Operators on multiset bitopological spaces

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Abstract The main purpose of this paper is to introduce and study some operators on multiset bitopological spaces such as $M^p$-closure, $M^p$-interior and $M^p$-boundary. Also, their properties are presented in detail. Moreover, the concept of $M^p$-continuous function is introduced in multiset bitopological spaces.

Key Words Mset, Multiset bitopological space, $M^p$-closure, $M^p$-interior and $M^p$-boundary

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1 Introduction

The notion of a multiset is well established both in mathematics and computer science [1, 2, 8, 15, 17, 19]. In mathematics, a multiset is considered to be the generalization of a set. In classical set theory, a set is a well-defined collection of distinct objects. If repeated occurrences of any object are allowed in a set, then a mathematical structure, that is known as multiset (mset, for short), is obtained [3, 9, 16, 18, 21]. For the sake of convenience an mset is written as $\{k_1/x_1, k_2/x_2, \ldots, k_n/x_n\}$ in which the element $x_i$ occurs $k_i$ times. We observe that each multiplicity $k_i$ is a positive integer. The number of occurrences of an object $x$ in an mset $A$, which is finite in most of the studies that involve msets, is called its multiplicity or characteristic value, usually denoted by $m_A(x)$ or $C_A(x)$ or simply by $A(x)$. One of the most natural and simplest examples is the mset of prime factors of a positive integer $n$. The number 504 has the factorization $504 = 2^3 3^2 7^1$ which gives the mset $X = \{3/2, 2/3, 1/7\}$ where $C_X(2) = 3$, $C_X(3) = 2$, $C_X(7) = 1$. Jena et. al. [10] studied the concept of bags and some properties and results about this concept. Girish et. al. [6] presented mset topologies induced by mset relations and studied the concepts of closure operator, interior operator and neighborhood operator on mset. In 2012 Girish et. al. [7] studied the notions of basis, sub-basis, closed sets, closure, interior and continuous mset function.

A bitopological space $(X, \tau_1, \tau_2)$ was introduced by Kelly [11] in 1963, as a method of generalizes topological spaces $(X, \tau)$. Every bitopological space $(X, \tau_1, \tau_2)$ can be regarded as a topological space $(X, \tau)$ if $\tau_1 = \tau_2 = \tau$. Furthermore, he extended some of the standard results of separation axioms and
mappings in a topological space to a bitopological space. The notion of connectedness in bitopological spaces has been studied by Pervin [13], Rely [14] and Swart [20]. In 2015, El-Sheikh et. al. [5] introduced initially the concept of multiset bitopological spaces and they presented some properties and results about this concept. In addition, they defined the notion of \(ij\)-operators on multiset bitopological spaces and studied the relationships among them.

In this paper, we firstly introduced some operators on multiset bitopological spaces such as \(MP^*\)-closure, \(MP^*\)-interior and \(MP^*\)-boundary. Additionally, their properties are presented in detail. Moreover, there exist many of deviations between multiset bitopological spaces and the previous work [4]. Finally, the concept of \(MP^*\)-continuous function is presented in multiset bitopological spaces.

2 Preliminaries

Definition 2.1. [10] An mset \(X\) drawn from the set \(U\) is represented by a count function \(X\) or \(C_X\) defined as \(C_X : U \rightarrow N\), where \(N\) represents the set of non-negative integers.

Here \(C_X(x)\) is the number of occurrences of the element \(x\) in the mset \(X\). We present the mset \(X\) drawn from the set \(U = \{x_1, x_2, x_3, ..., x_n\}\) as \(X = \{m_1/x_1, m_2/x_2, m_3/x_3, ..., m_n/x_n\}\) where \(m_i\) is the number of occurrences of the element \(x_i\), \(i = 1, 2, 3, ..., n\) in the mset \(X\).

Definition 2.2. [10] A domain \(U\), is defined as a set of elements from which mssets are constructed. The mset space \([U]^w\) is the set of all mssets whose elements are in \(U\) such that no element in the mset occurs more than \(w\) times.

The mset space \([U]^\infty\) is the set of all mssets over a domain \(U\) such that there is no limit on the number of occurrences of an element in a mset. If \(U = \{x_1, x_2, ..., x_k\}\), then \([U]^w = \{(m_1/x_1, m_2/x_2, ..., m_k/x_k):\) for \(i = 1, 2, ..., k; \ m_i \in \{0, 1, 2, ..., w\}\}\).

Definition 2.3. [10] Let \(X\) and \(Y\) be two mssets drawn from a set \(U\). Then,

1. \(X = Y\) if \(C_X(x) = C_Y(x)\) for all \(x \in U\),
2. \(X \subseteq Y\) if \(C_X(x) \leq C_Y(x)\) for all \(x \in U\),
3. \(P = X \cup Y\) if \(C_P(x) = \text{Max}(C_X(x), C_Y(x))\) for all \(x \in U\),
4. \(P = X \cap Y\) if \(C_P(x) = \text{Min}(C_X(x), C_Y(x))\) for all \(x \in U\),
5. \(P = X \oplus Y\) if \(C_P(x) = \text{Min}(C_X(x) + C_Y(x), w)\) for all \(x \in U\),
6. \(P = X \ominus Y\) if \(C_P(x) = \text{Max}(C_X(x) - C_Y(x), 0)\) for all \(x \in U\), where \(\oplus\) and \(\ominus\) represent mset addition and mset subtraction respectively.

Definition 2.4. [10] Let \(X\) be a mset drawn from the set \(U\). If \(C_X(x) = 0\ \forall \ x \in U\), then \(X\) is called an empty mset and denoted by \(\phi\), i.e., \(\phi(x) = 0\ \forall x\).
If $X$ is an ordinary set with $n$ distinct elements, then the power set $P(X)$ of $X$ contains exactly $2^n$ elements. If $X$ is a mset with $n$ elements (repetitions counted), then the power set $P(X)$ contains strictly less than $2^n$ elements because singleton submsets do not repeat in $P(X)$. In classical set theory, Cantor’s power set theorem fails for mssets. It is possible to formulate the following reasonable definition of a power mset of $X$ for finite mset $X$ that preserves Cantor’s power set theorem.

**Definition 2.5.** [1] (Power Mset) Let $X \in [U]^{w}$ be a mset. Then, the power mset $P(X)$ of $X$ is the set of all submsets of $X$. We have $Y \in P(X)$ if and only if $Y \subseteq X$. If $Y = \phi$, then $Y \in ^{1}P(X)$; and if $Y \neq \phi$, then $Y \in ^{k}P(X)$ where $k = \prod_{x} \left( \frac{[X]_{z}}{[Y]_{z}} \right)$, the product $\prod_{x}$ is taken over by distinct elements of $z$ of the mset $Y$ and $[X]_{z} = m$ iff $z \in^{n} X$, $[Y]_{z} = n$ iff $z \in^{n} Y$, then $\left( \frac{[X]_{z}}{[Y]_{z}} \right) = \left( \frac{m}{n} \right) = \frac{m^{1}}{n^{1}(m-n)}$.

The power set of a mset is the support set of the power mset and is denoted by $P^{*}(X)$. The following theorem was showed the cardinality of the power set of a mset.

**Theorem 2.6.** [18] Let $P(X)$ be a power mset whose members drawn from the mset $X = \{m_{1}/x_{1}, m_{2}/x_{2}, ..., m_{n}/x_{n}\}$ and $P^{*}(X)$ be the power set of a mset $X$. Then, Card$(P^{*}(X)) = \Pi_{c=1}^{n}(1+m_{c})$.

**Definition 2.7.** [6] Let $X \in [U]^{w}$ and $\tau \subseteq P^{*}(X)$. Then, $\tau$ is called a multiset topology (for short, M-topology) of $X$ if $\tau$ satisfies the following properties:

1. the mset $X$ and the empty mset $\phi$ are in $\tau$,
2. the mset union of the elements of any subcollection of $\tau$ is in $\tau$,
3. the mset intersection of the elements of any finite subcollection of $\tau$ is in $\tau$.

Hence, $(X, \tau)$ is called an M-topological space. Each element in $\tau$ is called an open mset. Additionally, $OM(X)$ is the set of all open submsets of $X$.

**Definition 2.8.** [7] Let $(X, \tau)$ be a M-topological space and $Y$ be a submset of $X$. The collection $\tau_{Y} = \{G' = Y \cap G; G \in \tau\}$ is a M-topology on $Y$, called the subspace M-topology.

**Remark 2.9.** [7] The complement of any submset $Y$ in a mset topological space $(X, \tau)$ is defined by : $Y^{c} = X \ominus Y$.

**Definition 2.10.** [7] A submset $Y$ of a M-topological space $X$ is said to be closed if the mset $X \ominus Y$ is open.

**Definition 2.11.** [7] Let $A$ be a submset of an M-topological space $(X, \tau)$. Then,

1. the interior of $A$ is defined as the union of all open mssets contained in $A$ and denoted by $int(A)$, i.e., $int(A) = \cup\{G \subseteq X : G$ is an open mset and $G \subseteq A\}$

and $C_{int(A)}(x) = max\{C_{G}(x) : G \in \tau, G \subseteq A\}$,
2. the closure of $A$ is defined as the intersection of all closed msets containing $A$ and denoted by $cl(A)$, i.e., $cl(A) = \cap\{K \subseteq X : K$ is a closed mset and $A \subseteq K\}$ and $C_{cl(A)}(x) = \min\{C_K(x) : K \in \tau^c, A \subseteq K\}$.

**Proposition 2.12.** [6, 7] If $(X, \tau)$ is a M-topological space and $A, B$ are two submsets of $X$. Then, the following properties are satisfied:

- $int(A^c) = (cl(A))^c$.
- $cl(A^c) = (int(A))^c$.
- $cl(A \cup B) = cl(A) \cup cl(B)$.
- $int(A \cap B) = int(A) \cap int(B)$.

**Definition 2.13.** [12] The boundary of a submset $A$ of $X$ is the intersection of closure of $A$ and closure of the complement of $A$. It is denoted by $b(A)$ and defined as $b(A) = cl(A) \cap cl(A^c)$.

**Definition 2.14.** [5] A multiset bitopological space is a triple $(X, \tau_1, \tau_2)$ where $X$ is a non-empty mset and $\tau_1, \tau_2$ are arbitrary M-topologies on $X$.

**Definition 2.15.** [5] Let $(X, \tau_1, \tau_2)$ be a multiset bitopological space over $X$ and $Y$ be a non-empty submset of $X$. Then, $\tau^Y_1 = \{Y \cap F : F \in \tau_1\}$ and $\tau^Y_2 = \{Y \cap G : G \in \tau_2\}$ are said to be the relative M-topologies on $Y$. Also, $(Y, \tau^Y_1, \tau^Y_2)$ is called a relative multiset bitopological subspace of $(X, \tau_1, \tau_2)$.

## 3 Operators on multiset bitopological spaces

In this section, some operators on multiset bitopological spaces are introduced such as $MP^*$-closure, $MP^*$-interior and $MP^*$-boundary. Further, their properties are presented. Moreover, the concept of $MP^*$-continuous function is studied.

**Definition 3.1.** Let $(X, \tau_1, \tau_2)$ be a multiset bitopological space. Then, the following operators are defined as:

1. an $MP^*$-closure operator $cl_{\tau^*} : P^*(X) \rightarrow P^*(X)$ is defined by:
   
   $cl_{\tau^*}(A) = cl_{\tau_1}(A) \cap cl_{\tau_2}(A)$, where $A \in P^*(X)$ and $P^*(X)$ is the support set of the power mset of $X$,

2. an $MP^*$-interior operator $int_{\tau^*} : P^*(X) \rightarrow P^*(X)$ is established by:
   
   $int_{\tau^*}(A) = int_{\tau_1}(A) \cup int_{\tau_2}(A)$, where $A \in P^*(X)$ and $\tau^* = \{A \subseteq X : int_{\tau^*}(A) = A\}$,

3. an $MP^*$-boundary operator $b_{\tau^*} : P^*(X) \rightarrow P^*(X)$ is described by:
   
   $b_{\tau^*}(A) = b_{\tau_1}(A) \cap b_{\tau_2}(A) = cl_{\tau^*}(A) \cap cl_{\tau^*}(A^c)$, where $A \in P^*(X)$.

**Remark 3.2.** Clearly from the above definition, $\tau^*$ is a supra M-topological space because a finite intersection of members of $\tau^*$ may not be a member of $\tau^*$ as shown in the following example.
Example 3.3. Let $X = \{3/a, 2/b, 1/c\}$ be an mset and $\tau_1 = \{X, \phi, \{3/a\}, \{2/b, 1/c\}\}$, $\tau_2 = \{X, \phi, \{2/b\}, \{3/a, 1/c\}\}$. Therefore, $\tau^* = \{X, \phi, \{3/a\}, \{2/b\}, \{3/a, 1/c\}, \{2/b, 1/c\}\}$. Assume that $A = \{3/a, 1/c\}$, $B = \{2/b, 1/c\}$. Then, $A, B \in \tau^*$. But, $A \cap B \neq \{1/c\} \notin \tau^*$.

Theorem 3.4. Let $(X, \tau_1, \tau_2)$ be a multiset bitopological space. Then, $MP^*$-closure operator has the following properties:

1. $A \subseteq cl_{\tau^*}(A) \forall A \in P^*(X)$,
2. if $A \subseteq B$, then $cl_{\tau^*}(A) \subseteq cl_{\tau^*}(B) \forall A, B \in P^*(X)$,
3. $cl_{\tau^*}(cl_{\tau^*}(A)) = cl_{\tau^*}(A)$,
4. $cl_{\tau^*}(A \cup B) \supseteq cl_{\tau^*}(A) \cup cl_{\tau^*}(B)$,
5. $cl_{\tau^*}(A \cap B) \subseteq cl_{\tau^*}(A) \cap cl_{\tau^*}(B)$,
6. $cl_{\tau^*}(\phi) = \phi$ and $cl_{\tau^*}(X) = X$,
7. $(int_{\tau^*}(A))^c = cl_{\tau^*}((A^c))$, where $A^c$ is the complement of $A$ with respect to the mset $X$,
8. $A \in \tau^{c*}$ if and only if $cl_{\tau^*}(A) = A$, where $\tau^{c*}$ is the family of $\tau^*$-closed submsets of $X$.

Proof.

1. Since, $cl_{\tau^*}(A) = cl_{\tau_1}(A) \cap cl_{\tau_2}(A)$, $A \subseteq cl_{\tau_1}(A)$ and $A \subseteq cl_{\tau_2}(A)$. Then, $A \subseteq cl_{\tau_1}(A) \cap cl_{\tau_2}(A) = cl_{\tau^*}(A)$.

2. Let $A \subseteq B$. Then, $cl_{\tau_i}(A) \subseteq cl_{\tau_i}(B)$ where $i = 1, 2$. Therefore, $cl_{\tau_1}(A) \cap cl_{\tau_2}(A) \subseteq cl_{\tau_1}(B) \cap cl_{\tau_2}(B)$. Thus, $cl_{\tau^*}(A) \subseteq cl_{\tau^*}(B)$.

3. Since, $cl_{\tau^*}(cl_{\tau^*}(A)) = cl_{\tau_1}(cl_{\tau_1}(A) \cap cl_{\tau_2}(A)) \cap cl_{\tau_2}(cl_{\tau_1}(A) \cap cl_{\tau_2}(A))$.

Then, $cl_{\tau^*}(cl_{\tau^*}(A)) \subseteq (cl_{\tau_1}(cl_{\tau_1}(A)) \cap cl_{\tau_1}(cl_{\tau_2}(A))) \cap (cl_{\tau_2}(cl_{\tau_1}(A)) \cap cl_{\tau_2}(cl_{\tau_2}(A)))$.

Therefore, $cl_{\tau^*}(cl_{\tau^*}(A)) \subseteq (cl_{\tau_1}(A) \cap cl_{\tau_1}(cl_{\tau_2}(A))) \cap (cl_{\tau_2}(cl_{\tau_1}(A)) \cap cl_{\tau_2}(A))$.

Hence, $cl_{\tau^*}(cl_{\tau^*}(A)) \subseteq cl_{\tau_i}(A) \cap cl_{\tau_i}(A)$. This implies that $cl_{\tau^*}(cl_{\tau^*}(A)) \subseteq cl_{\tau^*}(A)$. Conversely, from part (1) we have $A \subseteq cl_{\tau^*}(A)$. Then, $cl_{\tau^*}(A) \subseteq cl_{\tau^*}(cl_{\tau^*}(A))$ by part (2). Thus, $cl_{\tau^*}(cl_{\tau^*}(A)) = cl_{\tau^*}(A)$.

4. Since, $A, B \subseteq A \cup B$. Then, $cl_{\tau^*}(A) \subseteq cl_{\tau^*}(A \cup B)$ and $cl_{\tau^*}(B) \subseteq cl_{\tau^*}(A \cup B)$ from part (2). Thus, $cl_{\tau^*}(A) \cup cl_{\tau^*}(B) \subseteq cl_{\tau^*}(A \cup B)$.

5. Since, $A \cap B \subseteq A, B$. Then, $cl_{\tau^*}(A \cap B) \subseteq cl_{\tau^*}(A)$ and $cl_{\tau^*}(A \cap B) \subseteq cl_{\tau^*}(B)$ from part (2). Thus, $cl_{\tau^*}(A \cap B) \subseteq cl_{\tau^*}(A) \cap cl_{\tau^*}(B)$.


7. Since, $(int_{\tau^*}(A))^c = (int_{\tau_1}(A) \cup int_{\tau_2}(A))^c$. Then, $(int_{\tau^*}(A))^c = (int_{\tau_1}(A))^c \cap (int_{\tau_2}(A))^c$. Therefore, $(int_{\tau^*}(A))^c = (cl_{\tau_1}(A^c)) \cap (cl_{\tau_2}(A^c))$. Hence, $(int_{\tau^*}(A))^c = cl_{\tau^*}(A^c)$.
8. Immediate by using part (7).

Remark 3.5. In multiset bitopological space \((X, \tau_1, \tau_2)\), \(cl_(\tau)(A \cup B) \neq cl_(\tau)(A) \cup cl_(\tau)(B)\) in general as shown in the following example.

Example 3.6. From Example 3.3, Let \(A = \{2/a\}, B = \{1/b\}\). Then, \(cl_(\tau)(A) = \{3/a\}, cl_(\tau)(B) = \{2/b\}\), but \(cl_(\tau)(A \cup B) = X\). Hence, \(cl_(\tau)(A \cup B) \neq cl_(\tau)(A) \cup cl_(\tau)(B)\).

Theorem 3.7. Let \((X, \tau_1, \tau_2)\) be a multiset bitopological space. Then, \(MP\)-interior operator has the following properties:

1. \(int_(\tau)(A) \subseteq A \forall A \in P^*(X)\),
2. If \(A \subseteq B\), then \(int_(\tau)(A) \subseteq int_(\tau)(B) \forall A, B \in P^*(X)\),
3. \(int_(\tau)(int_(\tau)(A)) = int_(\tau)(A)\),
4. \(int_(\tau)(\phi) = \phi\) and \(int_(\tau)(X) = X\),
5. \(int_(\tau)(A \cap B) \subseteq int_(\tau)(A) \cap int_(\tau)(B)\),
6. \(int_(\tau)(A \cup B) \supseteq int_(\tau)(A) \cup int_(\tau)(B)\).

Proof.

1. Since, \(int_(\tau)(A) = int_(\tau_1)(A) \cup int_(\tau_2)(A)\), \(int_(\tau_1)(A) \subseteq A\) and \(int_(\tau_2)(A) \subseteq A\). Then, \(int_(\tau)(A) \subseteq A\).

2. Let \(A \subseteq B\). Then, \(int_(\tau_i)(A) \subseteq int_(\tau_i)(B)\) where \(i = 1, 2\). Therefore, \(int_(\tau_1)(A) \cup int_(\tau_2)(A) \subseteq int_(\tau_1)(B) \cup int_(\tau_2)(B)\). Thus, \(int_(\tau)(A) \subseteq int_(\tau)(B)\).

3. Since, \(int_(\tau)(int_(\tau)(A)) = int_(\tau_1)(int_(\tau_1)(A) \cup int_(\tau_2)(A)) \cup int_(\tau_2)(int_(\tau_1)(A) \cup int_(\tau_2)(A))\).

Then, \(int_(\tau)(int_(\tau)(A)) \supseteq (int_(\tau_1)(A) \cup int_(\tau_1)(int_(\tau_2)(A))) \cup (int_(\tau_2)(int_(\tau_1)(A)) \cup int_(\tau_2)(A))\). Therefore, \(int_(\tau)(int_(\tau)(A)) \supseteq int_(\tau_1)(A) \cup int_(\tau_2)(A)\).

Hence, \(int_(\tau)(int_(\tau)(A)) \supseteq int_(\tau)(A)\).

Conversely, from part (1) we have \(int_(\tau)(A) \subseteq A\). Then, \(int_(\tau)(int_(\tau)(A)) \subseteq int_(\tau)(A)\) by part (2).

Thus, \(int_(\tau)(int_(\tau)(A)) = int_(\tau)(A)\).


5. Since, \(A \cap B \subseteq A, B\). Then, \(int_(\tau)(A \cap B) \subseteq int_(\tau)(A)\) and \(int_(\tau)(A \cap B) \subseteq int_(\tau)(B)\) from part (2).

Thus, \(int_(\tau)(A \cap B) \subseteq int_(\tau)(A) \cap int_(\tau)(B)\).

6. Similarly.

Remark 3.8. The following example shows that:

1. \(cl_(\tau)(A) = A\) does not imply that \(A \in \tau^c_1\) or \(A \in \tau^c_2\),
2. \(int_(\tau)(A) = A\) does not imply that \(A \in \tau_1\) or \(A \in \tau_2\).
Example 3.9. From Example 3.3,

1. Let \( A = \{1/c\} \). Then, \( \text{cl}_{\tau_1}(A) = A \). But, \( \{1/c\} \) is neither \( \tau_1 \)-closed mset nor \( \tau_2 \)-closed mset.

2. Let \( A = \{3/a, 2/b\} \). Then, \( \text{int}_{\tau_1}(A) = A \). But, \( A \) is neither \( \tau_1 \)-open mset nor \( \tau_2 \)-open mset.

Theorem 3.10. If \( (X, \tau) \) is a multiset topological space and \( A \subseteq X \), then

1. \( \text{int}(A) \subseteq A \cap (b(A))^c \),

2. \( \text{cl}(A) \supseteq A \cup b(A) \).

Proof.

1. Since, \( b(A) = \text{cl}(A) \cap \text{cl}(A^c) \). Then, \( (b(A))^c = (\text{cl}(A))^c \cup \text{int}(A) \). Therefore, \( A \cap (b(A))^c = (A \cap (\text{cl}(A))^c) \cup (A \cap \text{int}(A)) \). Thus, \( A \cap (b(A))^c = (A \cap (\text{cl}(A))^c) \cup \text{int}(A) \). Hence, \( \text{int}(A) \subseteq A \cap (b(A))^c \).

2. Similarly.

Remark 3.11. The equality of Theorem 3.10 is not true in general as shown in the following example.

Example 3.12. Let \( X = \{3/a, 2/b, 1/c\} \) be an mset and \( \tau = \{X, \phi, \{1/a, 2/b\}, \{1/a, 1/b\}\} \) be an M-topology on \( X \).

1. If \( A = \{2/a, 1/c\} \). Then, \( \text{int}(A) = \phi \). But, \( A \cap (b(A))^c = \{1/a\} \neq \text{int}(A) \).

2. If \( A = \{2/a, 2/b\} \). Then, \( \text{cl}(A) = X \). But, \( A \cup b(A) = \{2/a, 2/b, 1/c\} \neq \text{cl}(A) \).

Theorem 3.13. Let \( (X, \tau_1, \tau_2) \) be a multiset bitopological space. Then, \( MP^* \)-boundary operator has the following properties:

1. \( b_{\tau_1}(X) = b_{\tau_2}(\phi) = \phi \),

2. \( b_{\tau_1}(A) = \text{cl}_{\tau_1}(A) \cap (\text{int}_{\tau_1}(A))^c \forall A \in P^*(X) \),

3. \( \text{int}_{\tau_1}(A) \subseteq A \cap (b_{\tau_1}(A))^c \),

4. \( \text{cl}_{\tau_1}(A) \supseteq A \cup b_{\tau_1}(A) \),

5. \( b_{\tau_1}(A^c) = b_{\tau_1}(A) \),

6. \( b_{\tau_1}(\text{cl}_{\tau_1}(A)) \subseteq b_{\tau_1}(A) \),

7. \( b_{\tau_1}(\text{int}_{\tau_1}(A)) \subseteq b_{\tau_1}(A) \).

Proof.

1. Immediate.

2. Since, \( b_{\tau_1}(A) = b_{\tau_1}(A) \cap b_{\tau_2}(A) \). Then, \( b_{\tau_1}(A) = (\text{cl}_{\tau_1}(A) \cap \text{cl}_{\tau_2}(A^c)) \cap (\text{cl}_{\tau_2}(A) \cap \text{cl}_{\tau_2}(A^c)) \). Therefore, \( b_{\tau_1}(A) = (\text{cl}_{\tau_1}(A) \cap \text{cl}_{\tau_2}(A)) \cap (\text{cl}_{\tau_2}(A^c) \cap \text{cl}_{\tau_2}(A^c)) \). Thus, \( b_{\tau_1}(A) = \text{cl}_{\tau_1}(A) \cap \text{cl}_{\tau_2}(A^c) \). By using Theorem 3.4, \( b_{\tau_1}(A) = \text{cl}_{\tau_1}(A) \cap (\text{int}_{\tau_1}(A))^c \).
3. Since, \(\text{int}_{\tau_1}(A) \subseteq A \cap (b_{\tau_1}(A))^c\) and \(\text{int}_{\tau_2}(A) \subseteq A \cap (b_{\tau_2}(A))^c\). Then, \(\text{int}_{\tau}(A) \subseteq (A \cap (b_{\tau_1}(A))^c) \cup (A \cap (b_{\tau_2}(A))^c)\). Therefore, \(\text{int}_{\tau}(A) \subseteq A \cap ((b_{\tau_1}(A))^c \cup (b_{\tau_2}(A))^c)\). Thus, \(\text{int}_{\tau}(A) \subseteq A \cap (b_{\tau_1}(A) \cap b_{\tau_2}(A))^c\). Hence, \(\text{int}_{\tau}(A) \subseteq A \cap (b_{\tau}(A))^c\).

4. Since, \(cl_{\tau_i}(A) \supseteq A \cup b_{\tau_i}(A)\) where \(i = 1, 2\). Then, \(cl_{\tau}(A) \supseteq (A \cup b_{\tau_1}(A)) \cap (A \cup b_{\tau_2}(A))\). Thus, \(cl_{\tau}(A) \supseteq A \cup (b_{\tau_1}(A) \cap b_{\tau_2}(A))\). Therefore, \(cl_{\tau}(A) \supseteq A \cup b_{\tau}(A)\).

5. Clearly from the definition of MP*-boundary operator.

6. Since, \(b_{\tau'}(A) = cl_{\tau'}(A) \cap cl_{\tau'}(A^c)\). Then, \(b_{\tau'}(cl_{\tau'}(A)) = cl_{\tau'}(cl_{\tau'}(A) \cap cl_{\tau'}(A^c))\). Therefore, \(b_{\tau'}(cl_{\tau'}(A)) \subseteq cl_{\tau'}(A) \cap cl_{\tau'}(A^c)\). Hence, \(b_{\tau'}(cl_{\tau'}(A)) \subseteq b_{\tau'}(A)\).

7. Since, \(b_{\tau'}(A) = cl_{\tau'}(A) \cap cl_{\tau'}(A^c)\). Then, \(b_{\tau'}(\text{int}_{\tau'}(A)) = cl_{\tau'}(\text{int}_{\tau'}(A) \cap cl_{\tau'}((\text{int}_{\tau'}(A))^c))\). Therefore, \(b_{\tau'}(\text{int}_{\tau'}(A)) \subseteq cl_{\tau'}(A) \cap cl_{\tau'}((\text{int}_{\tau'}(A))^c)\). So, \(b_{\tau'}(\text{int}_{\tau'}(A)) \subseteq cl_{\tau'}(A) \cap cl_{\tau'}(A^c)\). Hence, \(b_{\tau'}(\text{int}_{\tau'}(A)) \subseteq (b_{\tau'}(A))^c\).

**Theorem 3.14.** Let \((X, \tau_1, \tau_2)\) be a multiset bitopological space. If \(b_{\tau'}(A) = \phi\), then \(A \in \tau^* \cap (\tau^*)^c\).

**Proof.** Since \(b_{\tau'}(A) = \phi\). Then, \(cl_{\tau'}(A) \cap cl_{\tau'}(A^c) = \phi\). Therefore, \(cl_{\tau'}(A) \subseteq (cl_{\tau'}(A^c))^c = \text{int}_{\tau'}(A)\). Thus, \(A \subseteq cl_{\tau'}(A) \subseteq \text{int}_{\tau'}(A) \subseteq A\). Hence, \(A \in \tau^* \cap (\tau^*)^c\).

**Remark 3.15.** The converse of Theorem 3.14 is not true in general as shown in the following example.

**Example 3.16.** Let \(X = \{3/a, 2/b, 1/c\}\) be an mset and \(\tau_1 = \{X, \phi, \{2/a\}, \{1/b\}, \{2/a, 1/b\}\}\), \(\tau_2 = \{X, \phi, \{1/a\}, \{1/a, 1/c\}, \{1/b, 1/c\}, \{1/c\}, \{1/a, 1/b, 1/c\}\}\). Therefore, \(\tau^* = \{X, \phi, \{1/a\}, \{1/b\}, \{1/c\}, \{2/a\}, \{1/a, 1/b\}, \{1/a, 1/c\}, \{1/b, 1/c\}, \{2/a, 1/c\}, \{1/a, 1/b, 1/c\}, \{2/a, 1/b, 1/c\}\}\). Assume that \(A = \{1/a, 1/b\} \in \tau^* \cap \tau^{**}\). But, \(b_{\tau'}(A) = cl_{\tau'}(A) \cap cl_{\tau'}(A^c) = \{1/a, 1/b\} \cap \{2/a, 1/b, 1/c\} = \{1/a, 1/b\} \neq \phi\). Hence, \(A \in \tau^* \cap \tau^{**} \Rightarrow b_{\tau'}(A) = \phi\).

**Definition 3.17.** Let \((X, \tau_1, \tau_2)\) and \((Y, \eta_1, \eta_2)\) be two multiset bitopological spaces. Then, \(f : X \rightarrow Y\) is called an MP*-continuous function if and only if \(f^{-1}(V) \in \tau^* \forall V \in \eta^*\), where \(\eta^* = \{B \subseteq Y : \text{int}_{\eta'}(B) = B\}\).

**Theorem 3.18.** Let \((X, \tau_1, \tau_2)\) and \((Y, \eta_1, \eta_2)\) be two multiset bitopological spaces and \(f : X \rightarrow Y\) is an mset function. Then, the following conditions are equivalent:

1. \(f\) is an MP*-continuous function,
2. \(f^{-1}(H) \in \tau^* \forall H \in \eta^*\),
3. \(f(cl_{\tau'}(A)) \subseteq cl_{\eta'}(f(A)) \forall A \subseteq X\),
4. \(cl_{\tau'}(f^{-1}(B)) \subseteq f^{-1}(cl_{\eta'}(B)) \forall B \subseteq Y\),
5. \(f^{-1}(\text{int}_{\eta'}(B)) \subseteq \text{int}_{\tau'}(f^{-1}(B)) \forall B \subseteq Y\).

**Proof.**
(1 $\Rightarrow$ 2) Let $f$ is an $MP^*$-continuous function and $H \in \eta^*$. Then, $H^c \in \eta^*$. Thus, $f^{-1}(H^c) \in \tau^*$. Therefore, $(f^{-1}(H^c))^c \in \tau^{**}$. Hence, $f^{-1}(H) \in \tau^{**}$.

(2 $\Rightarrow$ 3) Let $A \subseteq X$. Since, $f(A) \subseteq cl_\eta^* (f(A))$. Then, $A \subseteq f^{-1}(cl_\eta^* (f(A)))$. But, $cl_\eta^* (f(A))$ is a closed set over $Y$. So, $f^{-1}(cl_\eta^* (f(A)))$ is also a closed set over $X$. Hence, $cl_\tau^* (A) \subseteq f^{-1}(cl_\eta^* (f(A)))$. Therefore, $f(cl_\tau^* (A)) \subseteq cl_\eta^* (f(A))$.

(3 $\Rightarrow$ 4) Let $B \subseteq Y$. Then, $f^{-1}(B) \subseteq X$. From part (3), $f(cl_\tau^* (f^{-1}(B))) \subseteq cl_\eta^*(ff^{-1}(B))$. Therefore, $f(cl_\tau^* (f^{-1}(B))) \subseteq cl_\eta^* (f(A))$. Hence, $cl_\tau^* (f^{-1}(B)) \subseteq f^{-1}(cl_\eta^* (f(A)))$.

(4 $\Rightarrow$ 5) Immediate by taking the complement to part (4).

(5 $\Rightarrow$ 1) Let $V \in \eta^*$. Then, $int_\eta^* (V) = V$. Thus, $f^{-1}(f^{-1}(V)) \subseteq int_\tau^* (f^{-1}(V))$ by using part (5). But, $int_\tau^* (f^{-1}(V)) \subseteq f^{-1}(V)$. Hence, $f^{-1}(V) \in \tau^*$. This implies that $f$ is an $MP^*$-continuous function.

References