South Asian Journal of Mathematics 2016 , Vol. 6 (1) : 1 ∼ 9
www.sajm-online.com ISSN 2251-1512
RESEARCH ARTICLE

Operators on multiset bitopological spaces

S. A. El-Sheikh①, R. A-K. Omar①, M. Raafat①*

① Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt
E-mail: dr_mahmoudraafat2011@yahoo.com

Received: Oct-18-2015; Accepted: Dec-26-2015 *Corresponding author

Abstract The main purpose of this paper is to introduce and study some operators on multiset bitopological spaces such as MP^*-closure, MP^*-interior and MP^*-boundary. Also, their properties are presented in detail. Moreover, the concept of MP^*-continuous function is introduced in multiset bitopological spaces.

Key Words Mset, Multiset bitopological space, MP^*-closure, MP^*-interior and MP^*-boundary
MSC 2010 54A05, 54B05

1 Introduction

The notion of a multiset is well established both in mathematics and computer science [1, 2, 8, 15, 17, 19]. In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object are allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained [3, 9, 16, 18, 21]. For the sake of convenience an mset is written as $\{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$ in which the element x_i occurs k_i times. We observe that each multiplicity k_i is a positive integer. The number of occurrences of an object x in an mset A, which is finite in most of the studies that involve msets, is called its multiplicity or characteristic value, usually denoted by $m_A(x)$ or $C_A(x)$ or simply by $A(x)$. One of the most natural and simplest examples is the mset of prime factors of a positive integer n. The number 504 has the factorization $504 = 2^3 3^2 7^1$ which gives the mset $X = \{3/2, 2/3, 1/7\}$ where $C_X(2) = 3$, $C_X(3) = 2$, $C_X(7) = 1$. Jena et. al. [10] studied the concept of bags and some properties and results about this concept. Girish et. al. [6] presented mset topologies induced by mset relations and studied the concepts of closure operator, interior operator and neighborhood operator on mset. In 2012 Girish et. al. [7] studied the notions of basis, sub-basis, closed sets, closure, interior and continuous mset function.

A bitopological space (X, τ_1, τ_2) was introduced by Kelly [11] in 1963, as a method of generalizes topological spaces (X, τ). Every bitopological space (X, τ_1, τ_2) can be regarded as a topological space (X, τ) if $\tau_1 = \tau_2 = \tau$. Furthermore, he extended some of the standard results of separation axioms and

Citation: S. A. El-Sheikh, R. A-K. Omar, M. Raafat, Operators on multiset bitopological spaces, South Asian J Math, 2016, 6(1), 1-9.

mappings in a topological space to a bitopological space. The notion of connectedness in bitopological spaces has been studied by Pervin [13], Reily [14] and Swart [20]. In 2015, El-Sheikh et. al. [5] introduced initially the concept of multiset bitopological spaces and they presented some properties and results about this concept. In addition, they defined the notion of ij-operators on multiset bitopological spaces and studied the relationships among them.

In this paper, we firstly introduced some operators on multiset bitopological spaces such as MP^*-closure, MP^*-interior and MP^*-boundary. Additionally, their properties are presented in detail. Moreover, there exist many of deviations between multiset bitopological spaces and the previous work [4]. Finally, the concept of MP^*-continuous function is presented in multiset bitopological spaces.

2 Preliminaries

Definition 2.1. [10] An mset X drawn from the set U is represented by a count function X or C_X defined as $C_X : U \to N$, where N represents the set of non-negative integers.

Here C_X (x) is the number of occurrences of the element x in the mset X. We present the mset X drawn from the set $U = \{x_1, x_2, x_3, ..., x_n\}$ as $X = \{m_1/x_1, m_2/x_2, m_3/x_3, ..., m_n/x_n\}$ where m_i is the number of occurrences of the element x_i, $i = 1, 2, 3, ..., n$ in the mset X.

Definition 2.2. [10] A domain U, is defined as a set of elements from which msets are constructed. The mset space $[U]^w$ is the set of all msets whose elements are in U such that no element in the mset occurs more than w times.

The mset space $[U]^\infty$ is the set of all msets over a domain U such that there is no limit on the number of occurrences of an element in a mset. If $U = \{x_1, x_2, ..., x_k\}$, then $[U]^w = \{\{m_1/x_1, m_2/x_2, ..., m_k/x_k\} : for\ i = 1, 2, ..., k;\ m_i \in \{0, 1, 2, ..., w\}\}$.

Definition 2.3. [10] Let X and Y be two msets drawn from a set U. Then,

1. $X = Y$ if $C_X(x) = C_Y(x)$ for all $x \in U$,
2. $X \subseteq Y$ if $C_X(x) \leqslant C_Y(x)$ for all $x \in U$,
3. $P = X \cup Y$ if $C_P(x) = Max\{C_X(x), C_Y(x)\}$ for all $x \in U$,
4. $P = X \cap Y$ if $C_P(x) = Min\{C_X(x), C_Y(x)\}$ for all $x \in U$,
5. $P = X \oplus Y$ if $C_P(x) = Min\{C_X(x) + C_Y(x), w\}$ for all $x \in U$,
6. $P = X \ominus Y$ if $C_P(x) = Max\{C_X(x) - C_Y(x), 0\}$ for all $x \in U$, where $\oplus$ and $\ominus$ represent mset addition and mset subtraction respectively.

Definition 2.4. [10] Let X be a mset drawn from the set U. If $C_X(x) = 0\ \forall\ x \in U$, then X is called an empty mset and denoted by ϕ, i.e., $\phi(x) = 0\ \forall x$.

If X is an ordinary set with n distinct elements, then the power set $P(X)$ of X contains exactly 2^n elements. If X is a mset with n elements (repetitions counted), then the power set $P(X)$ contains strictly less than 2^n elements because singleton submsets do not repeat in $P(X)$. In classical set theory, Cantor's power set theorem fails for msets. It is possible to formulate the following reasonable definition of a power mset of X for finite mset X that preserves Cantor's power set theorem.

Definition 2.5. [1] (Power Mset) Let $X \in [U]^w$ be a mset. Then, the power mset $P(X)$ of X is the set of all submsets of X. We have $Y \in P(X)$ if and only if $Y \subseteq X$. If $Y = \phi$, then $Y \in^1 P(X)$; and if $Y \neq \phi$, then $Y \in^k P(X)$ where $k = \prod_z \begin{pmatrix} \mid [X]_z \mid \\ \mid [Y]_z \mid \end{pmatrix}$, the product $\prod_z$ is taken over by distinct elements of z of the mset Y and $\mid [X]_z \mid = m$ iff $z \in^m X$, $\mid [Y]_z \mid = n$ iff $z \in^n Y$, then $\begin{pmatrix} \mid [X]_z \mid \\ \mid [Y]_z \mid \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix} = \frac{m!}{n!(m-n)!}$.

The power set of a mset is the support set of the power mset and is denoted by $P^*(X)$. The following theorem was showed the cardinality of the power set of a mset.

Theorem 2.6. *[18] Let $P(X)$ be a power mset whose members drawn from the mset $X = \{m_1/x_1, m_2/x_2, ..., m_n/x_n\}$ and $P^*(X)$ be the power set of a mset X. Then, Card$(P^*(X)) = \Pi_{i=1}^n$ $(1+m_i)$.*

Definition 2.7. [6] Let $X \in [U]^w$ and $\tau \subseteq P^*(X)$. Then, τ is called a multiset topology (for short, M-topology) of X if τ satisfies the following properties:

1. the mset X and the empty mset ϕ are in τ,
2. the mset union of the elements of any subcollection of τ is in τ,
3. the mset intersection of the elements of any finite subcollection of τ is in τ.

Hence, (X,τ) is called an M-topological space. Each element in τ is called an open mset. Additionally, $OM(X)$ is the set of all open submsets of X.

Definition 2.8. [7] Let (X,τ) be a M-topological space and Y be a submset of X. The collection $\tau_Y = \{G' = Y \cap G; G \in \tau\}$ is a M-topology on Y, called the subspace M-topology.

Remark 2.9. [7] The complement of any submset Y in a mset topological space (X,τ) is defined by : $Y^c = X \ominus Y$.

Definition 2.10. [7] A submset Y of a M-topological space X is said to be closed if the mset $X \ominus Y$ is open.

Definition 2.11. [7] Let A be a submset of an M-topological space (X, τ). Then,

1. the interior of A is defined as the union of all open msets contained in A and denoted by $int(A)$, i.e., $int(A) = \cup\{G \subseteq X : G \text{ is an open mset and } G \subseteq A\}$
   and $C_{int(A)}(x) = max\{C_G(x) : G \in \tau, G \subseteq A\}$,

2. the closure of A is defined as the intersection of all closed msets containing A and denoted by $cl(A)$, i.e., $cl(A) = \cap\{K \subseteq X : K \textit{ is a closed mset and } A \subseteq K\}$ and $C_{cl(A)}(x) = min\{C_K(x) : K \in \tau^c, A \subseteq K\}$.

Proposition 2.12. *[6, 7] If (X,τ) is a M-topological space and A, B are two submsets of X. Then, the following properties are satisfied:*

- $int(A^c) = (cl(A))^c$.
- $cl(A^c) = (int(A))^c$.
- $cl(A \cup B) = cl(A) \cup cl(B)$.
- $int(A \cap B) = int(A) \cap int(B)$.

Definition 2.13. [12] The boundary of a submset A of X is the intersection of closure of A and closure of the complement of A. It is denoted by $b(A)$ and defined as $b(A) = cl(A) \cap cl(A^c)$.

Definition 2.14. [5] A multiset bitopological space is a triple (X, τ_1, τ_2) where X is a non-empty mset and τ_1,τ_2 are arbitrary M-topologies on X.

Definition 2.15. [5] Let (X, τ_1, τ_2) be a multiset bitopological space over X and Y be a non-empty submset of X. Then, $\tau_1^Y = \{Y \cap F : F \in \tau_1\}$ and $\tau_2^Y = \{Y \cap G : G \in \tau_2\}$ are said to be the relative M-topologies on Y. Also, (Y, τ_1^Y, τ_2^Y) is called a relative multiset bitopological subspace of (X, τ_1, τ_2).

3 Operators on multiset bitopological spaces

In this section, some operators on multiset bitopological spaces are introduced such as MP^*-closure, MP^*-interior and MP^*-boundary. Further, their properties are presented. Moreover, the concept of MP^*-continuous function is studied.

Definition 3.1. Let (X, τ_1, τ_2) be a multiset bitopological space. Then, the following operators are defined as:

1. an MP^*-closure operator $cl_{\tau^*} : P^*(X) \to P^*(X)$ is defined by:
   $cl_{\tau^*}(A) = cl_{\tau_1}(A) \cap cl_{\tau_2}(A)$, where $A \in P^*(X)$ and $P^*(X)$ is the support set of the power mset of X,
2. an MP^*-interior operator $int_{\tau^*} : P^*(X) \to P^*(X)$ is established by:
   $int_{\tau^*}(A) = int_{\tau_1}(A) \cup int_{\tau_2}(A)$, where $A \in P^*(X)$ and $\tau^* = \{A \subseteq X : int_{\tau^*}(A) = A\}$,
3. an MP^*-boundary operator $b_{\tau^*} : P^*(X) \to P^*(X)$ is described by:
   $b_{\tau^*}(A) = b_{\tau_1}(A) \cap b_{\tau_2}(A) = cl_{\tau^*}(A) \cap cl_{\tau^*}(A^c)$, where $A \in P^*(X)$.

Remark 3.2. Clearly from the above definition, τ^* is a supra M-topological space because a finite intersection of members of τ^* may not be a member of τ^* as shown in the following example.

Example 3.3. Let $X = \{3/a, 2/b, 1/c\}$ be an mset and $\tau_1 = \{X, \phi, \{3/a\}, \{2/b, 1/c\}\}$, $\tau_2 = \{X, \phi, \{2/b\}, \{3/a, 1/c\}\}$. Therefore,
$\tau^* = \{X, \phi, \{3/a\}, \{2/b\}, \{3/a, 2/b\}, \{3/a, 1/c\}, \{2/b, 1/c\}\}$. Assume that $A = \{3/a, 1/c\}$, $B = \{2/b, 1/c\}$. Then, $A, B \in \tau^*$. But, $A \cap B = \{1/c\} \notin \tau^*$.

Theorem 3.4. *Let (X, τ_1, τ_2) be a multiset bitopological space. Then, MP^*-closure operator has the following properties:*

1. $A \subseteq cl_{\tau^*}(A)\ \forall A \in P^*(X)$,
2. *if* $A \subseteq B$, *then* $cl_{\tau^*}(A) \subseteq cl_{\tau^*}(B)\ \forall A, B \in P^*(X)$,
3. $cl_{\tau^*}(cl_{\tau^*}(A)) = cl_{\tau^*}(A)$,
4. $cl_{\tau^*}(A \cup B) \supseteq cl_{\tau^*}(A) \cup cl_{\tau^*}(B)$,
5. $cl_{\tau^*}(A \cap B) \subseteq cl_{\tau^*}(A) \cap cl_{\tau^*}(B)$,
6. $cl_{\tau^*}(\phi) = \phi$ *and* $cl_{\tau^*}(X) = X$,
7. $(int_{\tau^*}(A))^c = cl_{\tau^*}(A^c)$, *where* A^c *is the complement of* A *with respect to the mset* X,
8. $A \in \tau^{*c}$ *if and only if* $cl_{\tau^*}(A) = A$, *where* τ^{*c} *is the family of* τ^**-closed submsets of* X.

Proof.

1. Since, $cl_{\tau^*}(A) = cl_{\tau_1}(A) \cap cl_{\tau_2}(A)$, $A \subseteq cl_{\tau_1}(A)$ and $A \subseteq cl_{\tau_2}(A)$. Then, $A \subseteq cl_{\tau_1}(A) \cap cl_{\tau_2}(A) = cl_{\tau^*}(A)$.
2. Let $A \subseteq B$. Then, $cl_{\tau_i}(A) \subseteq cl_{\tau_i}(B)$ where $i = 1, 2$. Therefore, $cl_{\tau_1}(A) \cap cl_{\tau_2}(A) \subseteq cl_{\tau_1}(B) \cap cl_{\tau_2}(B)$. Thus, $cl_{\tau^*}(A) \subseteq cl_{\tau^*}(B)$.
3. Since, $cl_{\tau^*}(cl_{\tau^*}(A)) = cl_{\tau_1}(cl_{\tau_1}(A) \cap cl_{\tau_2}(A)) \cap cl_{\tau_2}(cl_{\tau_1}(A) \cap cl_{\tau_2}(A))$.
   Then, $cl_{\tau^*}(cl_{\tau^*}(A)) \subseteq (cl_{\tau_1}(cl_{\tau_1}(A)) \cap cl_{\tau_1}(cl_{\tau_2}(A))) \cap (cl_{\tau_2}(cl_{\tau_1}(A)) \cap cl_{\tau_2}(cl_{\tau_2}(A)))$.
   Therefore, $cl_{\tau^*}(cl_{\tau^*}(A)) \subseteq (cl_{\tau_1}(A) \cap cl_{\tau_1}(cl_{\tau_2}(A))) \cap (cl_{\tau_2}(cl_{\tau_1}(A)) \cap cl_{\tau_2}(A))$.
   Hence, $cl_{\tau^*}(cl_{\tau^*}(A)) \subseteq cl_{\tau_1}(A) \cap cl_{\tau_2}(A)$. This implies that $cl_{\tau^*}(cl_{\tau^*}(A)) \subseteq cl_{\tau^*}(A)$. Conversely, from part (1) we have $A \subseteq cl_{\tau^*}(A)$. Then, $cl_{\tau^*}(A) \subseteq cl_{\tau^*}(cl_{\tau^*}(A))$ by part (2). Thus, $cl_{\tau^*}(cl_{\tau^*}(A)) = cl_{\tau^*}(A)$.
4. Since, $A, B \subseteq A \cup B$. Then, $cl_{\tau^*}(A) \subseteq cl_{\tau^*}(A \cup B)$ and $cl_{\tau^*}(B) \subseteq cl_{\tau^*}(A \cup B)$ from part (2). Thus, $cl_{\tau^*}(A) \cup cl_{\tau^*}(B) \subseteq cl_{\tau^*}(A \cup B)$.
5. Since, $A \cap B \subseteq A, B$. Then, $cl_{\tau^*}(A \cap B) \subseteq cl_{\tau^*}(A)$ and $cl_{\tau^*}(A \cap B) \subseteq cl_{\tau^*}(B)$ from part (2). Thus, $cl_{\tau^*}(A \cap B) \subseteq cl_{\tau^*}(A) \cap cl_{\tau^*}(B)$.
6. Clear.
7. Since, $(int_{\tau^*}(A))^c = (int_{\tau_1}(A) \cup int_{\tau_2}(A))^c$. Then, $(int_{\tau^*}(A))^c = (int_{\tau_1}(A))^c \cap (int_{\tau_2}(A))^c$. Therefore, $(int_{\tau^*}(A))^c = (cl_{\tau_1}(A^c)) \cap (cl_{\tau_2}(A^c))$. Hence, $(int_{\tau^*}(A))^c = cl_{\tau^*}(A^c)$.

8. Immediate by using part (7).

Remark 3.5. In multiset bitopological space (X, τ_1, τ_2), $cl_{\tau^*}(A \cup B) \neq cl_{\tau^*}(A) \cup cl_{\tau^*}(B)$ in general as shown in the following example.

Example 3.6. From Example 3.3, Let $A = \{2/a\}$, $B = \{1/b\}$. Then, $cl_{\tau^*}(A) = \{3/a\}$, $cl_{\tau^*}(B) = \{2/b\}$, but $cl_{\tau^*}(A \cup B) = X$. Hence, $cl_{\tau^*}(A \cup B) \neq cl_{\tau^*}(A) \cup cl_{\tau^*}(B)$.

Theorem 3.7. *Let (X, τ_1, τ_2) be a multiset bitopological space. Then, MP^*-interior operator has the following properties:*

1. $int_{\tau^*}(A) \subseteq A\ \forall A \in P^*(X)$,
2. *if* $A \subseteq B$, *then* $int_{\tau^*}(A) \subseteq int_{\tau^*}(B)\ \forall A, B \in P^*(X)$,
3. $int_{\tau^*}(int_{\tau^*}(A)) = int_{\tau^*}(A)$,
4. $int_{\tau^*}(\phi) = \phi$ *and* $int_{\tau^*}(X) = X$,
5. $int_{\tau^*}(A \cap B) \subseteq int_{\tau^*}(A) \cap int_{\tau^*}(B)$,
6. $int_{\tau^*}(A \cup B) \supseteq int_{\tau^*}(A) \cup int_{\tau^*}(B)$.

Proof.

1. Since, $int_{\tau^*}(A) = int_{\tau_1}(A) \cup int_{\tau_2}(A)$, $int_{\tau_1}(A) \subseteq A$ and $int_{\tau_2}(A) \subseteq A$. Then, $int_{\tau^*}(A) \subseteq A$.
2. Let $A \subseteq B$. Then, $int_{\tau_i}(A) \subseteq int_{\tau_i}(B)$ where $i = 1, 2$. Therefore, $int_{\tau_1}(A) \cup int_{\tau_2}(A) \subseteq int_{\tau_1}(B) \cup int_{\tau_2}(B)$. Thus, $int_{\tau^*}(A) \subseteq int_{\tau^*}(B)$.
3. Since, $int_{\tau^*}(int_{\tau^*}(A)) = int_{\tau_1}(int_{\tau_1}(A) \cup int_{\tau_2}(A)) \cup int_{\tau_2}(int_{\tau_1}(A) \cup int_{\tau_2}(A))$.
   Then, $int_{\tau^*}(int_{\tau^*}(A)) \supseteq (int_{\tau_1}(A) \cup int_{\tau_1}(int_{\tau_2}(A))) \cup (int_{\tau_2}(int_{\tau_1}(A)) \cup int_{\tau_2}(A))$.
   Therefore, $int_{\tau^*}(int_{\tau^*}(A)) \supseteq int_{\tau_1}(A) \cup int_{\tau_2}(A)$.
   Hence, $int_{\tau^*}(int_{\tau^*}(A)) \supseteq int_{\tau^*}(A)$.
   Conversely, from part (1) we have $int_{\tau^*}(A) \subseteq A$. Then, $int_{\tau^*}(int_{\tau^*}(A)) \subseteq int_{\tau^*}(A)$ by part (2). Thus, $int_{\tau^*}(int_{\tau^*}(A)) = int_{\tau^*}(A)$.
4. Clear.
5. Since, $A \cap B \subseteq A, B$. Then, $int_{\tau^*}(A \cap B) \subseteq int_{\tau^*}(A)$ and $int_{\tau^*}(A \cap B) \subseteq int_{\tau^*}(B)$ from part (2). Thus, $int_{\tau^*}(A \cap B) \subseteq int_{\tau^*}(A) \cap int_{\tau^*}(B)$.
6. Similarly.

Remark 3.8. The following example shows that:

1. $cl_{\tau^*}(A) = A$ does not imply that $A \in \tau_1^c$ or $A \in \tau_2^c$,
2. $int_{\tau^*}(A) = A$ does not imply that $A \in \tau_1$ or $A \in \tau_2$.

Example 3.9. From Example 3.3,

1. Let $A = \{1/c\}$. Then, $cl_{\tau^*}(A) = A$. But, $\{1/c\}$ is neither τ_1-closed mset nor τ_2-closed mset.

2. Let $A = \{3/a, 2/b\}$. Then, $int_{\tau^*}(A) = A$. But, A is neither τ_1-open mset nor τ_2-open mset.

Theorem 3.10. *If (X, τ) is a multiset topological space and $A \subseteq X$, then*

1. $int(A) \subseteq A \cap (b(A))^c$,

2. $cl(A) \supseteq A \cup b(A)$.

Proof.

1. Since, $b(A) = cl(A) \cap cl(A^c)$. Then, $(b(A))^c = (cl(A))^c \cup int(A)$. Therefore, $A \cap (b(A))^c = (A \cap (cl(A))^c) \cup (A \cap int(A))$. Thus, $A \cap (b(A))^c = (A \cap (cl(A))^c) \cup int(A)$. Hence, $int(A) \subseteq A \cap (b(A))^c$.

2. Similarly.

Remark 3.11. The equality of Theorem 3.10 is not true in general as shown in the following example.

Example 3.12. Let $X = \{3/a, 2/b, 1/c\}$ be an mset and $\tau = \{X, \phi, \{1/a, 2/b\}, \{1/a, 1/b\}\}$ be an M-topology on X.

1. If $A = \{2/a, 1/c\}$. Then, $int(A) = \phi$. But, $A \cap (b(A))^c = \{1/a\} \neq int(A)$.

2. If $A = \{2/a, 2/b\}$. Then, $cl(A) = X$. But, $A \cup b(A) = \{2/a, 2/b, 1/c\} \neq cl(A)$.

Theorem 3.13. *Let (X, τ_1, τ_2) be a multiset bitopological space. Then, MP^*-boundary operator has the following properties:*

1. $b_{\tau^*}(X) = b_{\tau^*}(\phi) = \phi$,

2. $b_{\tau^*}(A) = cl_{\tau^*}(A) \cap (int_{\tau^*}(A))^c \ \forall A \in P^*(X)$,

3. $int_{\tau^*}(A) \subseteq A \cap (b_{\tau^*}(A))^c$,

4. $cl_{\tau^*}(A) \supseteq A \cup b_{\tau^*}(A)$,

5. $b_{\tau^*}(A^c) = b_{\tau^*}(A)$,

6. $b_{\tau^*}(cl_{\tau^*}(A)) \subseteq b_{\tau^*}(A)$,

7. $b_{\tau^*}(int_{\tau^*}(A)) \subseteq b_{\tau^*}(A)$.

Proof.

1. Immediate.

2. Since, $b_{\tau^*}(A) = b_{\tau_1}(A) \cap b_{\tau_2}(A)$. Then, $b_{\tau^*}(A) = (cl_{\tau_1}(A) \cap cl_{\tau_1}(A^c)) \cap (cl_{\tau_2}(A) \cap cl_{\tau_2}(A^c))$. Therefore, $b_{\tau^*}(A) = (cl_{\tau_1}(A) \cap cl_{\tau_2}(A)) \cap (cl_{\tau_1}(A^c) \cap cl_{\tau_2}(A^c))$. Thus, $b_{\tau^*}(A) = cl_{\tau^*}(A) \cap cl_{\tau^*}(A^c)$. By using Theorem 3.4, $b_{\tau^*}(A) = cl_{\tau^*}(A) \cap (int_{\tau^*}(A))^c$.

3. Since, $int_{\tau_1}(A) \subseteq A \cap (b_{\tau_1}(A))^c$ and $int_{\tau_2}(A) \subseteq A \cap (b_{\tau_2}(A))^c$. Then, $int_{\tau^*}(A) \subseteq (A \cap (b_{\tau_1}(A))^c) \cup (A \cap (b_{\tau_2}(A))^c)$. Therefore, $int_{\tau^*}(A) \subseteq A \cap ((b_{\tau_1}(A))^c \cup (b_{\tau_2}(A))^c)$. Thus, $int_{\tau^*}(A) \subseteq A \cap (b_{\tau_1}(A) \cap b_{\tau_2}(A))^c$. Hence, $int_{\tau^*}(A) \subseteq A \cap (b_{\tau^*}(A))^c$.

4. Since, $cl_{\tau_i}(A) \supseteq A \cup b_{\tau_i}(A)$ where $i = 1, 2$. Then, $cl_{\tau^*}(A) \supseteq (A \cup b_{\tau_1}(A)) \cap (A \cup b_{\tau_2}(A))$. Thus, $cl_{\tau^*}(A) \supseteq A \cup (b_{\tau_1}(A) \cap b_{\tau_2}(A))$. Therefore, $cl_{\tau^*}(A) \supseteq A \cup b_{\tau^*}(A)$.

5. Clearly from the definition of MP^*-boundary operator.

6. Since, $b_{\tau^*}(A) = cl_{\tau^*}(A) \cap cl_{\tau^*}(A^c)$. Then, $b_{\tau^*}(cl_{\tau^*}(A)) = cl_{\tau^*}(cl_{\tau^*}(A)) \cap cl_{\tau^*}(cl_{\tau^*}(A))^c$. Therefore, $b_{\tau^*}(cl_{\tau^*}(A)) = cl_{\tau^*}(A) \cap cl_{\tau^*}(int_{\tau^*}(A^c))$. So,
$b_{\tau^*}(cl_{\tau^*}(A)) \subseteq cl_{\tau^*}(A) \cap cl_{\tau^*}(A^c)$. Hence, $b_{\tau^*}(cl_{\tau^*}(A)) \subseteq b_{\tau^*}(A)$.

7. Since, $b_{\tau^*}(A) = cl_{\tau^*}(A) \cap cl_{\tau^*}(A^c)$. Then, $b_{\tau^*}(int_{\tau^*}(A)) = cl_{\tau^*}(int_{\tau^*}(A)) \cap cl_{\tau^*}((int_{\tau^*}(A))^c)$. Therefore, $b_{\tau^*}(int_{\tau^*}(A)) \subseteq cl_{\tau^*}(A) \cap cl_{\tau^*}(cl_{\tau^*}(A^c))$. So, $b_{\tau^*}(int_{\tau^*}(A)) \subseteq cl_{\tau^*}(A) \cap cl_{\tau^*}(A^c)$. Hence, $b_{\tau^*}(int_{\tau^*}(A)) \subseteq b_{\tau^*}(A)$.

Theorem 3.14. *Let (X, τ_1, τ_2) be a multiset bitopological space. If $b_{\tau^*}(A) = \phi$, then $A \in \tau^* \cap (\tau^*)^c$.*

Proof. Since, $b_{\tau^*}(A) = \phi$. Then, $cl_{\tau^*}(A) \cap cl_{\tau^*}(A^c) = \phi$. Therefore, $cl_{\tau^*}(A) \subseteq (cl_{\tau^*}(A^c))^c = int_{\tau^*}(A)$. Thus, $A \subseteq cl_{\tau^*}(A) \subseteq int_{\tau^*}(A) \subseteq A$. Hence, $A \in \tau^* \cap (\tau^*)^c$.

Remark 3.15. The converse of Theorem 3.14 is not true in general as shown in the following example.

Example 3.16. Let $X = \{3/a, 2/b, 1/c\}$ be an mset and
$\tau_1 = \{X, \phi, \{2/a\}, \{1/b\}, \{2/a, 1/b\}\}$, $\tau_2 = \{X, \phi, \{1/a\}, \{1/a, 1/c\}, \{1/b, 1/c\},$
$\{1/c\}, \{1/a, 1/b, 1/c\}\}$. Therefore,
$\tau^* = \{X, \phi, \{1/a\}, \{1/b\}, \{1/c\}, \{2/a\}, \{1/a, 1/b\}, \{1/a, 1/c\}, \{1/b, 1/c\}, \{2/a, 1/b\},$
$\{2/a, 1/c\}, \{1/a, 1/b, 1/c\}, \{2/a, 1/b, 1/c\}\}$. Assume that $A = \{1/a, 1/b\} \in \tau^* \cap \tau^{*c}$. But, $b_{\tau^*}(A) = cl_{\tau^*}(A) \cap cl_{\tau^*}(A^c) = \{1/a, 1/b\} \cap \{2/a, 1/b, 1/c\} = \{1/a, 1/b\} \neq \phi$. Hence, $A \in \tau^* \cap \tau^{*c} \nRightarrow b_{\tau^*}(A) = \phi$.

Definition 3.17. Let (X, τ_1, τ_2) and (Y, η_1, η_2) be two multiset bitopological spaces. Then, $f : X \to Y$ is called an MP^*-continuous function if and only if $f^{-1}(V) \in \tau^*\ \forall\ V \in \eta^*$, where $\eta^* = \{B \subseteq Y : int_{\eta^*}(B) = B\}$.

Theorem 3.18. *Let (X, τ_1, τ_2) and (Y, η_1, η_2) be two multiset bitopological spaces and $f : X \to Y$ is an mset function. Then, the following conditions are equivalent:*

1. *f is an MP^*-continuous function,*

2. *$f^{-1}(H) \in \tau^{*c}\ \forall H \in \eta^{*c}$,*

3. *$f(cl_{\tau^*}(A)) \subseteq cl_{\eta^*}(f(A))\ \forall A \subseteq X$,*

4. *$cl_{\tau^*}(f^{-1}(B)) \subseteq f^{-1}(cl_{\eta^*}(B))\ \forall B \subseteq Y$,*

5. *$f^{-1}(int_{\eta^*}(B)) \subseteq int_{\tau^*}(f^{-1}(B))\ \forall B \subseteq Y$.*

Proof.

(1 ⇒ 2) Let f is an MP^*-continuous function and $H \in \eta^{*c}$. Then, $H^c \in \eta^*$. Thus, $f^{-1}(H^c) \in \tau^*$. Therefore, $(f^{-1}(H^c))^c \in \tau^{*c}$. Hence, $f^{-1}(H) \in \tau^{*c}$.

(2 ⇒ 3) Let $A \subseteq X$. Since, $f(A) \subseteq cl_{\eta^*}(f(A))$. Then, $A \subseteq f^{-1}(cl_{\eta^*}(f(A)))$. But, $cl_{\eta^*}(f(A))$ is a closed mset over Y. So, $f^{-1}(cl_{\eta^*}(f(A)))$ is also a closed mset over X. Hence, $cl_{\tau^*}(A) \subseteq f^{-1}(cl_{\eta^*}(f(A)))$. Therefore, $f(cl_{\tau^*}(A)) \subseteq cl_{\eta^*}(f(A))$.

(3 ⇒ 4) Let $B \subseteq Y$. Then, $f^{-1}(B) \subseteq X$. From part (3), $f(cl_{\tau^*}(f^{-1}(B))) \subseteq cl_{\eta^*}(ff^{-1}(B))$. Therefore, $f(cl_{\tau^*}(f^{-1}(B))) \subseteq cl_{\eta^*}(B)$. Hence, $cl_{\tau^*}(f^{-1}(B)) \subseteq f^{-1}(cl_{\eta^*}(B))$.

(4 ⇒ 5) Immediate by taking the complement to part (4).

(5 ⇒ 1) Let $V \in \eta^*$. Then, $int_{\eta^*}(V) = V$. Thus, $f^{-1}(V) \subseteq int_{\tau^*}(f^{-1}(V))$ by using part (5). But, $int_{\tau^*}(f^{-1}(V)) \subseteq f^{-1}(V)$. Hence, $f^{-1}(V) \in \tau^*$. This implies that f is an MP^*-continuous function.

References

1 Blizard and D. Wayne, Multiset theory, *Notre Dame Journal of Formal Logic* 30 (1989) 36−66.

2 Blizard and D. Wayne, Real-valued multisets and fuzzy sets, *Fuzzy Sets and Systems* 33 (1989) 77−97.

3 Blizard and D. Wayne, The development of multiset theory, *Modern Logic* 1 (1991) 319−352.

4 S. A. El-Sheikh, Dimension Theory of Bitopological Spaces, Master Thesis, Ain Shams University, Cairo, Egypt (1987).

5 S. A. El-Sheikh, R. A-K. Omar and M. Raafat, Multiset bitopological spaces, *Asian Journal of Mathematics and Computer Research* 8 (2015) 103−115.

6 K. P. Girish and J. J. Sunil, Multiset topologies induced by multiset relations, *Inform. Sci.* 188 (2012) 298−313.

7 K. P. Girish and J. J. Sunil, On multiset topologies, *Theory and Applications of Mathematics* & *Computer Science* 2 (2012) 37−52.

8 K. P. Girish and J. J. Sunil, Relations and functions in multiset context, *Inform. Sci.* 179 (2009) 758−768.

9 K. P. Girish and J. J. Sunil, Rough multisets and information multisystems, *Advances in Decision Sciences* Article ID 495392 (2011) 1−17.

10 S. P. Jena, S. K. Ghosh and B. K. Tripathy, On the theory of bags and lists, *Inform. Sci.* 132 (2001) 241−254.

11 J. C. Kelly, Bitopological spaces, *Proc. London Math. Soc.* 13 (1963) 71−89.

12 J. Mahanta and D. Das, Boundary and exterior of a multiset topology, *arXiv* 1501.07193v1 (2015).

13 W. J. Pervin, Connectedness in bitopological spaces, *Indog Math. J.* 29 (1967) 369−372.

14 I. L. Reilly, On pairwise connected bitopological spaces, *Kyungpook Math. J.* 11 (1971) 25−28.

15 D. Singh, A note on the development of multiset theory, *Modern Logic* 4 (1994) 405−406.

16 D. Singh, A. M. Ibrahim, T. Yohana and J. N. Singh, An overview of the applications of multisets, *Novi Sad J. Math* 37 (2007) 73−92.

17 D. Singh, A. M. Ibrahim, T. Yohana and J. N. Singh, Complementation in multiset theory, *International Mathematical Forum* 6 (2011) 1877−1884.

18 D. Singh and J. N. Singh, Some combinatorics of multisets, *International Journal of Mathematical Education in Science and Technology* 34 (2003) 489−499.

19 J. Šlapal, Relations and topologies, *Czechoslovak Mathematical Journal* 43 (1993) 141−150.

20 J. Swart, Total disconectedness in bitopological spaces and product bitopological spaces, *Indagationes Mathematicae* 74 (1971) 135−145.

21 R. R. Yager, On the theory of bags, *Int. General Syst.* 13 (1986) 23−37.