Common fixed point theorems in intuitionistic fuzzy metric space using the property CLRg

Rajesh Shrivastava, Megha Shrivastava

Abstract  In this note, we prove common fixed point theorem in intuitionistic fuzzy metric spaces using the common limit in the range property of mappings called (CLR) property. We give an example which validates the main result. Our results improve and generalize the main result of Kumar et al. [S.Kumar, R.K.Vats, V.Singh and S.K.Garg, “Some common fixed point theorems in intuitionistic fuzzy metric spaces”, Int. journal of mathematical analysis, vol.4, 2010, no.26, pp. 1255-1270].

Key Words  IFMS, weakly compatible mappings, (E.A) property, (CLR) property

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1 Introduction

The concept of fuzzy sets was introduced initially by Zadeh [47] in the year 1965. As a generalization of fuzzy sets, Atanassove [6] introduced the concept of intuitionistic fuzzy sets. Further Coker [8] introduced the idea of the topology of intuitionistic fuzzy sets. Intuitionistic fuzzy sets deals with both degree of nearness and degree of non-nearness. In 2004, Park [32] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization fuzzy metric space due to George and Veeramani [14]. Alea et al. [4] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michale [24]. In 2006, Tukoglu et al. [45] extended the notion of compatible mappings to IFM-spaces. In 2002, Aamri and El-Moutawakil defined the notion of property (E.A) in metric spaces for self mappings which contained the class of non-compatible mappings in metric spaces. The property (E.A) requires completeness (or closedness) of the underlying subspaces for the existence of common fixed point.

Most recently, Sintunavarat and Kumam [41] defined the notion of ”Common limit in the range” property or CLR property in fuzzy metric spaces. In [41], they showed that the notion of (CLR) property...
never requires the condition of the closedness of the subspaces or completeness of the space. Many authors have proved a number of fixed point theorems for different contractions in IFM-spaces. The aim of this paper is to prove a common fixed point theorem in intuitionistic fuzzy metric spaces using the notion of (CLR) property.

2 Preliminary Notes

In this section, we first give some definitions.

Definition 2.1. ([37]) A binary operation \(* : [0, 1] \times [0, 1] \to [0, 1]\) is continuous t-norm if * is satisfying the following conditions:

(i) * is commutative and associative;
(ii) * is continuous;
(iii) \(a * 1 = a\) for all \(a \in [0, 1]\);
(iv) \(a * b \leq c * d\) whenever \(a \leq c\) and \(b \leq d\) for all \(a, b, c, d \in [0, 1]\).

Definition 2.2. ([37]) A binary operation \(\odot : [0, 1] \times [0, 1] \to [0, 1]\) is continuous t-conorm if \(\odot\) is satisfying the following conditions:

(i) \(\odot\) is commutative and associative;
(ii) \(\odot\) is continuous;
(iii) \(a \odot 0 = a\) for all \(a \in [0, 1]\);
(iv) \(a \odot b \leq c \odot d\) whenever \(a \leq c\) and \(b \leq d\) for all \(a, b, c, d \in [0, 1]\).

Definition 2.3. ([4]) A 5-tuple \((X, M, N, *,\odot)\) is said to an intuitionistic fuzzy metric space if \(X\) is an arbitrary set, * is a continuous t-norm, \(\odot\) is a continuous t-conorm and \(M, N\) are fuzzy sets on \(X^2 \times (0, \infty)\) satisfying the following conditions:

(i) \(M(x, y, t) + N(x, y, t) \leq 1\) for all \(x, y \in X\) and \(t > 0\);
(ii) \(M(x, y, t) = 0\) for all \(x, y \in X\);
(iii) \(M(x, y, t) = 1\) for all \(x, y \in X\) and \(t > 0\);
(iv) \(M(x, y, t) = M(y, x, t)\) for all \(x, y \in X\) and \(t > 0\);
(v) \(M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)\) for all \(x, y, z \in X\) and \(s, t > 0\);
(vi) For all \(x, y, z \in X\), \(M(x, y, \cdot) : [0, \infty) \to [0, 1]\) is continuous;
(vii) \(\lim_{n \to \infty} M(x, y, t) = 1\) for all \(x, y \in X\) and \(t > 0\);
(viii) \(N(x, y, 0) = 1\) for all \(x, y \in X\);
(ix) \(N(x, y, t) = 0\) for all \(x, y \in X\) and \(t > 0\) iff \(x = y\);
(x) \(N(x, y, t) = N(y, x, t)\) for all \(x, y \in X\) and \(t > 0\);
(xi) \(N(x, y, t) \odot N(y, z, s) \geq N(x, z, t + s)\) for all \(x, y, z \in X\) and \(s, t > 0\);
(xii) For all \(x, y \in X\), \(N(x, y, \cdot) : [0, \infty) \to [0, 1]\) is continuous;
(xiii) \(\lim_{n \to \infty} N(x, y, t) = 0\) for all \(x, y \in X\).

Example 2.1. ([26]) Let \(X = \{1/n : n \in N\}, u \in \{0\}\) with * continuous t-norm and \(\odot\) continuous t-conorm defined by \(a * b = ab\) and \(a \odot b = \min\{1, a + b\}\) respectively for all \(a, b \in [0, 1]\). For each \(t \in (0, \infty)\) and \(x, y \in X\), define \((M, N)\) by
\[ M(x, y, t) = \begin{cases} \frac{x}{|x| + y}, & t > 0 \\ 0, & t = 0 \\ 1, & t < 0 \end{cases} \quad \text{and} \quad N(x, y, t) = \begin{cases} \frac{|x-y|}{t + |x-y|}, & t > 0 \\ 0, & t = 0 \\ 1, & t < 0 \end{cases} \]

Then \((X, M, N, *, \circ)\) is an intuitionistic fuzzy metric space.

**Remark 2.1.** ([26]) An intuitionistic fuzzy metric spaces with continuous \(t\)-norm * and continuous \(t\)-conorm \(\circ\) defined by \(a * a \geq a\) and \((1-a) \circ (1-a) \leq (1-a)\) for all \(a \in [0, 1]\). Then for all \(x, y \in X, M(x, y, *)\) is non-decreasing and \(N(x, y, \circ)\) is non-increasing.

**Definition 2.4.** Let \((X, M, N, *, \circ)\) be an intuitionistic fuzzy metric space. Then

(a) A sequence \(\{X_n\}\) is said to be Cauchy sequence if, for all \(t > 0\) and \(p > 0\), \(\lim_{n \to \infty} M(X_{n+p}, X_n, t) = 1\), \(\lim_{n \to \infty} N(X_{n+p}, X_n, t) = 0\).

(b) A sequence \(\{X_n\}\) in \(X\) is said to be convergent to a point \(x \in X\) if, for all \(t > 0\), \(\lim_{n \to \infty} M(X_n, x, t) = 1\), \(\lim_{n \to \infty} N(X_n, x, t) = 0\). Since * and \(\circ\) are continuous, the limit it is uniquely determined from (v) and (xi) of definitions 3 respectively.

**Definition 2.5.** An intuitionistic fuzzy metric space \((X, M, N, *, \circ)\) is said to be complete if and only if every Cauchy sequence in \(X\) is convergent.

**Definition 2.6.** A pair of self-mappings \((f, g)\) of an intuitionistic fuzzy metric space \((X, M, N, *, \circ)\) is said to be compatible if \(\lim_{n \to \infty} M(fgX_n, gfX_n, t) = 1\) and \(\lim_{n \to \infty} N(fgX_n, gfX_n, t) = 0\) for every \(t > 0\), whenever \(\{X_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} fX_n = \lim_{n \to \infty} gX_n = z\) for some \(z \in X\).

**Definition 2.7.** A pair of self-mappings \((f, g)\) of an intuitionistic fuzzy metric space \((X, M, N, *, \circ)\) is said to be non-compatible if \(\lim_{n \to \infty} M(fgX_n, gfX_n, t) \neq 1\) or non-existent, \(\lim_{n \to \infty} N(fgX_n, gfX_n, t) \neq 0\) or non-existent for every \(t > 0\), whenever \(\{X_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} fX_n = \lim_{n \to \infty} gX_n = z\) for some \(z \in X\).

**Definition 2.9.** ([1]) A pair \((f, g)\) of self mappings defined on an intuitionistic fuzzy metric space \((X, M, N, *, \circ)\) is said to satisfy the (E.A) property if there exists a sequence \(\{X_n\}\) in \(X\) such that \(\lim_{n \to \infty} fX_n = \lim_{n \to \infty} gX_n = u\), for some \(u \in X\).

**Definition 2.10.** ([41]) A pair of self mappings \((f, g)\) of self mappings defined on an intuitionistic fuzzy metric space \((X, M, N, *, \circ)\) is said to satisfy the (CLRg) property if there exists a sequence \(\{X_n\}\) in \(X\) such that \(\lim_{n \to \infty} fX_n = \lim_{n \to \infty} gX_n = gu\), for some \(u \in X\).

**Example 2.2.** Let \(X = [0, \infty)\) consider \((X, M, N, *, \circ)\) be an intuitionistic fuzzy metric space, where \(M\) and \(N\) are two fuzzy sets defined by \(M(x, y, t) = t/|t + d(x, y)|\) and \(N(x, y, t) = d(x, y)/|t + d(x, y)|\) where \(d\) is usual metric. Define \(f, g : X \to [0, \infty)\) by \(f(x) = x + 5\) and \(g(x) = 6x\) for all \(x \in X\). Consider \(\{X_n\} = \{1 + 1/n\}\) in \(X\), we have \(\lim f(1 + 1/n) = \lim g(6 + 1/n) = 6\) \(= g(1) = \lim (6 + 6/n) = \lim g(1 + 1/n)\), which shows that the pair \((f, g)\) satisfy property \((f, g)\).

**Lemma 2.1.** ([3]) Let \((X, M, N, *, \circ)\) be an intuitionistic fuzzy metric space and for all \(x, y \in X, t > 0\) and if for a number \(k \in (0, 1), M(x, y, kt) \geq M(x, y, t)\) and \(N(x, y, kt) \leq N(x, y, t)\) then \(x = y\).

**Theorem A.** Let \(A, B, S\) and \(T\) be self maps of a complete intuitionistic fuzzy metric spaces \((X, M, N, *)\) with continuous \(t\)-norm * and continuous \(t\)-conorm \(\circ\) defined by \(a * a \geq a\) and \((1-a) \circ (1-a) \leq (1-a)\) for all \(a \in [0, 1]\) satisfying the following conditions:
For all \(a\) t

Theorem 3.1.

3 Main Result

We will give the main theorem in this section.

Theorem 3.1. Let \(A, B, S\) and \(T\) be self maps of a intuitionistic fuzzy metric spaces \((X, M, N, *, \diamond)\) with continuous \(t\)-norm * and continuous \(t\)-conorm \(\diamond\) defined by \(a * a \geq a\) and \((1 - a) \diamond (1 - a) \leq (1 - a)\) for all \(a \in [0, 1]\) satisfying the following conditions:

(A1) \(A(x) \subset T(x)\) and \(B(x) \subset S(x)\).

(A2) pairs \((A, S)\) and \((B, T)\) are weakly compatible.

(A3) \(M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)\})\) and \(N(Ax, By, t) \leq \psi\max\{N(Sx, Ty, t), N(Sx, Ay, t), N(Bx, Ty, t)\}\) for all \(x, y \in X\), where \(\phi, \psi : [0, 1] \rightarrow [0, 1]\) is a continuous function such that \(\phi(s) > s\) and \(\psi(s) < s\) for each \(0 < s < 1\) with \(M(x, y, t) > 0(x, y \in X, t > 0)\). Then \(A, B, S\) and \(T\) have a unique common fixed point.

Theorem B. Let \(A, B, S\) and \(T\) be self maps of a intuitionistic fuzzy metric spaces \((X, M, N, *, \diamond)\) with continuous \(t\)-norm * and continuous \(t\)-conorm \(\diamond\) defined by \(a * a \geq a\) and \((1 - a) \diamond (1 - a) \leq (1 - a)\) for all \(a \in [0, 1]\) satisfying the following conditions:

(B1) \(A(x) \subset T(x)\) and \(B(x) \subset S(x)\).

(B2) \(M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)\})\) and \(N(Ax, By, t) \leq \psi\max\{N(Sx, Ty, t), N(Sx, Ay, t), N(Bx, Ty, t)\}\) for all \(x, y \in X\), where \(\phi, \psi : [0, 1] \rightarrow [0, 1]\) is a continuous function such that \(\phi(s) > s\) and \(\psi(s) < s\) for each \(0 < s < 1\) with \(M(x, y, t) > 0(x, y \in X, t > 0)\).

(B3) One of \(A(x), B(x), S(x)\) and \(T(x)\) is a complete subspace of \(X\), then

(i) \(A\) and \(S\) have a point of coincidence.

(ii) \(B\) and \(T\) have point of coincidence.

Moreover, if the pairs \((A, S)\) and \((B, T)\) are weakly compatible, then \(A, B, S\) and \(T\) have a unique common fixed point.

Theorem C. Let \(A, B, S\) and \(T\) be self maps of a intuitionistic fuzzy metric spaces \((X, M, N, *, \diamond)\) satisfying the following conditions:

(C1) \(A(x) \subset T(x)\) and \(B(x) \subset S(x)\).

(C2) pairs \((A, S)\) and \((B, T)\) are weakly compatible.

(C3) pairs \((A, S)\) or \((B, T)\) satisfy E.A property.

(C4) \(M(Ax, By, t) \geq \phi(\min\{M(Sx, Ty, t), M(Sx, By, t), M(Ty, By, t)\})\) and \(N(Ax, By, t) \leq \psi\max\{N(Sx, Ty, t), N(Sx, By, t), N(Ty, By, t)\}\) for all \(x, y \in X\).

If any one of \(A(x), B(x), S(x)\) and \(T(x)\) is a complete subspace of \(X\), then \(A, B, S\) and \(T\) have a unique fixed point.

3 Main Result

We will give the main theorem in this section.

Theorem 3.1. Let \(A, B, S\) and \(T\) be self maps of a intuitionistic fuzzy metric spaces \((X, M, N, *, \diamond)\) with continuous \(t\)-norm * and continuous \(t\)-conorm \(\diamond\) defined by \(a * a \geq a\) and \((1 - a) \diamond (1 - a) \leq (1 - a)\) for all \(a \in [0, 1]\) satisfying the following conditions:

(3.11) \(B(x) \subset S(x)\) and the pair \((B, T)\) satisfies the property \(CLR_T\), or \(A(x) \subset T(x)\) and the pair \((A, S)\) satisfies the property \(CLR_S\).

(3.12) pairs \((A, S)\) and \((B, T)\) are weakly compatible.
(3.13) \( M(Ax, By, t) \geq \phi(\min\{M(Sx, Tx, t), M(Sy, Bx, t), M(By, Ty, t)\}) \) and \( N(Ax, By, t) \leq \psi \max\{N(Sx, Ty, t), N(Sx, By, t), N(By, Ty, t)\} \) for all \( x, y \in X \), where \( \phi, \psi : [0, 1] \rightarrow [0, 1] \) is a continuous function such that \( \phi(s) > s \) and \( \psi(s) < s \) for each \( 0 < s < 1 \) with \( M(x, y, t) > 0(x, y \in X, t > 0) \).

Then \( A, B, S \) and \( T \) have a unique common fixed point.

**Proof.** Without loss of generality, assume that \( B(x) \subset S(x) \) and the pair \((B, T)\) satisfies the property \( CLR_T \) then there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = Tx \) for some \( x \in X \). Since \( B(x) \subset S(x) \) there exists a sequence \( \{y_n\} \) in \( X \) such that \( \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Sy_n = z = Tx \) for some \( z \in X \). Now we shall show that \( \lim_{n \to \infty} Ay_n = Tx = z \).

Taking \( y = x_n \) and \( x = y_n \). From (3.13) , we have;

\[
M(Ay_n, Bx_n, t) \geq \phi(\min\{M(Sy_n, Tx_n, t), M(Sy_n, Bx_n, t), M(Tx_n, Bx_n, t)\})
\]

and

\[
N(Ay_n, Bx_n, t) \leq \psi \max\{N(Sy_n, Tx_n, t), N(Sy_n, Bx_n, t), N(Tx_n, Bx_n, t)\}
\]

proceeding lim as \( n \to \infty \), we have \( \lim_{n \to \infty} M(Ay_n, Bx_n, t) \geq \phi(\min\{1, 1, 1\}) \geq 1 \) and

\[
\lim_{n \to \infty} N(Ay_n, Bx_n, t) \leq \psi \max\{0, 0, 0\} \leq 0.
\]

Therefore \( \lim_{n \to \infty} Ay_n = \lim_{n \to \infty} Bx_n = z \). Subsequently, we have

\[
\lim_{n \to \infty} Ay_n = \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} Sy_n = Tx = Z.
\]

Now we shall show that \( Bx = Tx \). Taking \( x = y_n \) and \( y = x \) in (3.13)

\[
M(Ay_n, Bx, t) \geq \phi(\min\{M(Sy_n, Tx, t), M(Sy_n, Bx, t), M(Tx, Bx_n, t)\})
\]

and

\[
N(Ay_n, Bx, t) \leq \psi \max\{N(Sy_n, Tx, t), N(Sy_n, Bx, t), N(Tx, Bx, t)\}
\]

taking lim as \( n \to \infty \), we have \( \lim_{n \to \infty} M(Ay_n, Bx, t) \geq \phi(\min\{1, 1, 1\}) \geq 1 \) and

\[
\lim_{n \to \infty} N(Ay_n, Bx, t) \leq \psi \max\{0, 0, 0\} \leq 0.
\]

Therefore \( \lim_{n \to \infty} M(Ay_n, Bx, t) = 1 \) and \( \lim_{n \to \infty} N(Ay_n, Bx, t) = 0 \) which implies that \( Bx = Tx = z \) since the pair \((B, T)\) is weak compatible , it follows that

\[
Bz = Tz
\]

(1)

Also since \( B(x) \subset S(x) \), there exists some \( y \) in \( X \) such that \( Bx = Sy = (z) \). We next show that \( Sy = Ay = (z) \). Taking \( x = y \) and \( y = x \) in (3.13),

\[
M(Ay, Bx, t) \geq \phi(\min\{M(Sy, Tx, t), M(Sy, Bx, t), M(Tx, Bx, t)\})
\]

\[
\geq \phi(\min\{M(Bx, Bx, t), M(Bx, Bx, t), M(Bx, Bx, t)\})
\]

and

\[
N(Ay, Bx, t) \leq \psi \max\{N(Sy, Tx, t), N(Sy, Bx, t), N(Tx, Bx, t)\}
\]

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implies \( Ay = Bx \). Hence \( Ay = Sy = (z) \). Subsequently, we have \( Bx = Tx = Sy = Ay = z \). But the pair \((A, S)\) is weakly compatible. It follows that

\[
Az = Sz. \tag{2}
\]

Next we claim that \( Az = Bz \). Taking \( x = z, y = z \) in (3.13) and applying (1) and (2)

\[
M(Az, Bz, t) \geq \phi(\min\{M(Sz, Tz, t), M(Sz, Bz, t), M(Tz, Bz, t)\})
\]

\[
\geq \phi(\min\{M(Az, Bz, t), M(Az, Bz, t), M(Bz, Bz, t)\})
\]

\[
N(Az, Bz, t) \leq \psi(\max\{N(Sz, Tz, t), N(Sz, Bz, t), N(Tz, Bz, t)\})
\]

\[
\leq \psi(\max\{N(Az, Bz, t), N(Az, Bz, t), N(Bz, Bz, t)\})
\]

\[
M(Az, Bz, t) \geq \phi(M(Az, Bz, t)) > M(Az, Bz, t)
\]

and

\[
N(Az, Bz, t) \leq \psi\{N(Az, Bz, t)\} < N(Az, Bz, t)
\]

which implies that

\[
Az = Bz. \tag{3}
\]

Hence by (1), (2) and (3) we have

\[
Az = Bz = Sz = Tz. \tag{4}
\]

Now we show that \( z = Az \). Taking \( x = z, y = z \) in (3.13) and applying (4) and previously known results like \( Bx = Tx = z \), we have

\[
M(Az, Bz, t) \geq \phi(\min\{M(Sz, Tz, t), M(Sz, Bz, t), M(Tz, Bz, t)\})
\]

\[
\geq \phi(\min\{M(Az, z, t), M(Az, z, t), M(z, z)\})
\]

\[
\geq \phi(M(Az, z, t))
\]

\[
N(Az, Bz, t) \leq \psi(\max\{N(Sz, Tz, t), N(Sz, Bz, t), N(Tz, Bz, t)\})
\]

\[
\leq \psi(\max\{N(Az, z, t), N(Az, z, t), N(z, z)\})
\]

\[
\leq \psi(N(Az, z, t))
\]

implies that \( M(Az, z, t) \geq \phi(M(Az, z, t)) > M(Az, z, t) \) and \( N(Az, z, t) \leq \psi\{N(Az, z, t)\} < N(Az, z, t) \)

which implies that

\[
Az = z. \tag{5}
\]

Hence by (4) and (5) we have \( Az = Bz = Sz = Tz = z \), that is \( z \) is the common fixed point of the maps \( A, B, S \) and \( T \). Uniqueness follows easily. The proof is similar when we assume \( A(x) \subset T(x) \) and the pair \((A, S)\) satisfies the property (CLR). \(\square\)
Example 3.1. Let $X = [1, 21]$ with metric $d$ defined by $d(x, y) = |x - y|$ and define:

$$M(x, y, t) = \begin{cases} \frac{t}{1 + |x-y|}, & t > 0 \\ 0, & t = 0 \end{cases} \quad \text{and} \quad N(x, y, t) = \begin{cases} \frac{|x-y|}{1 + |x-y|}, & t > 0 \\ 1, & t = 0 \end{cases}$$

for all $x, y \in X$. Then $(X, M, N, *, \odot)$ is an intuitionistic fuzzy metric space. Where $*$ and $\odot$ are continuous $t$-norm and continuous $t$-conorm defined by $a * b = \min\{a, b\}$ and $a \odot b = \max\{a, b\}$ for all $a, b \in [0, 1]$.

Now we define the self mappings $B$ and $T$ on $X$ by:

$$B(x) = \begin{cases} 1 & \text{if } x \in \{1\} \cup \{3, 21\} \\ 7 & \text{if } x \in \{1, 3\} \end{cases} \quad \text{and} \quad T(x) = \begin{cases} 1 & \text{if } x = 1 \\ 6 & \text{if } x \in \{1, 3\} \\ \frac{x+1}{6} & \text{if } x \in \{3, 21\} \end{cases}$$

Consider a sequence $\{x_n\} = \{3 + \frac{1}{n}\}$ where $n \in \mathbb{N}$, or $\{x_n\} = 1$. Then we have: $\lim_{n \to \infty} B(x_n) = \lim_{n \to \infty} T(x_n) = 1 = T(1) \in X$. Hence the pair $(B, T)$ satisfies the $(CLR_T)$ property. It is noticed that $B(X) = \{1, 7\} \not\subseteq \{1, 4\} \cup \{6\} = T(X)$. Here $T(X)$ is not a closed subspace of $X$. Thus all the conditions of theorem 3.1 are satisfied for some $k \in (0, 1)$ and 1 is a unique common fixed point of the mappings $B$ and $T$.

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