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RESEARCH ARTICLE

# Common fixed point theorems in complex valued rectangular metric spaces

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**Abstract** In this paper we prove some common fixed point theorems for weakly compatible mappings in complex valued rectangular (generalized) metric spaces. Examples are provided to support the results.

**Key Words** weakly compatible mapping, (E.A) property, common limit range(CLR) property

**MSC 2010** 47H10, 54H25

## 1 Introduction

Very recently, complex valued rectangular(generalized) metric spaces were introduced by Abbas et.al[2]. They replaced the triangular inequality in the complex valued metric by the rectangular inequality involving four points and extended the concept of complex valued metric spaces introduced by Azam et.al.[1]. They proved fixed point theorems involving the rational type contractive conditions for weakly contractive mappings in these spaces. Singh et.al.[10] have obtained fixed point theorems in complex valued rectangular (generalized) metric spaces for mappings satisfying the (E.A) and the common limit range (CLR) properties.

We prove a fixed point theorem for four weakly compatible selfmaps in complex valued rectangular(generalized) metric spaces and also prove the same if the mappings satisfy the (E.A) property, the common limit in the range of $f$, $(CLR_f)$ property, common (E.A) property and common limit in the range of $f$ and $g$ i.e. the $(CLR_{fg})$ property.

## 2 Preliminaries

The following definitions and results will be needed in the sequel. Let us denote the set of complex numbers by $\mathbb{C}$. Let $z_1, z_2 \in \mathbb{C}$. We define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$, iff $Re(z_1) \leqslant Re(z_2)$ and $Im(z_1) \leqslant Im(z_2)$. Thus we can say that $z_1 \precsim z_2$ if one of the following holds:

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(1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (2),(3),(4) holds. Also we write $z_1 \prec z_2$ if only (4) is satisfied. Note that the following statements hold:

(i) $0 \precsim z_1 \precnsim z_2$ implies that $|z_1| < |z_2|$,

(ii) $z_1 \precsim z_2$ and $z_2 \prec z_3$ implies that $z_1 \prec z_3$,

(iii) $0 \precsim z_1 \precsim z_2$ implies that $|z_1| \leqslant |z_2|$,

(iv) $a, b \in \mathbb{R}$ and $a \leqslant b$ implies that $az \precsim bz$, for all $z \in \mathbb{C}$.

**Definition 2.1.** [1] *Let $X$ be a nonempty set. If the mapping $d : X \times X \to \mathbb{C}$ satisfies the conditions:*

*(i)* $0 \precsim d(x, y)$, *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

*(ii)* $d(x, y) = d(y, x)$, *for all* $x, y \in X$;

*(iii)* $d(x, y) \precsim d(x, z) + d(z, y)$, *for all* $x, y, z \in X$.

*Then $d$ is called a complex valued metric on $X$ and $(X, d)$ is called a complex valued metric space.*

**Example 2.2.** *Let $X = \mathbb{C}$, define the mapping $d : X \times X \to \mathbb{C}$ by $d(z_1, z_2) = 3i|z_1 - z_2|$ for all $z_1, z_2 \in X$. Then $(X, d)$ is a complex valued metric space.*

The concept of complex valued rectangular(generalized) metric space defined by Abbas et.al. is as follows:

**Definition 2.3.** [2] *Let $X$ be a nonempty set. If the mapping $d : X \times X \to \mathbb{C}$ satisfies the conditions:*

*(i)* $0 \precsim d(x, y)$, *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

*(ii)* $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

*(iii)* $d(x, y) \precsim d(x, u) + d(u, v) + d(v, y)$, *for all* $x, y \in X$ *and all distinct* $u, v \in X$, *each one is different from* $x$ *and* $y$.

*Then $d$ is called a complex valued rectangular (generalized) metric on $X$ and $(X, d)$ is called a complex valued rectangular(generalized) metric space.*

**Example 2.4.** *Let $X = \{i, -i, 1, -1\}$,and define $d : X \times X \to \mathbb{C}$ as follows:* $d(1, -1) = d(-1, 1) = 3e^{i\theta}$,
$d(-1, i) = d(i, -1) = d(1, i) = d(i, 1) = e^{i\theta}$,
$d(1, -i) = d(-i, 1) = d(-1, -i) = d(-i, -1) = d(i, -i) = d(-i, i) = 4e^{i\theta}$,
$d(1, 1) = d(-1, -1) = d(i, i) = d(-i, -i) = 0$.

Then $(X, d)$ is a complex valued rectangular(generalized) metric space when $\theta \in [0, \pi/2]$. But $(X, d)$ is not a complex valued metric space, since $d(-1, 1) = 3e^{i\theta} > d(-1, i) + d(i, 1) = 2e^{i\theta}$.

**Definition 2.5.** [2] *Let $(X, d)$ be a complex valued rectangular(generalized) metric space and let $\{x_n\}$ be a sequence in $X$.*
*(1)If for every $c \in C$,with $0 \prec c$, there exists $n_0 \in \mathbb{N}$, such that $d(x_n, x) \prec c$ for all $n > n_0$, then $\{x_n\}$ is said to be convergent to $x$. We denote this by $x_n \to x$ as $n \to \infty$ or $\lim_{x\to\infty} x_n = x$.*
*(2)If for every $c \in \mathbb{C}$, with $0 \prec c$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n > n_0, d(x_n, x_m) \prec c$, then $\{x_n\}$ is said to be Cauchy sequence in $X$.*
*(3)If every Cauchy sequence is convergent in $(X, d)$, then $(X, d)$ is called a complete complex valued rectangular(generalized) metric space.*

**Lemma 2.6.** [2] *Let $(X, d)$ be a complex valued rectangular(generalized) metric space and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ converges to $x$ if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

**Lemma 2.7.** [2] *Let $(X, d)$ be a complex valued rectangular(generalized) metric space and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_m)| \to 0$ as $n \to \infty$.*

**Definition 2.8.** *Let $A$ and $B$ be two self maps of a nonempty set $X$. If $Ax = Bx = y$ for some $x \in X$, then $x$ is called the coincidence point of $A$ and $B$ and $y$ is called the point of coincidence of $A$ and $B$.*

**Definition 2.9.** *Two self mappings $A$ and $B$ are said to be weakly compatible if they commute at their coincidence points, i.e. $Ax = Bx$ implies that $ABx = BAx$.*

## 3 A fixed point theorem for weakly compatible maps

In this section, we prove a common fixed point theorem for four weakly compatible mappings in a complex valued rectangular(generalized) metric space.

**Theorem 3.1.** *Let $A, B, f$ and $g$ be four selfmappings of a complete complex valued rectangular(generalized) metric space $(X, d)$ which satisfy the following:*

$$d(Ax, By) \precsim \alpha\ d(fx, gy) + \beta\ [d(fx, Ax) + d(gy, By)] + \gamma\ \frac{[1 + d(fx, Ax)]d(gy, By)}{1 + d(fx, gy)} \tag{3.1}$$

*where $\alpha, \beta$ and $\gamma$ are nonnegative reals such that $\alpha + 2\beta + \gamma < 1$ and if,*

*(i)* $A(X) \subseteq g(X)$ *and* $B(X) \subseteq f(X)$,

*(ii)* *the pairs* $(A, f)$ *and* $(B, g)$ *are weakly compatible,*

*(iii)* *the subspace* $f(X)$ *or* $g(X)$ *is closed,*

*then the mappings $A, B, f$ and $g$ have a unique common fixed point.*

**Proof.** We construct a sequence $\{y_n\}$ in $X$ such that,

$$y_{2n} = Ax_{2n} = gx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = fx_{2n+2}, n \geqslant 0$$

where $\{x_{2n}\}$ is another sequence in $X$. Using (3.1), we have,

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Ax_{2n}, Bx_{2n+1}) \\ &\precsim \alpha\ d(fx_{2n}, gx_{2n+1}) + \beta\ [d(fx_{2n}, Ax_{2n}) + d(gx_{2n+1}, Bx_{2n+1})] + \\ &\quad \gamma\ \frac{[1 + d(fx_{2n}, Ax_{2n})]d(gx_{2n+1}, Bx_{2n+1})}{1 + d(fx_{2n}, gx_{2n+1})} \\ &\precsim \alpha\ d(y_{2n-1}, y_{2n}) + \beta\ [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] + \gamma\ \frac{[1 + d(y_{2n-1}, y_{2n})]d(y_{2n}, y_{2n+1})}{1 + d(y_{2n-1}, y_{2n})} \\ &\precsim (\alpha + \beta)d(y_{2n-1}, y_{2n}) + (\beta + \gamma)d(y_{2n}, y_{2n+1}). \end{aligned}$$

Hence,

$$d(y_{2n}, y_{2n+1}) \precsim \frac{\alpha + \beta}{1 - \beta - \gamma} d(y_{2n-1}, y_{2n}).$$

Therefore,

$$d(y_{2n}, y_{2n+1}) \precsim \delta\ d(y_{2n-1}, y_{2n}),$$

where $\delta = \frac{\alpha+\beta}{1-\beta-\gamma} < 1$, since $\alpha + 2\beta + \gamma < 1$. Proceeding in a similar way we have,

$$d(y_{2n}, y_{2n+1}) \precsim \delta d(y_{2n-1}, y_{2n}) \precsim \delta^2 d(y_{2n-2}, y_{2n-1}) ... \precsim \delta^{2n} d(y_0, y_1).$$

Finally we conclude that $d(y_n, y_{n+1}) \precsim \delta^n d(y_0, y_1)$. For $m, n \in \mathbb{N}, m > n$, we have,

$$\begin{aligned} d(y_n, y_m) &\precsim d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d((y_{m-1}, y_m) \\ &\precsim \delta^n d(y_0, y_1) + \delta^{n+1} d(y_0, y_1) + \delta^{n+2} d(y_0, y_1) + ... + \delta^{m-1} d(y_0, y_1) \\ &\precsim \frac{\delta^n}{1 - \delta} d(y_0, y_1). \end{aligned}$$

Therefore $|d(y_n, y_m)| \leqslant \frac{\delta^n}{1-\delta}|d(y_0, y_1)|$ and as $m, n \to \infty, |d(y_n, y_m)| \to 0$. Thus $\{y_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists point $z$ in $X$ such that,

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = z.$$

Assuming $f(X)$ is closed, $z \in f(X)$ and $z = fu$ for some $u \in X$. We claim that $Au = fu = z$. Using the rectangular inequality [Definition 2.3 (iii)] we get,

$$\begin{aligned} d(Au, z) &\precsim d(Au, Bx_{2n+1}) + d(Bx_{2n+1}, gx_{2n+1}) + d(gx_{2n+1}, z) \\ &\precsim \alpha\ d(fu, gx_{2n+1}) + \beta\ [d(fu, Au) + d(gx_{2n+1}, Bx_{2n+1})] + \\ &\quad \gamma\ \frac{[1 + d(fu, Au)]d(gx_{2n+1}, Bx_{2n+1})}{1 + d(fu, gx_{2n+1})} + d(Bx_{2n+1}, gx_{2n+1}) + d(gx_{2n+1}, z). \end{aligned}$$

As $n \to \infty$, we get,

$$d(Au, z) \precsim \alpha\ d(z, z) + \beta\ [d(z, Au) + d(z, z)] + \gamma\ \frac{[1 + d(z, Au)]d(z, z)}{1 + d(z, z)} + d(z, z) + d(z, z).$$

Hence,

$$d(Au, z) \precsim \beta\ d(z, Au).$$

As $\beta < 1, |d(Au, z)| = 0$ implies that $Au = z$ i.e. $fu = Au = z$ and $u$ is a coincidence point of $f$ and $A$. Since $A(X) \subseteq g(X), Au = gv$ for some $v \in X$. Hence $fu = Au = gv = z$. We claim that $Bv = z$. By the inequality (3.1),

$$\begin{aligned} d(z, Bv) = d(Au, Bv) &\precsim \alpha\ d(fu, gv) + \beta\ [d(fu, Au) + d(gv, Bv)] + \gamma\ \frac{[1 + d(fu, Au)]d(gv, Bv)}{1 + d(fu, gv)} \\ &\precsim \alpha\ d(z, z) + \beta\ [d(z, z) + d(z, Bv)] + \gamma\ \frac{[1 + d(z, z)]d(z, Bv)}{1 + d(z, z)} \\ &\precsim (\beta + \gamma)\ d(z, Bv). \end{aligned}$$

Since $\beta + \gamma < 1$, $|d(z, Bv)| = 0$ and $Bv = z$, hence $Au = fu = Bv = gv = z$. As $A$ and $f$ are weakly compatible, $Afu = fAu$ i.e. $Az = fz$. We now prove that $Az = z$, suppose not, $Az \neq z$, then by (3.1),

$$\begin{aligned} d(Az, z) = d(Az, Bv) &\precsim \alpha\ d(fz, gv) + \beta\ [d(fz, Az) + d(gv, Bv)] + \gamma\ \frac{[1 + d(fz, Az)]d(gv, Bv)}{1 + d(fz, gv)} \\ &\precsim \alpha\ d(Az, z) + \beta\ [d(Az, Az) + d(z, z)] + \gamma\ \frac{[1 + d(Az, Az)]d(z, z)}{1 + d(Az, z)} \\ &\precsim \alpha\ d(Az, z). \end{aligned}$$

Hence as $\alpha < 1$, $|d(Az, z)| = 0$ and $Az = z$ i.e. $Az = fz = z$. Thus $z$ is a fixed point of $A$ and $f$. Also since $B$ and $g$ are weakly compatible, $Bgv = gBv$ i.e. $Bz = gz$. We now prove that $Bz = z$, suppose not,$Bz \neq z$, then by (3.1) again,

$$\begin{aligned} d(z, Bz) = d(Az, Bz) &\precsim \alpha\ d(fz, gz) + \beta\ [d(fz, Az) + d(gz, Bz)] + \gamma\ \frac{[1 + d(fz, Az)]d(gz, Bz)}{1 + d(fz, gz)} \\ &\precsim \alpha\ d(z, Bz) + \beta\ [d(z, z) + d(Bz, Bz)] + \gamma\ \frac{[1 + d(z, z)]d(Bz, Bz)}{1 + d(z, Bz)} \\ &\precsim \alpha\ d(z, Bz). \end{aligned}$$

Hence $|d(z, Bz)| = 0$ i.e. $Bz = z$ and $Az = fz = Bz = gz = z$, $z$ is a common fixed point of $A, B, f$ and $g$. To show that the fixed point is unique, suppose that there is another point $w \in X$ such that $Aw = Bw = fw = gw = w$.
From(3.1), we have,

$$\begin{aligned} d(w, z) = d(Aw, Bz) &\precsim \alpha\ d(fw, gz) + \beta\ [d(fw, Aw) + d(gz, Bz)] + \gamma\ \frac{[1 + d(fw, Aw)]d(gz, Bz)}{1 + d(fw, gz)} \\ &\precsim \alpha\ d(w, z) + \beta\ [d(w, w) + d(z, z)] + \gamma\ \frac{[1 + d(w, w)]d(z, z)}{1 + d(w, z)}. \end{aligned}$$

Thus,

$$d(w, z) \precsim \alpha\ d(w, z).$$

Therefore $|d(w, z)| = 0$ as $\alpha < 1$and $w = z$ which proves the uniqueness of the fixed point. Similar argument holds if $g(X)$ is assumed to be closed. Hence $A, B, f$ and $g$ have a unique common fixed point in $X$.□

By putting $\gamma = 0$, in Theorem 3.1 above, we get the following corollary.

**Corollary 3.2.** *Let $A, B, f$ and $g$ be four mappings of a complete complex valued rectangular(generalized) metric space $(X, d)$ which satisfy the following:*

$$d(Ax, By) \precsim \alpha\ d(fx, gy) + \beta\ [d(fx, Ax) + d(gy, By)] \tag{3.2}$$

*where $\alpha$ and $\beta$ are nonnegative reals such that $\alpha + \beta < 1$ and if,*

*(i)* $A(X) \subseteq g(X)$ *and* $B(X) \subseteq f(X)$,

*(ii) the pairs $(A, f)$ and $(B, g)$ are weakly compatible,*

*(iii) the subspace $f(X)$ or $g(X)$ is closed,*

*then $A, B, f$ and $g$ have a unique common fixed point.*

If we put $B = A$ and $g = f$, $\beta = \gamma = 0$ in Theorem 3.1, we get the following corollary.

**Corollary 3.3.** *Let $A$ and $f$ be two mappings of a complete complex valued rectangular(generalized) metric space $(X, d)$ which satisfy the following:*

$$d(Ax, Ay) \precsim \alpha\ d(fx, fy) \tag{3.3}$$

*where $\alpha$ is nonnegative real such that $\alpha < 1$ and if,*

*(i)* $A(X) \subseteq f(X)$,

*(ii) the pairs $(A, f)$ is weakly compatible,*

*(iii) the subspace $f(X)$ is closed,*

*then $A$ and $f$ have a unique common fixed point.*

**Example 3.4.** *Let $X = \{1, 2, 3, 4\}$ and define the generalized metric on $d : X \times X \to \mathbb{C}$ as*
*d(1,4)= d(4,1)=d(2,4)=d(4,2)=d(3,4)=d(4,3)= 0.6i,*
*d(1,2)=d(2,1)= d(2,3)=d(3,2)= 0.2i,*
*d(1,3)=d(3,1)= 0.3i,*
*d(1,1)=d(2,2)=d(3,3)=d(4,4)= 0.*
*It is easy to show that $(X, d)$ is a complex valued rectangular(generalized) metric space but it is not a complex valued metric space since it does not satisfy the triangular property, d(1,3)= 0.6i >d(1,2)+ d(2,3)= 0.2i + 0.2i = 0.4i We define mappings $A$ and $f$ as $Ax = 1$ for all $x \in X$ and*

$$f(x) = \begin{cases} 1 & \text{for } x \in \{1, 2, 3\}; \\ 2 & \text{if } x = 4; \end{cases}$$

*Here $A(X) \subseteq f(X)$,also $A$ and $f$ are weakly compatible. For all $x \in X$, it can be shown that $d(Ax, Ay) \precsim \alpha\ d(fx, fy)$ i.e. $d(Ax, Ay) = d(1,1) = 0 \leqslant \alpha\ d(fx, fy)$, where $\alpha$ is any nonnegative real such that $\alpha < 1$. Hence all the conditions of Corollary 3.3 are fulfilled and $1 \in X$ is the unique common fixed point of $A$ and $f$.*

## 4 Fixed point theorems for mappings satisfying the (E.A)property and (CLR) property

Common fixed point theorems for mappings satisfying the (E.A)property and the common limit in the range of $f$ ($CLR_f$) property in complex valued metric spaces have been studied by many researchers such as Verma et.al.[11], S.Manro[8] and Chandok et.al.[5]. The theorems were proved relaxing the completeness of the space X, any of the subspaces or continuity of the mappings involved and assuming closedness of the subspaces.We aim to obtain such results in complex valued rectangular(generalized)metric spaces. The notion of property (E.A) was given by M. Aamri and D. El Moutawakil [3] and later for complex valued metric spaces by Verma et.al.[11]. The definition of property (E.A) in the context of complex valued rectangular (generalized) metric spaces is as follows:

**Definition 4.1.** *Let $A, B : X \to X$ be two selfmappings of a complex valued rectangular(generalized)metric space $(X, d)$. The pair is said to satisfy (E.A) property if there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$, for some $t \in X$.*

The class of mappings satisfying the property (E.A) contains the class of non-compatible mappings. The (E.A) property and weak compatibility are shown to be independent in [11].
The Common limit range property (CLR)property, defined by Sintunavarat et.al.[9] is as follows:

**Definition 4.2.** [9] *The self mappings $A, B : X \to X$ of a complex valued rectangular(generalized) metric space $(X, d)$ are said to satisfy the common limit in the range of $B$ property ($CLR_B$) property if $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = Bx$, for some $x \in X$.*

**Example 4.3.** *Let $(X, d)$ be a complex valued rectangular (generalized) metric space as defined in Example 3.4. Let us define mappings $A, B : X \to X$ as $Ax = 2x + 1$ and $Bx = 3x$ for all $x \in X$ and sequence $\{x_n\} = \{1 + 1/n\}$. Then,*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} 2\{1 + 1/n\} + 1 = 3, \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} 3\{1 + 1/n\} = 3.$$

*Since $3 \in X$, the mappings $A$ and $B$ satisfy the (E.A) property.*
*Also,*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} 2\{1 + 1/n\} + 1 = 3, \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} 3\{1 + 1/n\} = 3 = B(1)$$

*where $1 \in X$. Hence $A$ and $B$ satisfy the common limit in the range of $B$ i.e.($CLR_B$) property.*

We prove in the following theorem, the existence of a unique common fixed point for four mappings of a complex valued rectangular(generalized) metric space $(X, d)$ satisfying property (E.A) and we try to use minimum other requirements to prove the results.

**Theorem 4.4.** *Let $A, B, f$ and $g$ be four selfmappings of a complex valued rectangular(generalized) metric space $(X, d)$ which satisfy the following:*

$$d(Ax, By) \precsim \alpha\ d(fx, gy) + \beta\ [d(fx, Ax) + d(gy, By)] + \gamma\ \frac{[1 + d(fx, Ax)]d(gy, By)}{1 + d(fx, gy)} \tag{4.1}$$

*where $\alpha, \beta$ and $\gamma$ are nonnegative reals such that $\alpha + \beta + \gamma < 1$ and if*

*(i) one of the pairs $(A, f)$ (or $(B, g)$) satisfies the property (E.A),*

*(ii) $A(X) \subseteq g(X)$ (or $B(X) \subseteq f(X)$),*

*(iii) the subspace $f(X)$ (or $g(X)$ is closed),*

*(iv) the pairs $(A, f)$ and $(B, g)$ are weakly compatible,*

*then the mappings $A, B, f$ and $g$ have a unique common fixed point.*

**Proof.** Suppose the pair $(B, g)$ satisfies the property $(E.A)$, then there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$. Since $B(X) \subseteq f(X)$, there exists a sequence $\{y_n\}$ in $X$ such that $Bx_n = fy_n, \lim_{n\to\infty} fy_n = t$. We claim that $\lim_{n\to\infty} Ay_n = t$. Suppose that $\lim_{n\to\infty} Ay_n = t^* \neq t$, then from (4.1) we get,

$$d(Ay_n, Bx_n) \precsim \alpha\ d(fy_n, gx_n) + \beta\ [d(fy_n, Ay_n) + d(gx_n, Bx_n)] + \gamma\ \frac{[1 + d(fy_n, Ay_n)]d(gx_n, Bx_n)}{1 + d(fy_n, gx_n)}.$$

As $n \to \infty$,

$$d(t^*, t) \precsim \alpha\ d(t, t) + \beta\ [d(t, t^*) + d(t, t)] + \gamma\ \frac{[1 + d(t, t*)]d(t, t)}{1 + d(t, t)}.$$

Hence,

$$d(t^*, t) \precsim \beta\ d(t^*, t).$$

As $\beta < 1$, $|d(t^*, t)| = 0$ and $t = t^*$. Thus $\lim_{n\to\infty} Ay_n = t$.
Hence $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n gx_n = \lim_{n\to\infty} fy_n = \lim_{n\to\infty} gx_n = t$. Suppose $g(X)$ is a closed. Then $t = gu$ for some $u \in X$. Therefore $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fy_n = \lim_{n\to\infty} Ay_n = t = gu$. We claim that $Bu = gu$. From (4.1), we have,

$$d(Ay_n, Bu) \precsim \alpha\ d(fy_n, gu) + \beta\ [d(fy_n, Ay_n) + d(gu, Bu)] + \gamma\ \frac{[1 + d(fy_n, Ay_n)]d(gu, Bu)}{1 + d(fy_n, gu)}.$$

As $n \to \infty$, we get,

$$d(t, Bu) \precsim \alpha\ d(t, t) + \beta\ [d(t, t) + d(t, t)] + \gamma\ \frac{[1 + d(t, t)]d(t, Bu)}{1 + d(t, t)}.$$

Thus,

$$d(t, Bu) \precsim \gamma\ d(t, Bu).$$

As $\gamma < 1$, $|d(t, Bu)| = 0$ and $Bu = t$ i.e.$Bu = gu = t$. Since $B$ and $g$ are weakly compatible, $Bgu = gBu$, i.e. $Bt = gt$ and $t$ is a coincidence point of $B$ and $g$. Since $B(X) \subseteq f(X)$, $Bu = fv$ for some $v \in X$. Hence $Bu = gu = fv = t$. We claim that $Av = fv = t$. From (4.1) we have,

$$d(Av, Bu) \precsim \alpha\ d(fv, gu) + \beta\ [d(fv, Av) + d(gu, Bu)] + \gamma\ \frac{[1 + d(fv, Av)]d(gu, Bu)}{1 + d(fv, gu)}.$$

Hence,

$$d(Av, t) \precsim \alpha\ d(t, t) + \beta\ [d(t, Av) + d(t, t)] + \gamma\ \frac{[1 + d(t, Av)]d(t, t)}{1 + d(t, t)},$$

i.e.

$$d(Av, t) \precsim \beta\ d(Av, t).$$

As $\beta < 1$,$|d(Av, t)| = 0$ and $Av = t$. Thus $Av = fv = t$. Since $A$ and $f$ are weakly compatible, $Afv = fAv$ i.e. $At = ft$. We claim that $At = t$. Consider, from (4.1),

$$\begin{aligned} d(At, t) = d(At, Bu) &\precsim \alpha\ d(ft, gu) + \beta\ [d(ft, At) + d(gu, Bu)] + \gamma\ \frac{[1 + d(ft, At)]d(gu, Bu)}{1 + d(ft, gu)} \\ &\precsim \alpha\ d(At, t) + \beta\ [d(At, At) + d(t, t)] + \gamma\ \frac{[1 + d(At, At)]d(t, t)}{1 + d(At, t)} \\ &\precsim \alpha\ d(At, t). \end{aligned}$$

As $\alpha < 1$, $|d(At, t)| = 0$ and $At = t$. Similarly we can show that $Bt = t$. Hence $At = Bt = ft = gt = t$ and the mappings $A, B, f$ and $g$ have a common fixed point. To prove uniqueness of the fixed point, assume that $w \in X$ is another point such that $Aw = Bw = fw = gw = w$. Then, using (4.1),

$$\begin{aligned} d(t, w) = d(At, Bw) &\precsim \alpha\ d(ft, gw) + \beta\ [d(ft, At) + d(gw, Bw)] + \gamma\ \frac{[1 + d(ft, At)]d(gw, Bw)}{1 + d(ft, gw)} \\ &\precsim \alpha\ d(t, w) + \beta\ [d(t, t) + d(w, w)] + \gamma\ \frac{[1 + d(t, t)]d(w, w)}{1 + d(t, w)} \\ &\precsim \alpha\ d(t, w). \end{aligned}$$

Since $\alpha < 1$, we have $|d(t, w)| = 0$ and $t = w$, which proves the uniqueness of the fixed point. Similar result can be obtained assuming that the pair $(A, f)$ satisfies the (E.A) property and $f(X)$ is closed. Hence the proof. □

Now we obtain fixed point theorem for mapping satisfying the common limit in the range of $f(CLR_f)$(or $(CLR_g)$) property in complex valued rectangular(generalized) metric spaces. Here the results are obtained without the closedness assumption of the subspaces.

**Theorem 4.5.** *Let $A, B, f$ and $g$ be four selfmappings of a complex valued rectangular(generalized) metric space $(X, d)$ which satisfy the following:*

$$d(Ax, By) \precsim \alpha\ d(fx, gy) + \beta\ [d(fx, Ax) + d(gy, By)] + \gamma\ \frac{[1 + d(fx, Ax)]d(gy, By)}{1 + d(fx, gy)} \tag{4.2}$$

*where $\alpha, \beta$ and $\gamma$ are nonnegative reals such that $\alpha + \beta + \gamma < 1$ and if* *(i)* $A(X) \subseteq g(X)$ *(or* $B(X) \subseteq f(X)$*),*

*(ii) the pairs* $(A, f)$ *and* $(B, g)$ *are weakly compatible,*

*(iii) the pair* $(A, f)$ *satisfies the* $(CLR_f)$ *property (or* $(B, g)$*) satisfies the* $(CLR_g)$ *property,*

*then the mappings* $A, B, f$ *and* $g$ *have a unique common fixed point.*

**Proof.** Suppose the mappings $A$ and $f$ satisfy the $(CLR_f)$ property, then there exists sequence $\{x_n\}$ such that,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} fx_n = fx = t, say$$

for some $x \in X$. Since $A(X) \subseteq g(X)$, there exists a sequence $\{y_n\}$ in $X$ such that $Ax_n = gy_n$. Therefore $\lim_{n\to\infty} gy_n = t$. We claim that $\lim_{n\to\infty} By_n = t$. Suppose that $\lim_{n\to\infty} By_n = t^* \neq t$, then from (4.2) we get,

$$d(Ax_n, By_n) \precsim \alpha\ d(fx_n, gy_n) + \beta\ [d(fx_n, Ax_n) + d(gy_n, By_n)] + \gamma\ \frac{[1 + d(fx_n, Ax_n)]d(gy_n, By_n)}{1 + d(fx_n, gy_n)}.$$

As $n \to \infty$, we have,

$$d(t, t^*) \precsim \alpha\ d(t, t) + \beta\ [d(t, t) + d(t, t^*)] + \gamma\ \frac{[1 + d(t, t)]d(t, t^*)}{1 + d(t, t)}.$$

Thus,

$$d(t, t^*) \precsim (\beta + \gamma)\ d(t, t^*).$$

Since $\beta + \gamma < 1$,$|d(t, t^*)| = 0$ and $t = t^*$. Thus $\lim_{n\to\infty} By_n = t$. Hence $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} gy_n = t = fx$. We claim that $Ax = fx$, suppose not, then from (4.2), we have,

$$d(Ax, By_n) \precsim \alpha\ d(fx, gy_n) + \beta\ [d(fx, Ax) + d(gy_n, By_n)] + \gamma\ \frac{[1 + d(fx, Ax)]d(gy_n, By_n)}{1 + d(fx, gy_n)}.$$

As $n \to \infty$, we have,

$$d(Ax, fx) \precsim \alpha\ d(fx, fx) + \beta\ [d(fx, Ax) + d(fx, fx)] + \gamma\ \frac{[1 + d(fx, Ax)]d(fx, fx)}{1 + d(fx, fx)},$$

i.e.

$$d(Ax, fx) \precsim \beta\ d(fx, Ax).$$

As $\beta < 1$, $|d(Ax, fx)| = 0$ and $Ax = fx$ i.e. $Ax = fx = t$. Since $A(X) \subseteq g(X)$, $Ax = gv$ for some $v \in X$. Hence $Ax = fx = gv = t$. We claim that $Bv = t$. From (4.2),

$$d(Ax, Bv) \precsim \alpha\ d(fx, gv) + \beta\ [d(fx, Ax) + d(gv, Bv)] + \gamma\ \frac{[1 + d(fx, Ax)]d(gv, Bv)}{1 + d(fx, gv)}.$$

Hence,

$$d(t, Bv) \precsim \alpha\ d(t, t) + \beta\ [d(t, t) + d(t, Bv)] + \gamma\ \frac{[1 + d(t, t)]d(t, Bv)}{1 + d(t, t)},$$

i.e.

$$d(t, Bv) \precsim (\beta + \gamma)\ d(t, Bv).$$

As $\beta + \gamma < 1$, we have $|d(t, Bv)| = 0$ and $Bv = t$. Hence $Ax = fx = gv = Bv = t$. As $A$ and $f$ are weakly compatible, $Afx = fAx$ and $At = ft$, and also since $B$ and $g$ are weakly compatible, $Bgv = gBv$ i.e. $Bt = gt$. We claim $At = t$. Using (4.2), consider,

$$\begin{aligned} d(At, t) = d(At, Bv) &\precsim \alpha\ d(ft, gv) + \beta\ [d(ft, At) + d(gv, Bv)] + \gamma\ \frac{[1 + d(ft, At)]d(gv, Bv)}{1 + d(ft, gv)} \\ &\precsim \alpha\ d(At, t) + \beta\ [d(At, At) + d(t, t)] + \gamma\ \frac{[1 + d(At, At)]d(t, t)}{1 + d(At, t)} \\ &\precsim \alpha\ d(At, t). \end{aligned}$$

As $\alpha < 1$, $|d(At, t)| = 0$ and $At = t$. Similarly we can prove that $Bt = t$ i.e. $Bt = gt = t$, hence $At = ft = Bt = gt = t$ and $t$ is a common fixed point of the mappings $A, f, B$ and $g$. Uniqueness of the common fixed point follows easily from (4.2). Similarly existence and uniqueness of the fixed point can be proved assuming that $(B, g)$ satisfies the $(CLR_g)$ property. □

**Remark 4.6.** *We can conclude that if mappings $A$ and $f$ satisfy the property (E.A) and $f(X)$ is closed, then $A$ and $f$ satisfy the Common limit in the range of $f(CLR_f)$ property.*

## 5 Fixed point theorem for mappings satisfying the common (E.A) property

The concept of common (E.A)property was introduced by Liu et.al.[7]

**Definition 5.1.** *Two pairs of self maps $(A, f)$ and $(B, g)$ of a complex valued rectangular(generalized)metric space $(X, d)$ are said to satisfy the Common(E.A)property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} gy_n = t$$

*for some $t \in X$.*

Next we prove a fixed point theorem in complex valued rectangular(generalized) metric space in which the two pairs of selfmappings of $X$ satisfy the common property (E.A).

**Theorem 5.2.** *Let $A, B, f$ and $g$ be four selfmappings of a complex valued rectangular(generalized)metric space $(X, d)$ which satisfy the following:*

$$d(Ax, By) \precsim \alpha\ d(fx, gy) + \beta\ [d(fx, Ax) + d(gy, By)] + \gamma\ \frac{[1 + d(fx, Ax)]d(gy, By)}{1 + d(fx, gy)} \tag{5.1}$$

*where $\alpha, \beta$ and $\gamma$ are nonnegative reals such that $\alpha + \beta + \gamma < 1$ and if*

*(i) the pairs $(A, f)$ and $(B, g)$ are weakly compatible,*

*(ii) The pairs $(A, f)$ and $(B, g)$ satisfy the common property (E.A),*

*(iii) the subspaces $f(X)$ and $g(X)$ are closed,*

*then the mappings $A, B, f$ and $g$ have a unique common fixed point.*

**Proof.** Since the pairs $(A, f)$ and $(B, g)$ satisfy the common property (E.A), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} gy_n = z,$$

for some $z \in X$. Since $f(X)$ is closed, and $t \in f(X)$, there exists $u \in X$ such that $z = fu$. We claim that $Au = fu = z$. From (5.1) we have,

$$d(Au, By_n) \precsim \alpha\ d(fu, gy_n) + \beta\ [d(fu, Au) + d(gy_n, By_n)] + \gamma\ \frac{[1 + d(fu, Au)]d(gy_n, By_n)}{1 + d(fu, gy_n)}.$$

As $n \to \infty$,

$$\begin{aligned} d(Au, z) &\precsim \alpha\ d(z, z) + \beta\ [d(z, Au) + d(z, z)] + \gamma\ \frac{[1 + d(z, Au)]d(z, z)}{1 + d(z, z)} \\ &\precsim \beta\ d(z, Au). \end{aligned}$$

As $\beta < 1, |d(Au, z)| = 0$ and $Au = z$ i.e. $Au = fu = z$. Since $A$ and $f$ are weakly compatible, $Afu = fAu$ and $Az = fz$ i.e. $z$ is a coincidence point of $A$ and $f$. Also since $g(X)$ is closed and $z \in g(X)$, $z = gv$ for some $v \in X$. Hence $Au = fu = gv = z$. We prove that $Bv = gv = z$. Consider from (5.1),

$$\begin{aligned} d(z, Bv) = d(Au, Bv) &\precsim \alpha\ d(fu, gv) + \beta\ [d(fu, Au) + d(gv, Bv)] + \gamma\ \frac{[1 + d(fu, Au)]d(gv, Bv)}{1 + d(fu, gv)} \\ &\precsim \alpha\ d(z, z) + \beta\ [d(z, z) + d(z, Bv)] + \gamma\ \frac{[1 + d(z, z)]d(z, Bv)}{1 + d(z, z)} \\ &\precsim (\beta + \gamma)\ d(z, Bv). \end{aligned}$$

Hence $|d(z, Bv)| = 0$ as $\beta + \gamma < 1$. Therefore $Bv = gv = z$. Since $B$ and $g$ are weakly compatible, we have $Bgv = gBv$ i.e. $Bz = gz$ and $z$ is a coincidence point of $B$ and $g$. To show that $z$ is a fixed point, we claim that $Az = z$. By (5.1),

$$d(Az, z) = d(Az, Bv) \precsim \alpha\ d(fz, gv) + \beta\ [d(fz, Az) + d(gv, Bv)] + \gamma\ \frac{[1 + d(fz, Az)]d(gv, Bv)}{1 + d(fz, gv)}.$$

Since $fz = Az$,

$$\begin{aligned} d(Az, z) &\precsim \alpha\ d(Az, z) + \beta\ [d(Az, Az) + d(z, z)] + \gamma\ \frac{[1 + d(Az, Az)]d(z, z)}{1 + d(Az, z)} \\ &\precsim \alpha\ d(Az, z). \end{aligned}$$

Since $\alpha < 1, |d(Az, z)| = 0$ and $Az = z$. Hence $Az = fz = z$. Similarly we can show that $Bz = gz = z$. Thus we have $Az = fz = Bz = gz = z$ and $z$ is a common fixed point of $A, B, f$ and $g$. Uniqueness of the fixed point follows easily using inequality (5.1). □

**Remark 5.3.** *It can be noted that in proving the existence and uniqueness of the unique fixed point of $A, B, f$ and $g$, when one of the pairs $(A, f)$ or $(B, g)$ satisfies the property (E.A), closedness of any one of $f(X)$ or $g(X)$ and containment of one pair of subspaces is required,whereas when the pairs $(A, f)$ and $(B, g)$ satisfy the common property (E.A),it is required that both $f(X)$ and $g(X)$ are closed. Here the assumption of containment of subspaces is omitted.*

## 6 Fixed point theorem for mappings satisfying $(CLR_{fg})$ property

The $(CLR_f)$[or $(CLR_g)$] property extended by Imdad et.al.[6] to the $(CLR_{fg})$ property involving two pairs of selfmappings is defined below. The $(CLR_{fg})$ property does not require closedness of the range subspaces nor their containment.

**Definition 6.1.** *Two pairs of selfmappings $(A, f)$ and $(B, g)$ in a complex valued rectangular(generalized) metric space $(X, d)$ are said to satisfy the $(CLR_{fg})$ property with respect to maps $f$ and $g$ if there exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} gy_n = t$$

*where $t \in f(X) \cap g(X)$.*

**Theorem 6.2.** *Let $A, B, f$ and $g$ be four selfmappings of a complex valued rectangular(generalized)metric space $(X, d)$ which satisfy the following:*

$$d(Ax, By) \precsim \alpha\ d(fx, gy) + \beta\ [d(fx, Ax) + d(gy, By)] + \gamma\ \frac{[1 + d(fx, Ax)]d(gy, By)}{1 + d(fx, gy)} \tag{6.1}$$

*where $\alpha, \beta$ and $\gamma$ are nonnegative reals such that $\alpha + \beta + \gamma < 1$ and*

*(i) the pairs $(A, f)$ and $(B, g)$ are weakly compatible,*

*(ii) The pairs $(A, f)$ and $(B, g)$ satisfy the $(CLR_{fg})$ property,*

*then the mappings $A, B, f$ and $g$ have a unique common fixed point.*

**Proof.** The pairs $(A, f)$ and $(B, g)$ satisfy the $(CLR_{fg})$ property, so there exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} gy_n = t$$

where $t \in f(X) \cap g(X)$. Since $t \in f(X)$,$t = fu$ for some $u \in X$. We prove that $fu = Au = t$, then using (6.1) we get,

$$d(Au, By_n) \precsim \alpha\ d(fu, gy_n) + \beta\ [d(fu, Au) + d(gy_n, By_n)] + \gamma\ \frac{[1 + d(fu, Au)]d(gy_n, By_n)}{1 + d(fu, gy_n)}.$$

As $n \to \infty$,

$$\begin{aligned} d(Au, t) &\precsim \alpha\ d(t, t) + \beta\ [d(t, Au) + d(t, t)] + \gamma\ \frac{[1 + d(t, Au)]d(t, t)}{1 + d(t, t)} \\ &\precsim \beta\ d(t, Au). \end{aligned}$$

Hence as $\beta < 1$, $|d(Au, t)| = 0$ and $Au = t$. Thus $Au = fu = t$ and $u$ is a coincidence point of $A$ and $f$. Since $A$ and $f$ are weakly compatible,$Afu = fAu$ and $At = ft$. Also $t \in g(X)$, there exists $v \in X$ such that $t = gv$. Hence $Au = fu = gv = t$. To prove that $Bv = gv = t$, suppose $Bv \neq gv$, consider using (6.1),

$$d(t, Bv) = d(Au, Bv) \precsim \alpha\ d(fu, gv) + \beta\ [d(fu, Au) + d(gv, Bv)] + \gamma\ \frac{[1 + d(fu, Au)]d(gv, Bv)}{1 + d(fu, gv)}$$

$$\precsim \alpha\ d(t,t) + \beta\ [d(t,t) + d(t,Bv)] + \gamma\ \frac{[1+d(t,t)]d(t,Bv)}{1+d(t,t)}.$$

Since $\beta + \gamma < 1$, $|d(t,Bv)| = 0$ and $Bv = t$ i.e. $Bv = gv = t$. As $B$ and $g$ are weakly compatible, $Bgv = gBv$ i.e. $Bt = gt$. In a similar way as in Theorem 4.4, we can prove that $At = t$ i.e. $At = ft = t$ and also $Bt = t$, $Bt = gt = t$. It implies that $At = Bt = ft = gt = t$ and $t$ is a common fixed point of $A, B, f$ and $g$. The uniqueness of the common fixed point follows easily from (6.1). □

**Remark 6.3.** *We can conclude that when the pairs* $(A, f)$ *and* $(B, g)$ *satisfy the common (E.A) property and if* $f(X)$ *and* $g(X)$*,both are closed, then the pairs satisfy the* $(CLR_{fg})$ *property.*

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