subseteq A$ and F is P-closed. We have $A^{'} \subseteq F^{'}$. By assumption, $cl_{12}(A^{'}) \subseteq F^{'} \cup U$ for some $U \in I$. This implies, $[F^{'} \cup U]^{'} \subseteq [cl_{12}(A^{'})]^{'}$ and hence, $F \backslash U \subseteq int_{12}(A)$.

Conversely, assume that $F \subseteq A$ and F is P-closed imply $F \backslash U \subseteq int_{12}(A)$, for some $U \in I$. Consider a P-open set G such that $A^{'} \subseteq G$. Then, $G^{'} \subseteq A$. By assumption, $G^{'} \backslash U \subseteq int_{12}(A) = [cl_{12}(A^{'})]^{'}$. Then we get $[G \cup U]^{'} \subseteq [cl_{12}(A^{'})]^{'}$. So, $cl_{12}(A^{'}) \subseteq G \cup U$ for some $U \in I$. This shows that $cl_{12}(A^{'} \backslash G \subseteq U)$, thus $cl_{12}(A^{'}) \backslash G \in I$. Hence, $A^{'}$ is Igp-closed. □

Theorem 4.5. *If A, B are P^*-separated and Igp-open sets, then $A \cup B$ is Igp-open set.*

Proof. Suppose A and B are P^*-separated Igp-open sets and let F be a P-closed set such that $F \subseteq A \cup B$. Then, $F \cap cl_{12}(A) \subseteq A$ and $F \cap cl_{12}(B) \subseteq B$. Therefore, $F \cap cl_{12}(A) \backslash U_1 \subseteq int_{12}(A)$ and $F \cap cl_{12}(B) \backslash U_2 \subseteq int_{12}(B)$ for some $U_1, U_2 \in I$. This means that $[F \cap cl_{12}(A)] \backslash int_{12}(A) \in I$ and $[F \cap cl_{12}(B)] \backslash int_{12}(B) \in I$. So, $[F \cap cl_{12}(A)] \backslash int_{12}(A) \cup F \cap cl_{12}(B)] \backslash int_{12}(B) \in I$. Then, $[F \cap cl_{12}(A) \cup cl_{12}(B)] \backslash [int_{12}(A) \cup int_{12}(B)] \in I$. But, $F = F \cap (A \cup B) \subseteq F \cap cl_{12}(A \cup B)$. Then, $[F \cap cl_{12}(A) \cup F \cap cl_{12}(B)] \backslash int_{12}(A) \cup int_{12}(B) \in I$. Now, $F \subseteq A \cup B \subseteq cl_{12}(A) \cup cl_{12}(B)$. So, $F = F \cap (cl_{12}(A) \cup cl_{12}(B)) = [F \cap cl_{12}(A)] \cup [F \cap cl_{12}(B)]$. Then, $F \backslash int_{12}(A \cup B) = F \cap [int_{12}(A \cup B)]^{'} \subseteq F \cap [int_{12}(A))^{'} \cap int_{12}(B))^{'}] \subseteq F \cap [int_{12}(A) \cup int_{12}(B)]^{'} \subseteq F \backslash [int_{12}(A) \cup int_{12}(B)] \in I$. So, $F \backslash int_{12}(A \cup B) \in I$. Let $F \backslash int_{12}(A \cup B) = U, U \in I$. Then $F \backslash int_{12}(A \cup B) \cap U^{'} = \phi$. So, $F \backslash U \subseteq int_{12}(A \cup B)$ for some $U \in I$. Hence, $A \cup B$ is Igp-open set. □

Corollary 4.2. *Let A and B be Igp-closed sets and suppose that $A^{'}$ and $B^{'}$ are P^*-separated. Then, $A \cap B$ is Igp-closed*

Remark 4.3. *The intersection of two Igp-open sets is not necessary Igp-open set, see Example4.4.*

Theorem 4.6. *Let (X, τ_1, τ_2)be a bts. If $A \subseteq B \subseteq X$ and A is an Igp-open relative to B and B is Igp-open relative to X, then A is Igp-open relative to X.*

Proof. Suppose $A \subseteq B \subseteq X$, A is Igp-open relative to B and B is Igp-open relative to X. Suppose that $F \subseteq A$, and F is P-closed. Since, A is Igp-open relative to B, by Theorem 4.4, $\exists U_1 \in I$ such that $F \backslash U_1 \subseteq int^B_{12}(A)$. Also, $\exists U_2 \in I; F \backslash U_2 \subseteq int_{12}(A)$, thus $(F \cap U^{'}_1) \cap (F \cap U^{'}_2) \subseteq int^B_{12}(A) \cap int_{12}(A) = A \cap int_{12}(A) \subseteq int_{12}(A)$. Then, $F \cap (U^{'}_1 \cap U^{'}_2) = F \backslash (U_1 \cup U_2) \subseteq int_{12}(A)$, but $U_1, U_2 \in I \Longrightarrow (U_1 \cup U_2) \in I$ take $(U_1 \cup U_2) = U$, this implies that, $F \backslash U \subseteq int_{12}(A)$ for some $U \in I$. Hence, A is an Igp-open relative to X. □

Corollary 4.3. *If $int_{12}(A) \subseteq B \subseteq A$ and if A is an Igp-open, then B is an Igp-open in X.*

Proof. Suppose $int_{12}(A) \subseteq B \subseteq A$ and A is Igp-open. Then, $A^{'} \subseteq B^{'} \subseteq cl_{12}(A^{'})$ and $A^{'}$ is Igp-closed. By Corollary 4.1, $B^{'}$ is Igp-closed and hence, B is Igp-open. □

Theorem 4.7. *A set A is Igp-closed set if and only if $cl_{12}(A)\backslash A$ is Igp-open set.*

Proof. Necessity. Let A be Igp-closed set and suppose that $F \subseteq cl_{12}(A)\backslash A$ and F be P-closed.By Theorem4.1 we get, $F \in I$. This implies that, $F\backslash U = \phi$ for some $U \in I$. Then, $F\backslash U \subseteq int_{12}(cl_{12}(A)\backslash A)$. By Theorem 4.4 we get, $cl_{12}(A)\backslash A$ is Igp-open set. Sufficiency. Suppose that $A \subseteq G$ and G is P-open. Then, $G^{'} \subseteq A^{'}$. So, $cl_{12}(A) \cap G^{'} \subseteq cl_{12}(A) \cap A^{'} = cl_{12}(A)\backslash A$. But, $cl_{12}(A)\backslash A$ is Igp-open and $cl_{12}(A) \cap G^{'}$ is P-closed. Therefore, $[cl_{12}(A) \cap G^{'}]\backslash U \subseteq int_{12}(cl_{12}(A)\backslash A) = \phi$ for some $U \in I$. Thus, $cl_{12}(A) \cap G^{'} \subseteq U \in I$. Then, $cl_{12}(A)\backslash G \in I$. Hence, A is Igp-closed set. □

Theorem 4.8. *Let $f : X \longrightarrow Y$ be P^{*}-continuous and P^{*}-closed. If $A \subseteq X$ is Igp-closed in X, then $f(A)$ is $f(I)$- gp-closed in Y, where $f(I) = \{f(U) : U \in I\}$.*

Proof. Suppose $A \subseteq X$ and A is Igp-closed. Suppose $f(A) \subseteq G$ and$G$ is P-open. Then, $A \subseteq f^{-1}(G)$. But, A is Igp- closed. Then, $cl_{12}(A)\backslash f^{-1}(G) \in I$ and hence, $f(cl_{12}(A))\backslash G \in f(I)$. Since, f is P^{*}-closed, $cl_{12}(f(A)) \subseteq cl_{12}(f(cl_{12}(A))) = f(cl_{12}(A))$. Then, $cl_{12}(f(A))\backslash G \subseteq f(cl_{12}(A))\backslash G \in f(I)$ and hence, $f(A)$ is $f(I) - gp$-closed. □

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