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Degree resistance distance of lollipop graphs

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Abstract The resistance distance $r_G(u,v)$ between two vertices u,v of a connected graph G is defined as the effective resistance between them in the corresponding electrical network constructed from G by replacing each edge of G with a unit resistor. Let G be a connected graph, the degree resistance distance of G is defined as $D_R(G) = \sum\limits_{\{u,v\}\subseteq V(G)} [d(u)+d(v)]r(u,v)$, where $d(u)$ (and $d(v)$) is the degree of the vertex u (and v). The lollipop graph $L_{n,k}$ is a class of special unicyclic graphs which comes from joining an endpoint of P_{n-k} to one vertex of the cycle C_k. In this paper, the authors firstly give the formula for computing the degree resistance distance of $L_{n,k}$, and then determine graphs in $L_{n,k}$ with the maximum and second maximum degree resistance distance.

Key Words Resistance distance; Kirchhoff index; Degree resistance distance; Lollipop graph
MSC 2010 05C35, 05C90

1 Introduction

All graphs considered here are both connected and simple if not stated in particular. The distance between vertices u and v of graph G, denoted by $d(u,v)$, is the length of a shortest path between them; $d(u)$ is the degree of the vertex u; n,m are the number of vertices and edges of G, respectively. The Wiener index was introduced by American chemistry Harold Wiener in 1947, defined as [1]

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u,v) \tag{1.1}$$

A modified version of the Wiener index is the degree distance, was introduce by A. A. Dobrynin and A. A. Kochetova[2], defined as

$$D(G) = \sum_{\{u,v\}\subseteq V(G)} [d(u)+d(v)]d(u,v) \tag{1.2}$$

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Resistance distance was introduced by Klein and Randić [3] in 1993, on the basis of electrical network theory. They viewed a graph G as an electrical network N such that each edge of G is assumed to be a unit resistor. The resistance distance between the vertices u and v of a graph G, are denoted by $r(u, v)$, is defined to be the effective resistance between nodes u, $v \in N$. The Kirchhoff index $Kf(G)$ of a graph G is defined as [3, 4]

$$Kf(G) = \sum_{\{u,v\}\subseteq V(G)} r(u, v) \tag{1.3}$$

If G is a tree, then $r(u, v) = d(u, v)$ for any two vertices u and v, the Kirchhoff and Wiener indices of trees coincide. The degree resistance distance was introduced by I. Gutman, L. Feng and G. Yu in [5]:

$$D_R(G) = \sum_{\{u,v\}\subseteq V(G)} [d(u) + d(v)]r(u, v) \tag{1.4}$$

They investigated the degree resistance distance of unicyclic graphs, determined the unicyclic graphs with the minimum and the second minimum degree resistance distance. Chen et al., [6] determined the unicyclic graphs with the maximum and the second maximum degree resistance distance. J. L. Palacios in [7] gave tight upper and lower bounds for the degree resistance distance of a connected undirected graph.

If G is a tree, then $r(u, v) = d(u, v)$ for any two vertices u and v. Consequently, the degree distances and additive degree Kirchhoff index coincide as well, i,e., $D_R(G) = 4W(G) - n(n - 1)$.

A graph G is called a unicyclic graph if it contains exactly one cycle. $\mathscr{U}(n)$ be the set of all unicyclic graphs with n vertices, S_n and P_n be the star and the path on n vertices, respectively. The lollipop graph $L_{n,k}$ is obtained by appending a k-cycle C_k to a pendant vertex of a path on $n - k$ vertices (see Fig. 1)

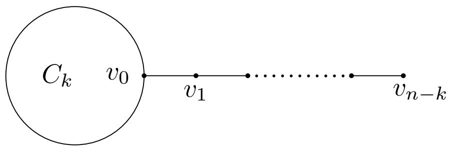

Figure 1. The lollipop graph $L_{n,k}$

The paper is organized as follows. In Section 2 we state some preparatory results, whereas in Section 3 we investigated the degree resistance distance $L_{n,k}$, and give the maximum and second maximum degree resistance distance of $L_{n,k}$.

2 Preliminary Results

For a graph G with $v \in V(G)$, $G - v$ denotes the graph obtained from G by deleting v (and its incident edges). For an edge uv of the graph G (the complement of G, respectively), $G - uv(G + uv$, respectively) denotes the graph resulting from G by deleting (adding, respectively) the edge uv.

Let $u \in V(G)$, and

$$r_u(G) = \sum_{v\in V(G)} r(v, u), \quad S'_u(G) = \sum_{v\in V(G)} d(v)r(v, u) \tag{2.1}$$

Let C_k be the cycle on $k \geqslant 3$ vertices, for any two vertices $v_i, v_j \in V(C_k)$ with $i < j$, by Ohm's law, we have $r(v_i, v_j) = \dfrac{(j-i)(k+i-j)}{k}$. For any vertex $u \in V(C_k)$, it's suffice to see that $r_u(C_k) = \dfrac{1}{6}(k^2-1)$, $S'_u(C_k) = \dfrac{1}{3}(k^2-1)$, $Kf(C_k) = \dfrac{1}{12}(k^3-k)$.

Lemma 2.1([8]). *Let T be any n vertices trees different from path P_n and S_n. Then $(n-1)^2 \leqslant W(T) \leqslant \dfrac{1}{6}(n^3-n)$, the left equality holds if and only if $G \cong S_n$ and the right does if and only if $G \cong P_n$.*

Lemma 2.2([2]). *Let x be a cut vertex of a connected graph and a and b be vertices occurring in different components which arise upon deletion of x, then $r_G(a,b) = r_G(a,x) + r_G(x,b)$.*

Lemma 2.3([5]). *Let G_1 and G_2 be connected graphs with disjoint vertex sets, with n_1 and n_2 vertices, and with m_1 and m_2 edges, respectively. Let $u_1 \in V(G_1)$, $u_2 \in V(G_2)$, constructing the graph G by identifying the vertices u_1 and u_2, and denote the so obtained vertex by u. Then*

$$D_R(G) = D_R(G_1) + D_R(G_2) + 2m_2 r_{u_1}(G_1) + 2m_1 r_{u_2}(G_2) + (n_2-1)S'_{u_1}(G_1) + (n_1-1)S'_{u_2}(G_2).$$

3 Main results

Firstly, we'll derive a formula for computing the degree resistance distance of $L_{n,k}$.

Theorem 3.1. *Let $G \in L_{n,k}$, then*

$$D_R(L_{n,k}) = k^3 - \frac{1}{3}(4n+3)k^2 + nk + \frac{2}{3}n^3 - \frac{1}{3}n.$$

Proof. Let $G_1 = C_k$, $G_2 = P_{n+1-k}$. By Lemma 2.3, one has,

$$\begin{aligned}
&D_R(L_{n,k}) \\
&= D_R(C_k) + D_R(P_{n+1-k}) + 2m_2 r_{v_0}(C_k) + 2m_1 r_{v_0}(P_{n+1-k}) + (n_2-1)S'_{v_0}(C_k) \\
&\quad + (n_1-1)S'_{v_0}(P_{n+1-k}).
\end{aligned}$$

Further, $D_R(C_k) = 4Kf(C_k) = \dfrac{k^3-k}{3}$,

$$\begin{aligned}
D_R(P_{n+1-k}) &= 4W(P_{n+1-k}) - (n+1-k)(n-k) \\
&= 4\binom{n+2-k}{3} - (n+1-k)(n-k) \\
&= \frac{2}{3}(n^3-k^3) - 2nk(n-k) + (n-k)^2 + \frac{1}{3}(n-k) \\
&= \frac{1}{3}(n-k)[2(n-k)^2 + 3(n-k) + 1]
\end{aligned}$$

$r_{v_0}(C_k) = \dfrac{k^2-1}{6}$, $r_{v_0}(P_{n+1-k}) = \dfrac{(n-k)(n+1-k)}{2}$; $S'_{v_0}(C_k) = \dfrac{k^2-1}{3}$, $S'_{v_0}(P_{n+1-k}) = (n-k)^2$.
Therefore,

$$D_R(L_{n,k}) = k^3 - \frac{1}{3}(4n+3)k^2 + nk + \frac{2}{3}n^3 - \frac{1}{3}n.$$

This completes the proof. □

Secondly, we give an order $L_{n,k}$ with the maximum and the second maximum degree resistance distance.

Theorem 3.2. *Let $G \in L_{n,k}$, one has*

(i) $D_R(G) \leqslant \frac{2n^3}{3} - \frac{28n}{3} + 18$, with the equality holds if and only if $G \cong L_{n,3}$;

(ii) if $G \ncong L_{n,3}$, then $D_R(G) \leqslant \frac{2n^3}{3} - \frac{53n}{3} + 48$, with the equality holds if and only if $G \cong L_{n,4}$.

Proof. Let $f(k) := D_R(L_{n,k}) = k^3 - \frac{1}{3}(4n+3)k^2 + nk + \frac{2}{3}n^3 - \frac{1}{3}n$, $4 \leqslant k \leqslant n$. In what follows, we'll discuss the monotonicity of $f(k)$ on interval $I := [3, 4, \cdots, n]$. The first derivative of $f(k)$ is

$$\frac{\partial f(k)}{\partial l} = 3k^2 - \frac{2}{3}(4n+3)k + n.$$

The roots of $\frac{\partial f(k)}{\partial k} = 0$ are $k_{1,2} = \frac{(4n+3) \mp \sqrt{16n^2 - 3n + 9}}{9}$. It is easy to see that for $n \geqslant 3$,

$$k_1 < \frac{4n + 3 - (4n - 24)}{9} = 3, \quad k_2 > \frac{4n + 3 + (24 - 4n)}{9} = 3.$$

It's easy to verify that $k_2 < n$. Then, one has

(i) when $k \in [3, k_2)$, $\frac{\partial f(k)}{\partial k} < 0$, which indicates that $f(k)$ is decreasing on $[3, k_2)$;

(ii) when $k \in [k_2, n]$, $\frac{\partial f(k)}{\partial k} > 0$, which indicates that $f(k)$ is increasing on $[k_2, n]$.

So, the maximum value of $f(k)$ must occurred between $f(3)$ and $f(n)$. It's suffice to see that $f(3) - f(n) = \frac{1}{3}(n^3 - 27n + 54)$. If $G \ncong L_{n,3}$, then the second maximum degree resistance distance of $D_R(G)$ is $D_R(L_{n,4})$ or $D_R(L_{n,n})$. By Theorem 3.1, one has, $D_R(L_{n,4}) = \frac{2}{3}n^3 - \frac{53}{3}n + 48$, $D_R(L_{n,n}) = \frac{1}{3}n^3 - \frac{1}{3}n$. Obviously, $D_R(L_{n,4}) > D_R(L_{n,n})$. □

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