Generalized closed sets in bitopological spaces


Abstract In this paper, we introduce the notions of generalized pairwise closed sets and I- generalized pairwise sets via ideal on bitopological spaces and study some of their properties. The properties of the space \((X, \tau_1, \tau_2)\) are studied through the study of the space \((X, \tau_{12})\) which is a supra topology associated to the bitopological space \((X, \tau_1, \tau_2)\). The relation between these approaches has studied. We note that the finite union (intersection) of \(gp-\) \((Igp)-\) closed sets may be not \(gp-\) \((Igp)-\) closed set. The study of the properties \((X, \tau_1, \tau_2)\) via properties of \((X, \tau_{12})\) is very important, since we are dealing with one family instead of two families and the class of supra topological spaces is wider than the class of bitopological spaces.

Key Words Bitopological space, Supra topology, P-closed, P-open, Generalized pairwise closed

MSC 2010 06D72, 54A40

1 Introduction

(Levine, 1970)[12] introduced the fundamental concept of generalized closed sets. He defined a set \(A\) to be generalized closed if its closure contained in every open superset of \(A\). A bitopological space \((X, \tau_1, \tau_2)\) was introduced by Kelly [11] in 1963, as a method of generalizes topological space \((X, \tau)\). Every bitopological space \((X, \tau_1, \tau_2)\) can be regarded as a topological space \((X, \tau)\) if \(\tau_1 = \tau_2 = \tau\). Furthermore, he extended some of the standard results of separation axioms and mappings in a topological space to a bitopological space. The notion of connectedness in bitopological space has been studied by Pervin[14], Reilly [15] and Swart [17]. Fukutake[7] introduce and investigate the concept of \((\tau_i, \tau_j) - g-\) closed in a bitopological space. In 1983 Mashhor et al. [13] introduced supra topological spaces by dropping only the intersection condition. Kandil,[10] generated a supra topological space \((X, \tau_{12})\) from the bitopological space \((X, \tau_1, \tau_2)\) and they studied some properties of the space \((X, \tau_1, \tau_2)\) via properties of the associated
space \((X, \tau_{12})\). Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting [2, 3, 4, 5, 10, 16].

In this paper we introduce the notion of generalized P-closed (P-open) sets and study some of their properties. Also, we introduce the notion of generalize pairwise closed (open) sets via ideals, study some properties and investigate the relation between two approaches.

2 Preliminaries

This section contains the basic concept properties of generalized closed sets, generalized open set, supra topological spaces and bitopological spaces.

Definition 2.1. [11] A bitopological spaces (bts, for short) is a triple \((X, \tau_1, \tau_2)\) where \(\tau_1\) and \(\tau_2\) are arbitrary topologies on \(X\).

Definition 2.2. [3]. Let \((X, \tau_1, \tau_2)\) be a bitopological space. Then, \(A \subseteq X\) is said to be pairwise open (P-open, for short) if \(A = U_1 \cup U_2, U_i \in \tau_i, (i = 1, 2)\). A set \(A\) is a P-closed if its complement \(A'\) is P-open.

Note that the notion of P-open sets as well as P-closed set has studied in [6, 10] under the name of \(P^*\)-open and \(P^*\)-closed.

Definition 2.3. [1]. A family \(\eta \subseteq P(X)\) is said to be a supra topology \(\eta\) contains \(X, \phi\) and closed under arbitrary union. The element of \(\eta\) are supraopen sets and their complements are said to be supraclosed sets.

Proposition 2.1. [6] Let \((X, \tau_1, \tau_2)\) be a bts. The family of all P-open subsets of \(X\), denoted by \(\tau_{12} = \{U_1 \cup U_2 : U_i \in \tau_i, i = 1, 2\}\) is a supratopology on \(X\) and \((X, \tau_{12})\) is the supra topological space associated to the bts \((X, \tau_1, \tau_2)\).

Definition 2.4. [10]. An operator \(C : P(X) \to P(X)\) is a supra closure closure operator if it satisfies the following conditions for all \(A, B \subseteq X\).

1. \(C(\phi) = \phi\).
2. \(A \subseteq C(A)\).
3. \(C(A \cup B) \supseteq C(A) \cup C(B)\).
4. \(C(C(A)) = C(A)\).

Proposition 2.2. [6]. Let \((X, \tau_1, \tau_2)\) be a bts. Then, the operator \(cl_{12} : P(X) \to P(X)\) defined by \(cl_{12}(A) = \overline{A^1} \cap \overline{A^2}\), is a supra closure operator such that, \(\tau_{12} = \{A \subseteq X : cl_{12}(A') = A'\}\) and \(\overline{A}\) is the closure of \(A\) with respect to \(\tau_i\) and \(i = 1, 2\).

Proposition 2.3. [6]. Let \((X, \tau_1, \tau_2)\) be a bts. Then the operator, \(int_{12} : P(X) \to P(X)\) defined by, \(int_{12}(A) = A^{\tau_1} \cup A^{\tau_2}\) is a supra interior operator such that \(\tau_{12} = \{A \subseteq X : int_{12}(A) = A\}\), where \(A^{\tau_i}, (i = 1, 2)\) is the \(\tau_i\)-interior with respect to \(\tau_i\).
It is easy to verify the following proposition.

**Proposition 2.4.** [6] Let \((X, \tau_1, \tau_2)\) be a bts and \(A \subseteq X\). Then

1. \(\tau_1, \tau_2 \subseteq \tau_{12}\).
2. \(\text{cl}_{12}(A) = X \setminus \text{int}_{12}(A)\).
3. \(\text{int}_{12}(A) = X \setminus \text{cl}_{12}(A)\).
4. \(A\) is \(P\)-open \(\iff\) \(A = \text{int}_{12}(A)\).
5. \(A\) is \(P\)-closed \(\iff\) \(A = \text{cl}_{12}(A)\).

**Definition 2.5.** [12] A subset \(A\) of a topological space \((X, \tau)\) is said to be generalized closed (briefly \(g\)-closed) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, \tau)\).

**Proposition 2.5.** [12] If \(A\) and \(B\) are \(g\)-closed sets in space \((X, \tau)\), then \(A \cup B\) is a \(g\)-closed set in \(X\).

**Definition 2.6.** [10] A mapping \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \gamma_1, \gamma_2)\) is said to be \(P^*\) continuous function (\(P^*\)-cts, for short) if and only if \(f^{-1}(B) \in \tau_{12}\) for all \(B \in \gamma_{12}\).

**Definition 2.7.** [10] A mapping \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \gamma_1, \gamma_2)\) is said to be \(P^*\) closed (\(P^*\)-open) if and only if \(f(A)\) is \(\gamma_{12}\)-closed (open) for all \(\tau_{12}\)-closed (open) set \(A\) in \(X\).

**Definition 2.8.** [9] An ideal \(I\) on a set \(X\) is a nonempty collection of subset of \(X\) (i.e \(I \subseteq \mathcal{P}(X)\)) such that

1. if \(A \in I\) and \(B \subseteq A\), then \(B \in I\),
2. if \(A, B \in I\), then \(A \cup B \in I\).

### 3 Generalized pairwise closed and generalized pairwise open sets

In this section we defined the concepts of generalized pairwise closed sets and generalized pairwise open sets and study some of their properties.

**Definition 3.1.** Let \((X, \tau_1, \tau_2)\) be a bts and \((X, \tau_{12})\) be its associated supra topological space. Then, \(A \subseteq X\) is called a generalized pairwise closed set (\(gp\)-closed, for short) if \(\text{cl}_{12}(A) \subseteq O\) whenever \(A \subseteq O\), \(O\) is a \(P\)-open.

**Theorem 3.1.** A set \(A\) is a \(gp\)-closed set if and only if \(\text{cl}_{12}(A) \setminus A\) contains no non empty \(P\)-closed sets.

**Proof.** Let \(\phi \neq F\) be a \(P\)-closed set such that \(F \subseteq \text{cl}_{12}(A) \setminus A\). It is follows that \(A \subseteq F'\) but \(A\) is a \(gp\)-closed set, \(F'\) is \(P\)-open, then \(\text{cl}_{12}(A) \subseteq F'\). This implies that \(F \subseteq (\text{cl}_{12}(A))^\prime\), this means \(F \subseteq (\text{cl}_{12}(A))^\prime \cap \text{cl}_{12}(A) \setminus A = \phi\). Then, \(F = \phi\) which is a contradiction.

Conversely, let \(A \subseteq X\) such that \(\text{cl}_{12}(A) \setminus A\) be contains no non empty \(P\)-closed set, \(O \in \tau_{12}\) such that \(A \subseteq O\). Suppose \(\text{cl}_{12}(A) \not\subseteq O\). Then \(\text{cl}_{12}(A) \cap O'\) is \(P\)-closed set. Thus, \(\text{cl}_{12}(A) \cap O' \subseteq \text{cl}_{12}(A) \setminus A\) which is a contradiction. Therefore, \(\text{cl}_{12}(A) \subseteq O\). Hence, \(A\) is a \(gp\)-closed set.

**Corollary 3.1.** If \(A\) is a \(gp\)-closed set, then \(A\) is \(P\)-closed if and only if \(\text{cl}_{12}(A) \setminus A\) is a \(P\)-closed set.
Let $A$ be P-closed. Then, $cl_{12}(A) = A$. Since, $cl_{12}(A) \setminus A = \emptyset$ which is a P-closed set. Sufficiency. Let $cl_{12}(A) \setminus A$ be P-closed. Since, $A$ is $gp$-closed set. Then, $cl_{12}(A) \setminus A = \emptyset$ gives $cl_{12}(A) \subseteq A$, but we have $A \subseteq cl_{12}(A)$. Therefore, $cl_{12}(A) = A$. Hence, $A$ is P-closed set. □

**Remark 3.1.**

1. If $A$ is $P$-closed set, then $A$ is $gp$-closed set.

2. If $A$ and $B$ are $gp$-closed sets, then $A \cup B$ and $A \cap B$ are not necessary $gp$-closed sets.

**Example 3.1.** Let $X = \{a, b, c, d, e\}$, $\tau_1 = \{X, \phi, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{c, b\}\}$. Then, $\tau_{12} = \{X, \phi, \{a, b\}\}$, $\{b, c\}, \{a, b, c\}$}. Take $A = \{a, b, d\}$ and $B = \{b, e, c\}$. Then $A$ and $B$ are $gp$-closed sets but $A \cap B = \{b\}$ is not $gp$-closed set, since $\{b\} \subseteq \{b, c\}$ but $cl_{12}\{b\} = X \notin \{b, c\}$.

**Example 3.2.** Let $X = \{1, 2, 3, 4, 5\}$, $\tau_1 = \{X, \phi, \{1, 2, 3\}, \{3, 4, 5\}, \{3\}\}$, $\tau_2 = \{X, \phi, \{1, 2, 4\}\}$. Then, $\tau_{12} = \{X, \phi, \{1, 2, 3\}, \{3\}, \{1, 2, 4\}, \{1, 2, 3, 4\}\}$. Now, $\{4, 5\}$ and $\{3, 5\}$ are $gp$-closed sets but $\{4, 5\} \cup \{3, 5\} = \{3, 4, 5\}$ is not a $gp$-closed set.

**Definition 3.2.** Let $(X, \tau_1, \tau_2)$ be a bts, $A \subseteq X$ and $(A, \tau_1|_A, \tau_2|_A)$ be a bitopological subspace of $(X, \tau_1, \tau_2)$. Then, $\tau_{12}(A) = \tau_1|_A \cup \tau_2|_A = \{U_1 \cup U_2 : U_1 \in \tau_1|_A, U_2 \in \tau_2|_A\}$ such that $\tau_{12}|_A = \{A \cap U_i : U_i \in \tau_i, i = 1, 2\}$.

**Theorem 3.2.** Let $(X, \tau_1, \tau_2)$ be a bts, $A \subseteq X$. Then, $\tau_{12}|_A = \tau_{12}(A)$, where $\tau_{12}|_A = \{A \cap U : U \in \tau_{12}\}$.

**Proof.** Let $V \in \tau_{12}|_A$. Then, $\exists V_1 \in \tau_{12}$ such that $V = V_1 \cap A$. Also $\exists U_i \in \tau_i$; $(i = 1, 2)$ such that $V_1 = U_1 \cup U_2$. Thus, $V = (U_1 \cup U_2) \cap A = (U_1 \cap A) \cup (U_2 \cap A)$. But $U_i \cap A \subseteq \tau_{12}|_A \forall i = 1, 2$, then we get $V \in \tau_{12}(A)$. Hence, $\tau_{12}|_A \subseteq \tau_{12}(A)$. On the other hand, Let $F \in \tau_{12}(A)$. Then $\exists F_1, F_2$ such that $F = F_1 \cup F_2, F_1 \in \tau_1|_A, F_2 \in \tau_2|_A$. Also, $F_1 = U_1 \cap A, F_2 = U_2 \cap A$, such that $U_i \in \tau_i, i = 1, 2$. Then we get, $F = F_1 \cup F_2 = (U_1 \cap A) \cup (U_2 \cap A) = (U_1 \cup U_2) \cap A$, but $U_1 \cup U_2 \in \tau_{12}$. Then $F \in \tau_{12}|_A$. Hence, $\tau_{12}(A) \subseteq \tau_{12}|_A$, and consequently $\tau_{12}(A) = \tau_{12}|_A$. □

**Theorem 3.3.** Let $(X, \tau_1, \tau_2)$ be a bts. Suppose that $B \subseteq A \subseteq X$, $B$ is a $gp$-closed set relative to $A$ and $A$ is a $gp$-closed set of $X$. Then, $B$ is a $gp$-closed set relative to $X$.

**Proof.** Let $B \subseteq O$, $O$ be P-open in $X$. Then, $B \cap A \subseteq O \cap A$. Hence, $B \subseteq O \cap A$, but $B$ is $gp$-closed relative to $A$, then $cl_{12}(B) \subseteq A \cap O$. $(cl_{12}^A$ the closure is taken w.r.t $A$). Then $cl_{1A}(B) \cap cl_{2A}(B) \subseteq A \cap O$. $\Rightarrow [A \cap cl_{1}(B)] \cap [A \cap cl_{2}(B)] \subseteq A \cap O$. $\Rightarrow cl_{1}(B) \cap cl_{2}(B) \subseteq A \cap O$. $\Rightarrow cl_{12}(B) \subseteq A \subseteq A \cap O$. $\Rightarrow A \subseteq O \cup [cl_{12}(B)]'$. $O \cup [cl_{12}(B)]'$ is $P$-open in $X$. Then $cl_{12}(A) \subseteq O \cup [cl_{12}(B)]'$. Since $B \subseteq A$, then $cl_{12}(B) \subset cl_{12}(A) \subseteq O \cup [cl_{12}(B)]'$. Hence, $cl_{12}(B) \cap (O \cup [cl_{12}(B)]')' = \phi \Rightarrow cl_{12}(B) \cap O' \cap cl_{12}(B) = \phi$. $\Rightarrow cl_{12}(B) \subseteq O$. It is follows that $B$ is a $gp$-closed set in $X$. □

**Corollary 3.2.** If $A$ is a $gp$-closed set and $F$ is $P$-closed set, then $A \cap F$ is a $gp$-closed set.

**Theorem 3.4.** Let $(X, \tau_1, \tau_2)$ be a bts and $A \subseteq X$. If $A$ is a $gp$ closed set and $A \subseteq B \subseteq cl_{12}(A)$. Then, $B$ is $gp$-closed set.

**Proof.** Let $A$ be a $gp$-closed set, $A \subseteq B \subseteq cl_{12}(A)$ and let $B \subseteq O$, $O$ be $P$-open set. Then, $A \subseteq O$. But, $A$ is $gp$-closed, this implies that $cl_{12}(A) \subseteq O$, but $cl_{12}(B) \subseteq cl_{12}(cl_{12}(A)) = cl_{12}(A)$, thus $cl_{12}(B) \subseteq O$. Hence, $B$ is a $gp$-closed set. □
Theorem 3.5. Let \((X, \tau_1, \tau_2)\) be a bts and \(A \subseteq Y \subseteq X\). Suppose that \(A\) is gp-closed in \(X\). Then, \(A\) is gp-closed relative to \(Y\).

**Proof.** Let \(A \subseteq Y \subseteq X\), \(A\) be a gp-closed set in \(X\). Let \(A \subseteq O^*, O^*\) be P-open in \(Y\). Then, \(\exists O \in \tau_{12}\) such that \(O \cap Y = O^*\). Now, \(A \subseteq O \cap Y \subseteq O\), but \(A\) is gp-closed relative to \(X\). So, \(cl_{12}(A) \subseteq O \Rightarrow cl_{12}(A) \cap Y \subseteq O \cap Y\), then \(cl_{12}'(A) \subseteq O^*\). Hence, \(A\) is gp-closed relative to \(Y\). \(\square\)

Theorem 3.6. Let \((X, \tau_1, \tau_2)\) be a bts, \(\tau_{12}\) be supra topology on \(X\) induced by \(\tau_1, \tau_2\). Then, \(\tau_{12} = \tau_{12}'\) iff every subset of \(X\) is a gp-closed set.

**Proof.** “\(\Rightarrow\)" : Let \(\tau_{12} = \tau_{12}'\), \(A \subseteq X\) and \(A \subseteq O\), \(O\) be a P-open set in \(X\). Then, \(cl_{12}(A) = cl_{12}(O) = O\), since \(\tau_{12} = \tau_{12}'\). Hence, \(A\) is a gp-closed set.

“\(\Leftarrow\)" : Let \(A \subseteq X\); \(A \in \tau_{12}\) and let every subset of \(X\) be a gp-closed set. Then, \(cl_{12}(A) \subseteq A\), but we have \(A \subseteq cl_{12}(A)\), so \(cl_{12}(A) = A\). Hence, \(\tau_{12} \subseteq \tau_{12}'\). On the other hand, let \(F \in \tau_{12}'\). Then, \(F' \in \tau_{12}, F' \subseteq F'\) (by hypothesis), \(cl_{12}(F') \subseteq F'\) but, \(F' \subseteq cl_{12}(F')\), then \(F' = cl_{12}(F')\). Thus, \(F \in \tau_{12}\) and consequently \(\tau_{12}' \subseteq \tau_{12}\). Hence, \(\tau_{12} = \tau_{12}'\). \(\square\)

Definition 3.3. A set \(A\) is called a generalized pairwise open (for short, gp-open) set if and only if \(A'\) is a gp-closed set.

Theorem 3.7. Let \((X, \tau_1, \tau_2)\) be a bts. Then, \(A\) set \(A\) is a gp-open set if and only if \(F \subseteq \text{int}_{12}(A)\) whenever \(F\) is a P-closed set and \(F \subseteq A\).

**Proof.** “\(\Rightarrow\)" Let \(A\) be a gp-open set and \(F \subseteq A\) such that \(F\) is P-closed set. Then, \(A'\) is gp-closed, \(A' \subseteq F'\) and \(F'\) is P-open thus, \(cl_{12}(A') \subseteq F'\), implies \(F \subseteq [cl_{12}(A')]' = \text{int}_{12}(A)\). Hence, \(F \subseteq \text{int}_{12}(A)\).

“\(\Leftarrow\)" Let \(A \subseteq X, F\) be a P-closed set and \(F \subseteq \text{int}_{12}(A)\) whenever \(F \subseteq A\). We prove that \(A\) is a gp-open or \(A'\) is a gp-closed. So, let \(A' \subseteq O; O\) is P-open. Then \(O' \subseteq A; O'\) is P-closed. By hypothesis, \(O' \subseteq \text{int}_{12}(A)\), thus, \([\text{int}_{12}(A)]' \subseteq O\) and \(cl_{12}(A') \subseteq O\). Hence, \(A'\) is gp-closed and consequently \(A\) is a gp-open. \(\square\)

Definition 3.4. \([4]\) Let \((X, \tau_1, \tau_2)\) be a bts and \(A \subseteq B \subseteq X\). Then \(A\) and \(B\) are \(P^*\)–separated in \(X\) if \(A \cap cl_{12}(B) = \phi\) and \(cl_{12}(A) \cap B = \phi\).

Not that if \(A\) and \(B\) are \(P^*\)-separated and \(C \subseteq A, D \subseteq B\), then \(C \cap D\) are \(P^*\)–separated.

Theorem 3.8. Let \((X, \tau_1, \tau_2)\) be a bts. If \(A\) and \(B\) are \(P^*\)-separated and gp-open sets, then \(A \cup B\) is a gp-open set.

**Proof.** Let \(F\) be a P-closed set such that \(F \subseteq A \cup B\). Also, let \(A, B\) be gp-open and \(P^*\)-separated. Then \(F \cap cl_{12}(A) \subseteq (A \cup B) \cap cl_{12}(A) = (A \cap cl_{12}(A)) \cup (B \cap cl_{12}(A))\), since \(A, B\) are P-separated, so, \(F \cap cl_{12}(A) \subseteq (A)\). Similarly, \(F \cap cl_{12}(B) \subseteq B\). Now, \(F \cap cl_{12}(A)\) and \(F \cap cl_{12}(B)\) are P-closed and \(A, B\) are gp-open, then, \(F \cap cl_{12}(A) \subseteq \text{int}_{12}(A)\) and \(F \cap cl_{12}(B) \subseteq \text{int}_{12}(B)\). It is follows \(F \cap cl_{12}(A) \cup F \cap cl_{12}(B) \subseteq \text{int}_{12}(A) \cup \text{int}_{12}(B) \subseteq \text{int}_{12}(A \cup B)\), but \(F = F \cap (A \cup B) \subseteq F \cap (cl_{12}(A) \cup cl_{12}(B))\), this implies that \(F \subseteq \text{int}_{12}(A \cup B)\), then \(A \cup B\) is gp-open set. \(\square\)
Corollary 3.3. Let \((X, \tau_1, \tau_2)\) be a bts and let \(A\) and \(B\) be two \(gp\)-closed sets and suppose that \(A', B'\) are \(P^*\) separated. Then, \(A \cap B\) is a \(gp\)-closed set.

Proof. Let \(A\) and \(B\) be a \(gp\)-closed subsets. Then, \(A', B'\) are \(gp\)-open, by Theorem 3.8 we get \(A' \cup B'\) is \(gp\)-open. Hence, \((A' \cup B')'= A \cap B\) is a \(gp\)-closed set.

\[\square\]

Theorem 3.9. Let \((X, \tau_1, \tau_2)\) be a bts. Then, a set \(A\) is \(gp\)-open if and only if \(O = X\) whenever \(O\) is \(P\)-open and \(int_{12}(A) \cup A' \subseteq O\).

Proof. Necessity. Let \(A\) be a \(gp\)-open set and \(O\) be a \(P\)-open set such that \(int_{12}(A) \cup A' \subseteq O\). Then \(O' \subseteq (int_{12}(A) \cup A')'= (int_{12}(A))' \cap A = cl_{12}(A') \setminus A\), but \(O'\) is \(P\)-closed set and \(A'\) is \(gp\)-closed set, then by Theorem 3.1 we get \(O' = \phi\). Hence, \(O = X\).

Sufficiency. Let \(F \subseteq A, F\) be a \(P\)-closed set. Then, \(A' \subseteq F'\), thus \(int_{12}(A) \cup A' \subseteq int_{12}(A) \cup F' = X\) (by hypothesis), then \(int_{12}(A) \supseteq F\). Hence, \(A\) is \(gp\)-open.

\[\square\]

Theorem 3.10. Let \((X, \tau_1, \tau_2)\) be a bts and \(A, B \in P(X)\). If \(\text{int}_{12}(A) \subseteq B \subseteq A\) and \(A\) is \(gp\)-open set, then \(B\) is \(gp\)-open set.

Proof. Let \(A, B \in P(X)\) and \(A\) be \(gp\)-open set such that \(\text{int}_{12}(A) \subseteq B \subseteq A\). Then, \(A' \subseteq B' \subseteq [\text{int}_{12}(A)'] = cl_{12}(A')\), by Theorem 3.4 we get \(A'\) is \(gp\)-closed, then \(B'\) is \(gp\)-closed set. Hence, \(B\) is \(gp\)-open set.

\[\square\]

Theorem 3.11. Let \((X, \tau_1, \tau_2)\) be a bts. Then a set \(A\) is a \(gp\)-closed if and only if \(cl_{12}(A) \setminus A\) is \(gp\)-open.

Proof. Necessity. Let \(A\) be a \(gp\)-closed set, \(F \subseteq cl_{12}(A) \setminus A\) such that \(F\) be a \(P\)-closed set. By Theorem 3.1 we get, \(cl_{12}(A) \setminus A\) has no non empty subset, then \(F = \phi\), thus \(\phi \subseteq \text{int}_{12}(cl_{12}(A) \setminus A)\). Hence, \(cl_{12}(A) \setminus A\) is \(gp\)-open set.

Sufficiency. Let \(cl_{12}(A) \setminus A\) be a \(gp\)-open set and \(A \subseteq O\) such that \(O\) be a \(P\)-open set. Then, \(O' \subseteq A'\), so we get \(cl_{12}(A) \cap O' \subseteq cl_{12}(A) \cap A' = cl_{12}(A) \setminus A\), but \(cl_{12}(A) \setminus A\) is \(gp\)-open set and \(cl_{12}(A) \cap O'\) is \(P\)-closed set. Thus, \(cl_{12}(A) \cap O' \subseteq \text{int}_{12}(cl_{12}(A) \setminus A) = \phi\) so, \(cl_{12}(A) \cap O' = \phi\) then, \(cl_{12}(A) \subseteq O\). Hence \(A\) is \(gp\)-closed set.

\[\square\]

Definition 3.5. Let \((X, \tau_1, \tau_2)\) be a bts. A space \((X, \tau_1, \tau_2)\) is said to be a pairwise symmetric (\(P^*\)-symmetric for short) if for all \(x, y\) in \(X\), then \(x \in cl_{12}\{y\} \Rightarrow y \in cl_{12}\{x\}\).

Theorem 3.12. A space \((X, \tau_1, \tau_2)\) is a \(P^*\)-symmetric if and only if \(\{x\}\) is a \(gp\)-closed \(\forall x \in X\).

Proof. Necessity. Let \((X, \tau_1, \tau_2)\) be a \(P^*\)-symmetric space, \(\{x\} \subseteq O\) such that \(O \in \tau_{12}\), we want to prove \(cl_{12}\{x\} \subseteq O\). Suppose that \(cl_{12}\{x\} \not\subseteq O\), thus \(cl_{12}\{x\} \cap O' \neq \phi\). Let \(y \in cl_{12}\{x\} \cap O'\). Then \(x\) is \(P\)-symmetric and \(x \neq y\), then \(x \in cl_{12}\{y\}\), therefore \(cl_{12}\{y\} \subseteq cl_{12}\{x\} \cap O'\), so \(x \in O'\) but \(x \in O\) which is contradiction, then \(cl_{12}\{x\} \subseteq O\). Hence, \(\{x\}\) is \(gp\)-closed set.

Sufficiency. Let \(x, y \in X, x \neq y\) and \(x \in cl_{12}\{y\}\). Suppose that \(y \notin cl_{12}\{x\}\), then \(\{y\} \subseteq (cl_{12}\{x\})'\), as \(\{y\}\) is \(gp\)-closed set, implies \(cl_{12}\{y\} \subseteq (cl_{12}\{x\})'\), then \(x \in (cl_{12}\{x\})'\) which is contradiction. Hence, \(y \in cl_{12}\{x\}\).

\[\square\]
Theorem 3.13. If $A$ is gp-closed set in $X$ and $f : X \rightarrow Y$ is $P^{*}$-cts and is $P^{*}$-closed function, then $f(A)$ is a gp-closed set.

Proof. Let $A$ be a gp-closed set in $X$, $f(A) \subseteq O$ such that $O$ is P-open subset of $Y$. So, $A \subseteq f^{-1}(O)$, but $f$ is $P^{*}$-cts, then $f^{-1}(O)$ is P-open set in $X$. As $A$ is a gp-closed set, then $cl_{12}(A) \subseteq f^{-1}(O)$. Since $f$ is $P^{*}$-closed, then $cl_{12}(f(A)) \subseteq f(cl_{12}(A))$. It follows that $cl_{12}[f(A)] \subseteq f(cl_{12}(A)) \subseteq ff^{-1}(O) \subseteq O$. Hence, $f(A)$ is gp-closed set in $Y$. 

Remark 3.2. If $f$ is $P^{*}$- continuous and $P^{*}$-closed function and $A$ is gp-open set, then $f(A)$ is not necessary to be a gp-open set, as shown in the following example.

Example 3.3. Let $X = \{a\}, Y = \{b, c\}, \tau_1 = \{X, \phi\}, \tau_2 = \{Y, \phi\}, \nu_1 = \{Y, \phi, \{b\}\}, \nu_2 = \{Y, \phi, \{b\}\}, f\{a\} = c, f$ is $P^{*}$-closed mapping and $P^{*}$- cts but $\{a\}$ is gp-open and $f\{a\} = \{c\}$ is not a gp-open set.

Theorem 3.14. If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \nu_1, \nu_2)$ is $P^{*}$-cts and $P^{*}$-closed function, $B$ is a gp-closed(or gp-open) subset of $Y$, then $f^{-1}(B)$ is gp-closed (or gp-open) set in $X$.

Proof. Suppose that $B$ is gp-closed subset of $Y$ and $f^{-1}(B) \subseteq O, O$ is P-open in $X$. We will show that $cl_{12}[f^{-1}(B)] \subseteq O$ or $cl_{12}[f^{-1}(B)] \cap O = \phi$, therefore $f[cl_{12}[f^{-1}(B)] \cap f^{-1}(O)] \subseteq f(cl_{12}[f^{-1}(B)]) \cap f(O) \subseteq cl_{12}(B) \cap f^{-1}(O) \subseteq cl_{12}(A) \subseteq cl_{12}(B) \subseteq cl_{12}(A) \cap B = \phi$, that is by Theorem 3.1, then $f[cl_{12}[f^{-1}(B)] \cap O] = \phi$, so $cl_{12}[f^{-1}(B)] \subseteq O$. Hence, $f^{-1}(B)$ is gp-closed. By taking complements we can show that if $B$ is gp-open in $Y$, then $f^{-1}[B]$ is gp-open in $X$. 

4 Generalized pairwise open and pairwise closed sets via Ideals

In this section we induced and study the concept of gp- closed sets and gp-open sets with respect to an ideal, which is the extension of the concept of gp-closed sets and gpopen sets.

Definition 4.1. Let $(X, \tau_1, \tau_2)$ be a bts, $(X, \tau_{12})$ its associated supra topological space and $I$ be an ideal on $X$. Then $A \subseteq X$ is called a generalized pairwise closed set with respect to an ideal (Igp-closed, for short) iff $cl_{12}(A) \backslash B \in I$ whenever $A \subseteq B$, $B$ is P-open.

Remark 4.1. Every gp-closed set is an Igp-closed set, but the converse need not be true as shown in the following example.

Example 4.1. Let $X = \{a, b, c, d, e\}, \tau_1 = \{X, \phi, \{a\}\}, \tau_2 = \{X, \phi, \{b, c\}\}$. Then $\tau_{12} = \{X, \phi, \{a\}, \{b, c\}, I = \{\phi, \{a\}, \{c\}, \{e\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}\}$. We have $\{b\}$ is not gp-closed set but it is an Igp-closed set.

Example 4.2. Let $X = \{a, b, c, d\}, \tau_1 = \{X, \phi, \{b\}\}, \tau_2 = \{X, \phi, \{c\}\}$. Then $\tau_{12} = \{X, \phi, \{b\}, \{c\}, I = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. We have here $\{b\}$ is an Igp-closed sets, but not a gp-closed set.

Theorem 4.1. Let $(X, \tau_1, \tau_2)$ be a bts. A set $A$ is an Igp-closed set if and only if $F \subseteq cl_{12}(A) \backslash A$ and $F$ is a P-closed set $\implies F \in I$. 

78
\textbf{Proof.} Assume that $A$ is a $Igp$-closed. Let $F \subseteq cl_{12}(A) \setminus A$. Suppose $F$ is a P-closed set. Then, $A \subseteq F'$, but $A$ is $Igp$-closed. By our assumption, $cl(A) \setminus F' \in I$. But $F \subseteq cl_{12}(A) \setminus F' \Rightarrow F \in I$.

Conversely, assume that $F \subseteq cl_{12}(A) \setminus A$ and $F$ is P-closed in $X$ implies that $F \in I$. Suppose $A \subseteq U$ and $U$ is $P$-open. Then, $cl_{12}(A) \setminus U = cl_{12}(A) \cap U'$ which is a $P$-closed set and contained in $cl_{12}(A) \setminus A$, since, $A \subseteq U$. By assumption, $cl_{12}(A) \setminus U \in I$. Hence, $A$ is $Igp$-closed.

\textbf{Remark 4.2.} The intersection and union of two an $Igp$-closed sets need not be an $Igp$-closed set as shown by next examples.

\textbf{Example 4.3.} Let $X = \{1, 2, 3, 4, 5\}$, $\tau_1 = \{X, \phi, \{1, 2, 3\}, \{3, 4, 5\}, \{3\}\}$, $\tau_2 = \{X, \phi, \{1, 2, 4\}\}$. Then $\tau_{12} = \{X, \phi, \{1, 2, 3\}, \{3, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4\}\}$, $I = \{\phi\}$, we have here $\{4, 5\}$ and $\{3, 5\}$ are $Igp$-closed sets, but $\{4, 5\} \cup \{3, 5\} = \{3, 4, 5\}$ is not an $Igp$-closed set.

\textbf{Example 4.4.} Let $X = \{a, b, c, d, e\}$, $\tau_1 = \{X, \phi, \{a, b\}\}$, $\tau_2 = \{X, \phi, \{a, b\}\}$. Then $\tau_{12} = \{X, \phi, \{a, b\}, \{a, b\}, \phi\}$ we have here $\{a, b, d\}$ and $\{d, e, c\}$ are $Igp$-closed sets, but $\{a, b, d\} \cap \{d, e, c\} = \{d\}$ is not $Igp$-closed set.

\textbf{Example 4.5.} In example 4.2 we can see that $\{b\}$ and $\{d\}$ are $Igp$-closed sets, but $\{b\} \cup \{d\} = \{b, d\}$ is not a $Igp$-closed set.

\textbf{Theorem 4.2.} Let $(X, \tau_1, \tau_2)$ be a bts. Let $A \subseteq Y \subseteq X$ and suppose that $A$ is an $Igp$-closed. Then $A$ is an $Igp$-closed relative to the subspace $Y$ of $X$, with respect to the ideal $I_Y = \{F \subseteq Y : F \in I\}$.

\textbf{Proof.} Suppose $A \subseteq U \cap Y$ and $U$ is $P$-open in $(X, \tau_{12})$. Then $A \subseteq U$. Since, $A$ is $Igp$-closed relative to $X$, we have $cl_{12}(A) \setminus U \in I$. Now, $cl_{12}(A) \setminus (U \setminus Y) = Y \cap (cl_{12}(A) \cap (Y \cap U')) \cap Y \cap cl_{12}(A) \cap U' = cl_{12}(A) \cap Y \in I$. Then, $cl_{12}(A) \setminus (U \cap Y) \in I_Y$. Hence, $A$ is $Igp$-closed in $Y$.

\textbf{Theorem 4.3.} Let $(X, \tau_1, \tau_2)$ be a bts, $A$ be an $Igp$-closed set and $F$ be a $P$-closed set. Then, $A \cap F$ is an $Igp$-closed set.

\textbf{Proof.} Let $A$ be an $Igp$-closed and $F$ be a $P$-closed set. Suppose $A \cap F \subseteq U$ and $U$ is a $P$-open. Then, $A \subseteq U \cup F'$. Since, $A$ is $Igp$-closed and $U \cup F'$ is a $P$-open, then $cl_{12}(A) \setminus U \cup F' \in I$. Now, $cl_{12}(A \cap F) \subseteq cl_{12}(A) \cap F = cl_{12}(A) \cap F' \in I$. Therefore, $cl_{12}(A \cap F) \setminus U = [cl_{12}(A \cap F)] \setminus U \in I$ since, $cl_{12}(A) \cap F \cup (U \setminus F') \subseteq cl_{12}(A) \setminus U \cup F' \in I$. Hence, $A \cap F$ is an $Igp$-closed.

\textbf{Corollary 4.1.} Let $(X, \tau_1, \tau_2)$ be a bts. If $A$ is an $Igp$-closed and $A \subseteq B \subseteq cl_{12}(A)$, then $B$ is an $Igp$-closed set.

\textbf{Proof.} Suppose $A$ is an $Igp$-closed and $A \subseteq B \subseteq cl_{12}(A)$. Suppose that $B \subseteq G$ and $G$ is $P$-open. Then, $A \subseteq G$. Since, $A$ is $Igp$-closed. Then $cl_{12}(A) \setminus G \in I$, we have, $B \subseteq cl_{12}(B)$. This implies that $cl_{12}(B) \subseteq cl_{12}(A)$. Now, $cl_{12}(B) \setminus G \subseteq cl_{12}(A) \setminus G$, but $cl_{12}(A) \setminus G \in I \Rightarrow cl_{12}(B) \setminus G \in I$. Hence, $B$ is an $Igp$-closed.
Definition 4.2. Let \((X, \tau_1, \tau_2)\) be a bts, \((X, \tau_{12})\) its associated supra topological space and \(I\) be an ideal on \(X\). Then, \(A \subseteq X\) is called a generalized pairwise open set with respect to an ideal (\(I\)gp-open for short) if and only if \(A'\) is \(I\)gp-closed.

Theorem 4.4. Let \((X, \tau_1, \tau_2)\) be a bts. Then, a set \(A\) is \(I\)gp-open if and only if \(F \setminus U \subseteq \text{int}_{A}(A)\), for some \(U \in I\) whenever \(F \subseteq A\) and \(F\) is a P-closed.

Proof. Suppose \(A\) is an \(I\)gp-open. Suppose \(F \subseteq A\) and \(F\) is P-closed. We have \(A' \subseteq F'\). By assumption, \(\text{cl}_{A}(A') \subseteq F' \cup U\) for some \(U \in I\). This implies, \([F' \cup U] \subseteq \text{cl}_{A}(A')\) and hence, \(F \setminus U \subseteq \text{int}_{A}(A)\).

Conversely, assume that \(F \subseteq A\) and \(F\) is P-closed imply \(F \setminus U \subseteq \text{int}_{A}(A)\), for some \(U \in I\). Consider a P-open set \(G\) such that \(A' \subseteq G\). Then, \(G' \subseteq A\). By assumption, \((F \setminus U) \subseteq \text{int}_{A}(A)\). Then we get \([G \cup U] \subseteq \text{cl}_{A}(A')\). So, \(\text{cl}_{A}(A') \subseteq G \cup U\) for some \(U \in I\). This shows that \(\text{cl}_{A}(A') \subseteq U\), thus \(\text{cl}_{A}(A') \subseteq G \subseteq I\). Hence, \(A'\) is \(I\)gp-closed.

Theorem 4.5. If \(A, B\) are \(P^*\)-separated and \(I\)gp-open sets, then \(A \cup B\) is \(I\)gp-open set.

Proof. Suppose \(A\) and \(B\) are \(P^*\)-separated \(I\)gp-open sets and let \(F\) be a P-closed set such that \(F \subseteq A \cup B\). Then, \(F \cap \text{cl}_{A}(A) \subseteq A\) and \(F \cap \text{cl}_{B}(B) \subseteq B\). Therefore, \(F \cap \text{cl}_{A}(A) \setminus U_1 \subseteq \text{int}_{A}(A)\) and \(F \cap \text{cl}_{B}(B) \setminus U_2 \subseteq \text{int}_{B}(B)\) for some \(U_1, U_2 \in I\). This means that \([F \cap \text{cl}_{A}(A)] \setminus [F \cap \text{cl}_{B}(B)] \subseteq \text{int}_{A}(A)\) and \([F \cap \text{cl}_{B}(B)] \setminus [F \cap \text{cl}_{A}(A)] \subseteq \text{int}_{B}(B)\). So, \(F = F \cap (A \cup B) \subseteq F \cap \text{cl}_{A}(A) \cup F \cap \text{cl}_{B}(B)\). Then, \(F \cap \text{cl}_{A}(A) \cup F \cap \text{cl}_{B}(B)\) is an \(I\)gp-open set for some \(U \in I\). Hence, \(A \cup B\) is \(I\)gp-open set.

Corollary 4.2. Let \(A\) and \(B\) be \(I\)gp-closed sets and suppose that \(A'\) and \(B'\) are \(P^*\)-separated. Then, \(A \cap B\) is \(I\)gp-closed.

Remark 4.3. The intersection of two \(I\)gp-closed sets is not necessary \(I\)gp-open set, see Example 4.4.

Theorem 4.6. Let \((X, \tau_1, \tau_2)\) be a bts. If \(A \subseteq B \subseteq X\) and \(A\) is an \(I\)gp-open relative to \(B\) and \(B\) is \(I\)gp-open relative to \(X\), then \(A\) is \(I\)gp-open relative to \(X\).

Proof. Suppose \(A \subseteq B \subseteq X\), \(A\) is \(I\)gp-open relative to \(B\) and \(B\) is \(I\)gp-open relative to \(X\). Suppose that \(F \subseteq A\) and \(F\) is P-closed. Since, \(A\) is \(I\)gp-open relative to \(B\), by Theorem 4.4, \(\exists U_1 \in I\) such that \(F \setminus U_1 \subseteq \text{int}_{B}(A)\). Also, \(\exists U_2 \in I\); \(F \setminus U_2 \subseteq \text{int}_{B}(A)\), thus \((F \setminus U_1) \cap (F \setminus U_2) \subseteq \text{int}_{B}(A) \cap \text{int}_{B}(A) = A \subseteq \text{int}_{B}(A) \subseteq \text{int}_{B}(A)\). Then, \(F \setminus (U_1 \cap U_2) = F \setminus (U_1 \cup U_2) \subseteq \text{int}_{B}(A)\), but \(U_1, U_2 \in I \implies (U_1 \cup U_2) \in I\). This implies that, \(\text{int}_{B}(A) \subseteq (U_1 \cup U_2)\) for some \(U \in I\). Hence, \(A\) is an \(I\)gp-open relative to \(X\).

Corollary 4.3. If \(\text{int}_{B}(A) \subseteq B \subseteq A\) and if \(A\) is an \(I\)gp-open, then \(B\) is an \(I\)gp-open in \(X\).
**Proof.** Suppose \( \text{int}_{12}(A) \subseteq B \subseteq A \) and \( A \) is \( Igp \)-open. Then, \( A' \subseteq B' \subseteq \text{cl}_{12}(A') \) and \( A' \) is \( Igp \)-closed. By Corollary 4.1, \( B' \) is \( Igp \)-closed and hence, \( B \) is \( Igp \)-open. \( \Box \)

**Theorem 4.7.** A set \( A \) is \( Igp \)-closed set if and only if \( cl_{12}(A) \backslash A \) is \( Igp \)-open set.

**Proof.** Necessity. Let \( A \) be \( Igp \)-closed set and suppose that \( F \subseteq cl_{12}(A) \backslash A \) and \( F \) be \( P \)-closed. By Theorem 4.1 we get, \( F \cup U = \emptyset \) for some \( U \in I \). Then, \( F \cup U \subseteq \text{int}_{12}(cl_{12}(A) \backslash A) \).

By Theorem 4.4 we get, \( cl_{12}(A) \backslash A \) is \( Igp \)-open set. Sufficiency. Suppose that \( A \subseteq G \) and \( G \) is \( P \)-open. Then, \( G' \subseteq A' \). So, \( cl_{12}(A) \cap G' \subseteq cl_{12}(A) \cap A' = cl_{12}(A) \backslash A \). But, \( cl_{12}(A) \backslash A \) is \( Igp \)-open and \( cl_{12}(A) \backslash G' \) is \( P \)-closed. Therefore, \( [cl_{12}(A) \cap G'] \cup U \subseteq \text{int}_{12}(cl_{12}(A) \backslash A) = \emptyset \) for some \( U \in I \). Thus, \( cl_{12}(A) \backslash G' \subseteq U \subseteq I \). Then, \( cl_{12}(A) \backslash G \in I \). Hence, \( A \) is \( Igp \)-closed set. \( \Box \)

**Theorem 4.8.** Let \( f : X \rightarrow Y \) be \( P^* \)-continuous and \( P^* \)-closed. If \( A \subseteq X \) is \( Igp \)-closed in \( X \), then \( f(A) \) is \( f(I) \)-gp-closed in \( Y \), where \( f(I) = \{f(U) : U \in I\} \).

**Proof.** Suppose \( A \subseteq X \) and \( A \) is \( Igp \)-closed. Suppose \( f(A) \subseteq G \) and \( G \) is \( P \)-open. Then, \( A \subseteq f^{-1}(G) \). But, \( A \) is \( Igp \)-closed. Then, \( cl_{12}(A) \backslash f^{-1}(G) \in I \) and hence, \( f(cl_{12}(A)) \backslash G \in f(I) \). Since, \( f \) is \( P^* \)-closed, \( cl_{12}(f(A)) \subseteq cl_{12}(f(cl_{12}(A))) = f(cl_{12}(A)) \). Then, \( cl_{12}(f(A)) \backslash G \subseteq f(cl_{12}(A)) \backslash G \in f(I) \) and hence, \( f(A) \) is \( f(I) \)-gp-closed. \( \Box \)

**References**