Semi compactness and semi-$I$-compactness in ditopological texture spaces

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Abstract  In this paper we generalize the notion of Semi-continuity and MS-continuity and go on to study Semi-compactness, Semi-cocompactness, Semi-stability and Semi-costability in a ditopological texture space. We also extend the notion of Semi-compactness and Semi-cocompactness to a ditopological texture space modulo an ideal [13].

Key Words  Texturing, Texture space, Bitopology, Ditopological texture space, Ditopological texture space

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1 Introduction

The notion of a texture space, under the name of fuzzy structure, was introduced by L. M. Brown in [1], as a means of representing a lattice of fuzzy sets as a lattice of crisp subsets of some base set. The notion of semi-open sets in ditopological texture spaces was initiated by S. Dost in [11], which is being extended to a ditopological texture space modulo an ideal in [14]. The purpose of this paper is to generalize the notion of Semi-continuity and MS-continuity and go on to study Semi-compactness, Semi-cocompactness, Semi-stability and Semi-costability in a ditopological texture space. We also extend the notion of Semi-compactness and Semi-cocompactness to a ditopological texture space modulo an ideal.

2 Preliminaries

This section contains the notions which are needed in the sequel. For more details see [1, 2, 6, 7, 8, 11, 13, 14, 15].

Definition 2.1. [1]. Let $X$ be a set. Then $L \subseteq P(X)$ is called texturing of $X$ and $X$ is said to be textured by $L$ if $L$ separates the points of $X$, complete, completely distributive lattice with respect to inclusion, which contains $X$, $\phi$, and for which arbitrary meet coincides with intersection and finite joins coincide with unions. The pair $(X,L)$ is then known as a texture.
The internal definition of textural concepts are expressed using the p-sets and q-sets. That is, for each $x \in X$ the sets $p_x = \bigcap \{ A \in L : x \in A \}$ and $q_x = \bigvee \{ A \in L : x \notin A \}$. A surjection $\sigma : L \to L$ is called a complementation if $\sigma^2(A) = A \forall A \in L$ and $A \subseteq B$ in $L$ implies $\sigma(B) \subseteq \sigma(A)$. A texture with a complementation is said to be complemented. We recall from [5] that for $A \in L$ the core of $A$ is the set $\text{core}(A) = \bigcap \{ \bigcup \{ A_i : i \in I \} : A = \bigvee \{ A_i : A_i \in L, i \in I \} \}$. Clearly $\text{core}(A) \subseteq A$ and in general we can have $\text{core}(A) \notin L$. We will generally denote $\text{core}(A)$ by $A^\flat$.

**Definition 2.2.**[7]. Let $(X, L), (T, J)$ be textures, $(X \times T, L \otimes J)$ be the product of $(X, L)$, where the product texturing $L \otimes J$ of $X \times T$ consists of arbitrary intersections of sets of the form $(A \times T) \cup (X \times B)$, $p_x, q_x, x \in X$, be the P-sets and q-sets for the texture $(X, L)$, $P_t, Q_t, t \in T$, be the P-sets and q-sets for the texture $(T, J)$, $P(x, t), Q(x, t), x \in X, t \in T$, be the P-sets and q-sets in $(X \times T, L \otimes J)$, $A \in L$, $B \in J$, and $\bar{Q}(x, t)$, $P(x, t), x \in X, t \in T$, be the P-sets and q-sets in $(X \times T, P(X) \otimes J)$. For $x \in X, t \in T$ we clearly have $P(x, t) = p_x \times P_t$ and $Q(x, t) = (Q_x \times T) \cup (X \times Q_t)$. Also,

1. $r \in P(X) \otimes J$ is called a relation from $(X, L)$ to $(T, J)$ if it satisfies the following two conditions:
   - **R1** $r \nsubseteq \bar{Q}(x, t), P_x \nsubseteq Q_x \Rightarrow r \nsubseteq \bar{Q}(x', t)$.
   - **R2** $r \nsubseteq \bar{Q}(x, t) \Rightarrow \exists x' \in X \text{ s.t. } P_x \nsubseteq Q_{x'} \text{ and } r \nsubseteq \bar{Q}(x', t)$.

2. $R \in P(X) \otimes J$ is called a correlation from $(X, L)$ to $(T, J)$ if it satisfies the following two conditions:
   - **CR1** $\bar{P}(x, t) \nsubseteq R, P_x \nsubseteq Q_{x'} \Rightarrow \bar{P}(x', t) \nsubseteq R$.
   - **CR2** $\bar{P}(x, t) \nsubseteq R \Rightarrow \exists x' \in X \text{ s.t. } P_x \nsubseteq Q_x \text{ and } \bar{P}(x', t) \nsubseteq R$.

3. A pair $(r, R)$, where $r$ is a relation and $R$ a correlation from $(X, L)$ to $(T, J)$, is called a direlation from $(X, L)$ to $(T, J)$.

**Corollary 2.1.**[7]. Let $(r, R)$ be a direlation from $(X, L)$ to $(T, J)$, $A_j \in L \forall j \in J$ and $B_j \in J \forall j \in J$. Then

1. $r^-((\bigcup_{j \in J} B_j)) = \bigcap_{j \in J}(r^-(B_j))$ and $R^-((\bigcup_{j \in J} A_j)) = \bigcap_{j \in J}(R^-(A_j))$.
2. $r^-((\bigvee_{j \in J} A_j)) = \bigvee_{j \in J}(r^-(A_j))$ and $R^-((\bigvee_{j \in J} B_j)) = \bigvee_{j \in J}(R^-(B_j))$.

**Definition 2.3.**[7]. Let $(X, L), (T, J)$ and $(U, U)$ be textures.

1. If $c$ is a relation from $(X, L)$ to $(T, J)$ and $d$ is a relation from $(T, J)$ to $(U, U)$. Then their composition is the relation $d \circ c$ from $(X, L)$ to $(U, U)$ defined by $d \circ c = \bigvee \{ \bar{P}(x, u) : \exists t \in T \text{ with } c \nsubseteq \bar{Q}(x, t) \text{ and } d \nsubseteq \bar{Q}(t, u) \}$.
2. If $C$ is a correlation from $(X, L)$ to $(T, J)$ and $D$ a correlation from $(T, J)$ to $(U, U)$. Then their composition is the relation $D \circ C$ from $(X, L)$ to $(U, U)$ defined by $D \circ C = \bigcap \{ \bar{Q}(x, u) : \exists t \in T \text{ with } \bar{P}(x, t) \nsubseteq C \text{ and } \bar{P}(t, u) \nsubseteq D \}$.
3. With $c, d, C, D$ as above, the composition of the direlations $(c, C)$, $(d, D)$ is the direlation $(d, D) \circ (c, C) = (d \circ c, D \circ C)$.
Note that $d \circ c$ is a relation and $D \circ C$ is a corelation from $(X, L)$ to $(U, U)$.

**Definition 2.4.** [?] Let $(f, F)$ be a direlation from $(X, L)$ to $(T, J)$. Then $(f, F)$ is called a difunction from $(X, L)$ to $(T, J)$ if it satisfies the following two conditions:

DF1 For $x, x' \in X$, $P_x \not\subseteq Q_{x'}$ $\implies \exists t \in T$ with $f \not\subseteq \bar{Q}(x, t)$ and $\bar{P}(x', t) \not\subseteq F$.

DF2 For $t, t' \in T$ and $x \in X$, $f \not\subseteq \bar{Q}(x, t)$ and $\bar{P}(x, t') \not\subseteq F$ $\implies P_v \not\subseteq Q_t$. The complement of the difunction $(f, F)$ is denoted by $(f, F)'$. If $(f, F) = (f, F)'$, then the difunction $(f, F)$ is called complemented.

**Definition 2.5.** [?] Let $(f, F)$ be a difunction from $(X, L)$ to $(T, J)$.

(1) For $A \subseteq L$, the image $f^{-}(A)$ and the co-image $F^{-}(A)$ are defined by

$$f^{-}(A) = \bigvee \{Q_t : \forall x, f \not\subseteq \bar{Q}(x, t) \implies A \subseteq Q_x\},$$

$$F^{-}(A) = \bigcap \{P_t : \forall x, \bar{P}(x, t) \not\subseteq F \implies P_x \subseteq A\}.$$  

(2) For $B \subseteq J$, the inverse image $f^{-}(B)$ and the inverse co-image $F^{-}(B)$ are defined by

$$f^{-}(B) = \bigvee \{P_x : \forall t, f \not\subseteq \bar{Q}(x, t) \implies P_t \subseteq B\} \in X,$$

$$F^{-}(B) = \bigcap \{Q_x : \forall t, \bar{P}(x, t) \not\subseteq F \implies B \subseteq Q_t\} \in X.$$  

For a difunction, the inverse image and the inverse co-image are usually not equal, but the image and co-image are usually not.

**Theorem 2.1.** [?] Let $(f, F)$ be a direlation from $(X, L)$ to $(T, J)$. Then the following are equivalent:

(1) $(f, F)$ is a difunction.

(2) The following inclusions hold:

(a) $f^{-}(F^{-}(A)) \subseteq A \subseteq F^{-}(f^{-}(A)) \forall A \in L$, and

(b) $f^{-}(F^{-}(B)) \subseteq B \subseteq F^{-}(f^{-}(B)) \forall B \in J$.

(c) $f^{-}(B) = F^{-}(B) \forall B \in J$.

**Proposition 2.1.** [?]

(1) Let $(f, F)$ be a difunction from $(X, L)$ to $(T, J)$. Then

(a) For $A \in L$, $A = \phi \iff f^{-}(A) = \phi$.

(b) For $A \in L$, $A = \phi \iff F^{-}(A) = T$.

(c) $f^{-}(\phi) = F^{-}(\phi) = \phi$ and $f^{-}(T) = F^{-}(T) = X$

(2) Let $(X, L)$, $(T, J)$ and $(U, U)$ be textures, $(f, F) : (X, L) \to (T, J)$, $(g, G) : (T, J) \to (U, U)$ difunctions. Then $(g, G) \circ (f, F) : (X, L) \to (U, U)$ is a difunction.

**Definition 2.6.** [2]. $(L, \tau, K)$ is called a ditopological texture space on $X$ if

(1) $\tau \subseteq L$ satisfies

(a) $X, \phi \in \tau$,

(b) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$, and

(c) $G_i \in \tau, i \in I \Rightarrow \bigvee_{i \in I} G_i \in \tau$.
\( K \subseteq L \) satisfies

(a) \( X, \phi \in K \),

(b) \( F_1, F_2 \in K \Rightarrow F_1 \cup F_2 \in K \), and

(c) \( F_i \in K, i \in I \Rightarrow \bigwedge_{i \in I} F_i \in K \).

The elements of \( \tau \) are called open and those of \( K \) are called closed. We refer to \( \tau \) as the topology and to \( K \) as the cotopology of \( (\tau, K) \).

In general there is no a priori relation between \( \tau \) and \( K \), but if \( \sigma \) is a complementation on \((X, L, \tau)\), then we call \((\tau, K)\) a complemented ditopology on \((X, L, \sigma)\).

Finally, let \( Z \subseteq X \). Then the closure of \( Z \) is the set \( \text{cl}(Z) = \bigcap \{ F \in K : Z \subseteq F \} \), the interior of \( Z \) is \( \text{int}(Z) = \bigcup \{ G \in \tau : G \subseteq Z \} \), the exterior of \( Z \) is \( \text{ext}(Z) = \bigcup \{ G \in \tau : G \cap Z = \emptyset \} \) and \( Z \) is called dense in \( X \) if \( \text{cl}(Z) = X \). Also, if \( A \nsubseteq F \ \forall F \in K - \{ X \} \), we say \( A \) is co-dense.

A texturing \( L \) need not be closed under the operation of taking the set complement. Now, suppose that \((X, L, \tau)\) has a complementation \( \sigma \), that is an involution \( \sigma : L \rightarrow L \) satisfying \( A, B \in L \), \( A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A) \). Then if \( \tau \) and \( K \) are related by \( K = \sigma(\tau) \) we say that \((\tau, K)\) is a complemented ditopology on \((X, L, \sigma)\).

In this case we have \( \sigma(\text{cl}(A)) = \text{int}(\sigma(A)) \) and \( \sigma(\text{int}(A)) = \text{cl}(\sigma(A)) \). We denote the set of all open sets by \( O(X, L, \tau, K) \), or when there can be no confusion by \( O(X) \) and the set of all closed sets by \( C(X, L, \tau, K) \), or when there can be no confusion by \( C(X) \).

**Definition 2.7.** [8]. Let \((X_j, L_j, \tau_j, K_j), j=1, 2\), be ditopological texture spaces and \((f, F)\) a difunction from \((X_1, L_1, \tau_1, K_1)\) to \((X_2, L_2, \tau_2, K_2)\). Then

1. \((f, F)\) is continuous if \( G \in \tau_2 \Rightarrow F^-(G) \in \tau_1 \).
2. \((f, F)\) is cocontinuous if \( H \in K_2 \Rightarrow f^-(H) \in K_1 \).
3. \((f, F)\) is bicontinuous if it is continuous and cocontinuous.

For complemented difunction these two properties are equivalent. On the other hand \((f, F)\) is open (co-open) if \( A \in \tau_1 \Rightarrow f^-(A) \in \tau_2 \ \text{and} \ \sigma(f^-(A)) \in \tau_2 \). Also, \((f, F)\) is closed (co-closed) if \( A \in K_1 \Rightarrow f^-(A) \in K_2 \ \text{and} \ \sigma(f^-(A)) \in K_2 \).

**Definition 2.8.** [8]. Let \((X_j, L_j, \tau_j, K_j), j=1, 2\), be ditopological texture spaces and \((f, F)\) a difunction from \((X_1, L_1, \tau_1, K_1)\) to \((X_2, L_2, \tau_2, K_2)\). Then \((f, F)\) is called surjective if it satisfies the condition,

\[
\text{SUR.} \quad \text{For} \ t, t' \in T \text{ and } P_t \nsubseteq Q_t \Rightarrow \exists x \in X \text{ with } f \nsubseteq Q(x, t) \text{ and } P(x, t') \nsubseteq F.
\]

\((f, F)\) is called injective if it satisfies the condition,

\[
\text{INJ.} \quad \text{For} \ x, x' \in X \text{ and } t \in T, f \nsubseteq Q(x, t) \text{ and } P(x, t') \nsubseteq F \Rightarrow P_x \nsubseteq Q_{x'}.
\]

If \((f, F)\) is both injective and surjective, then it is called bijective.

**Theorem 2.2.** [7]. Let \((f, F)\) be a difunction from \((X, L)\) to \((T, J)\). Then

1. \((f, F)\) is surjective if and only if \((f, F)^{-}\) satisfies \(DF1\).
2. \((f, F)\) is injective if and only if \((f, F)^{-}\) satisfies \(DF2\).
(f, F) is bijective if and only if (f, F) is a difunction from \((T, J)\) to \((X, L)\) and in this case \(f'(f)(x)\) is also bijective.

**Corollary 2.2.** [7]. Let \((f, F)\) be a difunction from \((X, L)\) to \((T, J)\).

1. If \((f, F)\) is surjective, then \(f'(F(B)) = B = f'(f'(B)) \forall B \in \mathcal{J}\). In particular
   \(a) F^{-}(A) \subseteq f^{-}(A) \forall A \in L\), and
   \(b) \forall B_1, B_2 \in \mathcal{J} \; \text{and} \; f^{-}(B_1) \subseteq f^{-}(B_2) \implies B_1 \subseteq B_2.\)

2. If \((f, F)\) is injective, then \(f'(F^{-}(A)) = A = f'(f'(A)) \forall A \in L\). In particular
   \(a) F^{-}(A) \subseteq F^{-}(A) \forall A \in L\), and
   \(b) \forall A_1, A_2 \in L \; \text{and} \; F^{-}(A_1) \subseteq F^{-}(B_2) \implies A_1 \subseteq A_2.\)

**Definition 2.9.** [4]. Let \((\tau, K)\) be a ditopology on the texture space \((X, L)\) and take \(A \in L\). The family \(\{G_i : i \in I\}\) is said to be an open cover of \(A\) if \(G_i \in \tau \forall i \in I\) and \(A \subseteq \bigcup_{i \in I} G_i\). Dually we may speak of a closed co-cover of \(A\), namely a family \(\{F_i : i \in I\}\) with \(F_i \in K \forall i \in I\) satisfying \(\bigcap_{i \in I} F_i \subseteq A\).

**Definition 2.10.** [1].

1. A subset \(D\) of \(L \times L\) in a ditopological texture space \((X, L, \tau, K)\) is called a difamily on \((X, L)\).

2. A difamily \(D\) satisfying \(D \subseteq \tau \times K\) is open and co-closed, one satisfying \(D \subseteq K \times \tau\) is closed and co-open.

**Definition 2.11.** [1]. A difamily \(D = \{(F_i, G_i) : i \in I\}\) is called a dicover of \((X, L)\) if for all partitions \(I_1, I_2\) of \(I\) (including the trivial partitions) we have \(\bigcap_{i \in I_1} F_i \subseteq \bigcup_{i \in I_2} G_i\).

**Definition 2.12.** [3]. A difamily \(D\) has the finite exclusion property (fep) if whenever \((F_i, G_i) \in D, i = 1, 2, \ldots, n,\) we have \(\bigcap_{i=1}^n F_i \not\subseteq \bigcup_{i=1}^n G_i\).

**Definition 2.13.** [4]. Let \((L, \tau, K)\) be a ditopological texture space on \(X\). Then \((\tau, K)\) is called

1. Compact if whenever \(X = \bigcup_{j \in J} G_j, G_j \in \tau, j \in J,\) there is a finite subset \(J_o\) of \(J\) with \(X = \bigcup_{j \in J_o} G_j\).

2. Co-compact if whenever \(\bigcap_{i \in I} F_i = \phi, F_i \in K, i \in I,\) there is a finite subset \(I_o\) of \(I\) with \(\bigcap_{i \in I_o} F_i = \phi\). In general compactness and co-compactness are independent, but for a complemented ditopological texture space \((X, L, \tau, K, \sigma)\) they are equivalent see [4].

**Definition 2.14.** [11]. For a ditopological texture space \((X, L, \tau, K)\) and \(A \in L\). Then

1. \(A\) is called pre-open (resp. semi-open, \(\beta\)-open) if \(A \subseteq \text{int}(\text{cl}(A))\) (resp. \(A \subseteq \text{cl}((\text{int}(A)))\), \(A \subseteq \text{cl}(\text{int}(\text{cl}(A))))\).

2. \(B \in L\) is called pre-closed (resp. semi-closed, \(\beta\)-closed) if \(\text{cl}(\text{int}(B)) \subseteq B\) (resp. \(\text{int}(\text{cl}(B)) \subseteq B, \text{int}(\text{cl}(\text{int}(B))) \subseteq B)\).

We denote the set of all pre-open sets by \(PO(X, L, \tau, K)\), or when there can be no confusion by \(IO(X)\), the set of all pre-closed sets by \(PC(X, L, \tau, K)\), or \(IC(X)\), the set of all semi-open sets by \(SO(X, L, \tau, K)\), or \(SO(X)\), the set of all semi-closed sets by \(SC(X, L, \tau, K)\), or \(SC(X)\), the set of all \(\beta\)-open sets by \(\beta O(X, L, \tau, K)\), or \(\beta O(X)\), the set of all \(\beta\)-closed sets by \(\beta C(X, L, \tau, K)\), or \(\beta C(X)\).
Definition 2.15. [12]. A ditopology $(\tau, K)$ on $(X, L)$ is said to be:

1. $\beta$-compact if every cover of $X$ by $\beta$-open sets has a finite subcover.
2. $\beta$-cocompact if every cocover of $\phi$ by $\beta$-closed sets has a finite sub-cocover.

Definition 2.16. [4]. Let $(L, \tau, K)$ be a ditopological texture space on $X$. Then $(\tau, K)$ is called,

1. Stable if every $F \in K$ with $F \neq X$ is compact, i.e., whenever $F \subseteq \bigcup_{j \in J} G_j$, $G_j \in \tau$, $j \in J$, there is a finite subset $J_o$ of $J$ with $F \subseteq \bigcup_{j \in J_o} G_j$.
2. Co-stable if every $G \in \tau$ with $G \neq \phi$ is co-compact, i.e., whenever $\bigcap_{i \in I} F_i \subseteq G$, $F_i \in K$, $i \in I$, there is a finite subset $I_o$ of $I$ with $\bigcap_{i \in I_o} F_i \subseteq G$.

Ditopological texture space which is compact, stable, co-compact and co-stable is said to be dicompact.

Definition 2.17. [13]. Let $(\tau, K)$ be a ditopological space on any texture space $(X, L)$. Then

1. define the local function $()^\ast: P(X) \to P(X)$ by $A^\ast(I, \tau) = \{ x \in X : O_x \cap A \notin I \forall O_x \in \tau \} \forall A \in P(X)$. A Kuratowski closure operator $\text{Cl}_\tau^\ast(\cdot)$ for the topology $\tau^\ast(I, \tau)$, called the $\ast$-topology, finer than $\tau$, induced by $\text{Cl}_\tau^\ast(A) = A \cup A^\ast(I, \tau)$, where $\tau^\ast = \{ G \subseteq X : \text{Cl}_\tau^\ast(G^\ast) = G^\ast \}$.
2. let $K^\prime = \{ X - F : F \in K \}$, which is a topology on $X$, so we again define a local function $()_{K^\prime}^\ast: P(X) \to P(X)$, where $A^\ast(I, K^\prime)$ is the local function of $A$ w.r.t $I$, $K^\prime$. Also a Kuratowski closure operator $\text{Cl}_{K^\prime}^\ast(\cdot)$ for the topology $K^\ast(I, K^\prime)$, called the $\ast$-topology, finer than $K^\prime$. Hence $K^\ast = K^\ast$ is a family of closed subsets of $X$ finer than $K$.
3. let $(X, L^\ast)$ be the smallest texture structure space containing $L$, $\tau^\ast$ and $K^\ast$. Hence $(\tau^\ast, K^\ast)$ is called the $\ast$-ditopology on $(X, L^\ast)$, finer than $(\tau, K)$ on $(X, L)$.

Finally, let $Z \subseteq X$. Then the $\ast$-closure of $Z$ is the set $\text{cl}_{K^\ast}(Z) = \bigcap\{ F \in K^\ast : Z \subseteq F \}$, the $\ast$-interior is $\text{int}_{\tau^\ast}(Z) = \bigvee\{ G \in \tau^\ast : G \subseteq Z \}$, the $\ast$-exterior is $\text{ext}_{\tau^\ast}(Z) = \bigvee\{ G \in \tau^\ast : G \cap Z = \phi \}$ and $Z$ is called $\ast$-dense in $X$ if $\text{cl}_{K^\ast}(Z) = X$. Also if $A \notin F \forall F \in K^\ast - \{ X \}$, we say $A$ is $\ast$-co-dense.

Definition 2.18. [14]. Let $\gamma: L \to L^\ast$ be a mapping, then $O(\gamma) = \{ V : V \subseteq \gamma(V), V \in L \}$ is the family of all $\gamma$-open sets and the complemented of $\gamma$-open set is called $\gamma$-closed set, i.e $C(\gamma) = \{ A^\prime : A = \gamma - open, A \in L \}$ is the family of all $\gamma$-closed sets.

Examples 2.1. [14]. Let $(X, L, \tau, K, I)$ be a ditopological texture space with an ideal, $\gamma: L \to L^\ast$ be an operation on $L$ and $A \in L$.

1. If $\gamma = \text{int}_{\tau^\ast}(\text{cl}_{K^\ast})$, then $\gamma$ is called pre-$I$-open operator. We denote the set of all pre-$I$-open sets by $\text{PIO}(X, L, \tau, K, I)$, or when there can be no confusion by $\text{PIO}(X)$. It is obvious that every open set is pre-$I$-open set.
2. If $I_{K^\prime}(A) = A^\ast(I, K^\prime)$ and $\gamma = \text{int}_{\tau}(I_{K^\prime})$, then $\gamma$ is called $I$-open operator. We denote the set of all $I$-open sets by $\text{IO}(X, L, \tau, K, I)$, or $\text{IO}(X)$.
3. If $\gamma = \text{int}_{\ast}(\text{cl}_{K^\ast}(\text{int}_{\ast}))$, then $A$ is called $\alpha$-$I$-open. We denote the set of all $\alpha$-$I$-open sets by $\alpha\text{IO}(X, L, \tau, K, I)$, or $\alpha\text{IO}(X)$.
(4) If \( \gamma = cl_{K}^{\prime}(int_{\gamma}) \), then \( A \) is called semi-I-open. We denote the set of all semi-I-open sets by \( SIO(X, L, \tau, K, I) \), or \( SIO(X) \).

(5) If \( \gamma = cl_{K}(int_{\gamma}(cl_{K}^{\prime})) \), then \( A \) is called \( \beta \)-I-open. We denote the set of all \( \beta \)-I-open sets by \( \beta IO(X, L, \tau, K, I) \), or \( \beta IO(X) \). It is obvious that \( O(X) \subseteq \beta IO(X) \subseteq \beta O(X) \).

3 Semi compactness in ditopological texture spaces

**Definition 3.1.** Let \( (f, F) : (X_1, L_1, \tau_1, K_1) \to (X_2, L_2, \tau_2, K_2) \) be a difunction. Then

1. \( (f, F) \) is called semi-continuous (resp. MS-continuous) if \( A \in O(X_2) \) (resp. \( A \in SO(X_2) \)) \( \Rightarrow f^{-}(A) \in SO(X_1) \).

2. \( (f, F) \) is called semi-cocontinuous (resp. MS-cocontinuous) if \( A \in C(X_2) \) (resp. \( A \in SC(X_2) \)) \( \Rightarrow F^{-}(A) \in SC(X_1) \).

3. \( (f, F) \) is semi-bicontinuous (resp. MS-bicontinuous) if it is semi-continuous and semi-cocontinuous (resp. MS-continuous and MS-cocontinuous).

Clearly MS-continuity (resp. MS-cocontinuity, MS-bicontinuity) is stronger than S-continuity (resp. S-cocontinuity, S-bicontinuity).

**Definition 3.2.** Let \( (X, L, \tau, K) \) be a ditopological texture space, \( A \subseteq X \) and \( x \in X \). Then,

1. \( x \) is called an semi-interior point of \( A \) if \( \exists \ G \in SO(X) \) s.t \( x \in G \subseteq A \), the set of all semi-interior points of \( A \) is called the semi-interior of \( A \) and is denoted by \( Sint(A) \), i.e
   \[
   Sint(A) = \bigvee \{G : G \in SO(X) \text{ and } G \subseteq A\}.
   \]

2. \( x \) is called an semi-cluster point of \( A \) if \( A \cap H \neq \emptyset \forall H \in SO(X) \). The set of all semi-cluster points of \( A \) is called semi-closure of \( A \) and is denoted by \( Scl(A) \), i.e
   \[
   Scl(A) = \bigcap \{H : H \in SC(X) \text{ and } A \subseteq H\}.
   \]
   It obvious that \( Sint(A) \in SO(X) \) and \( A \in SO(X) \Leftrightarrow A = Sint(A) \), also we have \( Scl(A) \in SC(X) \) and \( A \in SC(X) \Leftrightarrow A = Scl(A) \).

**Theorem 3.1.** Let \( (f, F) : (X_1, L_1, \tau_1, K_1) \to (X_2, L_2, \tau_2, K_2) \) be a difunction. Then the following are equivalent:

1. \( (f, F) \) is semi-continuous.

2. \( int_{\tau_2}(F^{-}(A)) \subseteq F^{-}(Sint_{\tau_1}(A)) \forall A \in L_1 \).

3. \( f^{-}(int_{\tau_2}(B)) \subseteq Sint_{\tau_1}(f^{-}(B)) \forall B \in L_2 \).

**Proof.**

(1) \( \Rightarrow \) (2) Let \( A \in L_1 \). Then \( f^{-}(int_{\tau_2}(F^{-}(A))) \subseteq f^{-}(F^{-}(A)) \subseteq A \) from Theorem 2.1(2(a)) and Definition 3.2. Since \( f^{-}(int_{\tau_2}(F^{-}(A))) = F^{-}(int_{\tau_2}(F^{-}(A))) \) from Theorem 2.1(c), then \( f^{-}(int_{\tau_2}(F^{-}(A))) = F^{-}(int_{\tau_2}(F^{-}(A))) \in SO(X_1) \) from Definition 3.1. Since \( Sint_{\tau_1}(A) \) is the largest semi-open set contained in \( A \) from Definition 3.2, then \( f^{-}(int_{\tau_2}(F^{-}(A))) \subseteq Sint_{\tau_1}(A) \), so \( F^{-}(f^{-}(int_{\tau_2}(F^{-}(A)))) \subseteq F^{-}(Sint_{\tau_1}(A)) \). Thus \( int_{\tau_2}(F^{-}(A)) \subseteq F^{-}(f^{-}(int_{\tau_2}(F^{-}(A)))) \subseteq F^{-}(Sint_{\tau_1}(A)) \) from Theorem 2.1(2(b)). Hence \( int_{\tau_2}(F^{-}(A)) \subseteq F^{-}(Sint_{\tau_1}(A)) \).
(2) $\Rightarrow$ (3) Let $B \in L_2$ and $A = f^{-1}(B) \in L_1$. Then $\text{int}_{\tau_2}(F^-((f^{-1}(B)))) \subseteq F^-(\text{Sint}_{\tau_1}(f^{-1}(B)))$ from (2). By Theorem 2.1(2(b)) $\text{int}_{\tau_2}(F^-((f^{-1}(B)))) \subseteq \text{int}_{\tau_2}(F^-(\text{Sint}_{\tau_1}(f^{-1}(B))))$. Thus $f^{-1}((\text{int}_{\tau_2}(B))) \subseteq f^{-1}(F^-(\text{Sint}_{\tau_1}(f^{-1}(B)))) \subseteq \text{Sint}_{\tau_1}(f^{-1}(B)))$ from Theorem 2.1(2(a)). Hence $f^{-1}((\text{int}_{\tau_2}(B))) \subseteq \text{Sint}_{\tau_1}(f^{-1}(B)))$.

(3) $\Rightarrow$ (1) Let $B \in \mathcal{O}(X_2)$. Then $\text{int}_{\tau_2}(B) = B$ and $f^{-1}(B) = f^{-1}(\text{int}_{\tau_2}(B)) \subseteq \text{Sint}_{\tau_1}(f^{-1}(B))$. But we have $\text{Sint}_{\tau_1}(f^{-1}(B)) \subseteq f^{-1}(B)$, then $f^{-1}(B) = F^-((f^{-1}(B))) = \text{Sint}_{\tau_1}(f^{-1}(B))$ from Theorem 2.1(3). Thus $f^{-1}(B) \in \mathcal{O}(X_1)$. Hence $(f, F)$ is $S$-continuous.

Theorem 3.2. Let $(f, F) : (X_1, L_1, \tau_1, K_1) \to (X_2, L_2, \tau_2, K_2)$ be a difunction. Then the following are equivalent:

(1) $(f, F)$ is semi-cocontinuous.

(2) $f^{-1}(\text{Scl}_{\tau_1}(A)) \subseteq \text{Cl}_{\tau_2}(f^{-1}(A)) \forall A \in L_1$.

(3) $\text{Scl}_{\tau_1}(F^-((B))) \subseteq F^-((\text{Cl}_{\tau_2}(B))) \forall B \in L_2$.

Proof. The proof is similar to Theorem 3.1.

Proposition 3.1. Let $(f, F) : (X_1, L_1, \tau_1, K_1) \to (X_2, L_2, \tau_2, K_2)$ be a difunction. Then the following are equivalent:

(1) $(f, F)$ is MS-continuous.

(2) $\text{Sint}_{\tau_2}(F^-((A))) \subseteq F^-((\text{Sint}_{\tau_1}(A))) \forall A \in L_1$.

(3) $f^-((\text{Sint}_{\tau_2}(B))) \subseteq \text{Sint}_{\tau_1}(f^{-1}(B))) \forall B \in L_2$.

Proof. Immediate from Theorem 3.1.

Proposition 3.2. Let $(f, F) : (X_1, L_1, \tau_1, K_1) \to (X_2, L_2, \tau_2, K_2)$ be a difunction. Then the following are equivalent:

(1) $(f, F)$ is MS-cocontinuous.

(2) $f^-((\text{Scl}_{\tau_1}(A)) \subseteq \text{Scl}_{\tau_2}(f^{-1}(A)) \forall A \in L_1$.

(3) $\text{Scl}_{\tau_1}(F^-((B))) \subseteq F^-((\text{Scl}_{\tau_2}(B))) \forall B \in L_2$.

Proof. Immediate from Theorem 3.2.

Definition 3.3. A ditopology $(\tau, K)$ on $(X, L)$ is said to be:

(1) Semi-compact if every cover of $X$ by semi-open sets has a finite subcover.

(2) Semi-cocompact if every cocover of $\phi$ by semi-closed sets has a finite sub-cocover.

Here we say that $C = \{G_\alpha : \alpha \in \Lambda\}$, $G_\alpha \in L$ is a cover of $X$ (resp. a cocover of $\phi$) if $\bigvee C = X$ (resp. $\bigwedge C = \phi$).

Proposition 3.3. For a ditopological texture space $(X, L, \tau, K)$:

(1) $\beta$-compact $\Rightarrow$ semi-compact $\Rightarrow$ compact.
(2) $\beta$-cocompact $\implies$ semi-cocompact $\implies$ cocompact.

**Proof.** It follows directly from the fact that, $O(X) \subseteq SO(X) \subseteq \beta O(X)$.

**Remark 3.1.** Note that the notions of semi-compactness and semi-cocompactness are independent, as shown in the following examples.

**Examples 3.1.** (1) Let $X = (0, 1]$, $L = \{(0, r] : r \in X\}$, $\tau = \{X, \phi\}$, $K = L$. Since the only semi-open sets are $X, \phi$, so we see that $(\tau, K)$ is S-compact. However, it is not semi-cocompact since it is not cocompact, for the family $\mathcal{C} = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$ of closed sets satisfies $\bigcap \mathcal{C} = \phi$, but no finite subset of $\mathcal{C}$ has an empty intersection i.e $(\tau, K)$ is not semi-cocompact.

(2) Dually let $X = (0, 1]$, $L = \{(0, r] : r \in X\}$, $\tau = L$, $K = \{X, \phi\}$. Then the ditopology $(\tau, K)$ is semi-cocompact but not semi-compact.

On the other hand, for complemented ditopological texture spaces these two properties are equivalent, as shown in the following theorem.

**Theorem 3.3.** Let $(X, L, \tau, K, \sigma)$ be a complemented ditopological texture space. Then $(X, L, \tau, K, \sigma)$ is semi-compact if and only if it is semi-cocompact.

**Proof.** Suppose that $(X, L, \tau, K, \sigma)$ is S-compact and let $\mathcal{C} = \{F_\alpha : \alpha \in \Lambda\}$ be a family of semi-closed sets with $\bigcap \mathcal{C} = \phi$. Clearly $\mathcal{D} = \{\sigma(F_\alpha) : \alpha \in \Lambda\}$ is a family of semi-open sets. Moreover, $\bigvee \mathcal{D} = \bigvee \{\sigma(F_\alpha) : \alpha \in \Lambda\} = \sigma(\bigcap \mathcal{F}_\alpha) = \sigma(\phi) = X$, then by semi-compactness $\exists \Lambda_0 \subseteq \Lambda$ finite with $\bigvee \{\sigma(F_\alpha) : \alpha \in \Lambda_0\} = X$. Hence $\sigma(\bigvee \{F_\alpha : \alpha \in \Lambda_0\}) = \sigma(X)$ i.e $\bigcap \{F_\alpha : \alpha \in \Lambda_0\} = \phi$. Therefore $(X, L, \tau, K, \sigma)$ is S-compact. Conversely, if $(X, L, \tau, K, \sigma)$ is S-cocompact, then by a similar way we can prove that $(X, L, \tau, K, \sigma)$ is S-compact.

**Example 3.1.** Let $X = [0, 1]$, $L = \{(0, r] : r \in X\} \cup \{(0, r] : r \in X\} \cup \{X\}$, $\tau = \{[0, r] : r \in X\} \cup \{\phi\}$ and let $\sigma$ be a complementation $\sigma([0, r] = [0, 1 - r], \sigma([0, r] = [0, 1 - r])$. It is clear that $(X, L, \tau, K, \sigma)$ is compact and cocompact.

Since $SO(X) = \tau$ and $SC(X) = K$, then $(X, L, \tau, K)$ is semi-compact and semi-cocompact.

**Theorem 3.4.** Let $(f, F) : (X_1, \tau_1, K_1) \to (X_2, \tau_2, K_2)$ be an MS-continuous difunction. Then if $A \subseteq L_2$ is semi-compact, then $f^{-1}(A) \subseteq L_1$ is semi-compact.

**Proof.** Let $f^{-1}(A) \subseteq \bigvee_{j \in J} (G_j)$. $G_j \in SO(X_2)$. Then $A \subseteq F^{-1}(f^{-1}(A)) \subseteq F^{-1}(\bigvee_{j \in J} (G_j)) = \bigvee_{j \in J} (F^{-1}(G_j))$, where $F^{-1}(G_j) \in SO(X_1)$ by MS-continuity, from Corollary 2.1 and Theorem 2.1(2(a)). Since $A$ is S-compact, then $\exists J_0 \subseteq J$ finite s.t $A \subseteq \bigvee_{j \in J_0} (F^{-1}(G_j)) = \bigcup_{j \in J_0} (F^{-1}(G_j))$. Thus $f^{-1}(A) \subseteq \bigvee_{j \in J_0} (F^{-1}(G_j)) = \bigcup_{j \in J_0} (F^{-1}(G_j)) \subseteq \bigcup_{j \in J_0} (G_j)$ from Theorem 2.1(2(b)). Hence $f^{-1}(A) \subseteq L_2$ is semi-compact.

**Corollary 3.1.** Let $(f, F) : (X_1, \tau_1, K_1) \to (X_2, \tau_2, K_2)$ be a surjective MS-continuous difunction. Then if $X_1$ is semi-compact, then $X_2$ is semi-compact.

**Proof.** Immediate from Corollary 2.1(1), Proposition 2.1(c) and Theorem 3.4.

**Theorem 3.5.** Let $(f, F) : (X_1, \tau_1, K_1) \to (X_2, \tau_2, K_2)$ be an MS-cocontinuous difunction. Then if $A \subseteq L_1$ is semi-cocontinuous, then $F^{-1}(A) \subseteq L_2$ is semi-cocontinuous.

**Proof.** The proof is similar to Theorem 3.4.
Corollary 3.2. Let \((f,F) : (X_1,L_1,\tau_1,K_1) \to (X_2,L_2,\tau_2,K_2)\) be a surjective MS-cocontinuous difunction. Then if \(X_1\) is semi-cocompact, then \(X_2\) is semi-cocompact.

**Proof.** Immediate from Corollary 2.1(1), Proposition 2.1(c) and Theorem 3.5.

4 Semi stability in ditopological texture spaces

The notion of stability for bitopological spaces was introduced by Ralph Kopperman in [17]. The analogous notion, and its dual, were given for ditopologies in [4], and studied in greater detail in [9]. We now wish to generalize these concepts for semi-open and semi-closed sets.

**Definition 4.1.** A ditopology \((\tau,K)\) on \((X,L)\) is said to be:

1. **Semi-stable** if every semi-closed set \(F \in L\setminus \{X\}\) is semi-compact in \(X\) i.e.
   \[
   \forall G_j \in SO(X), \; j \in J \quad \exists J_0 \subseteq J \quad \exists F \subseteq \bigcup_{j \in J_0} G_j.
   \]

2. **Semi-costable** if every semi-open set \(G \in L\setminus \phi\) is S-cocompact in \(X\) i.e.
   \[
   \forall F_j \in SC(X), \; j \in J \quad \exists F \subseteq \bigcup_{j \in J_0} G_j \quad \exists F \subseteq \bigcup_{j \in J_0} F_j.
   \]

**Proposition 4.1.** For a ditopological texture space \((X,L,\tau,K)\):

1. \(\beta\)-stable \(\Rightarrow\) semi-stable \(\Rightarrow\) stable.
2. \(\beta\)-costable \(\Rightarrow\) semi-costable \(\Rightarrow\) costable.

**Proof.** It follows directly from the fact that, \(C(X) \subseteq SC(X) \subseteq \beta C(X)\).

**Remark 4.1.** Note that semi-stability (resp. semi-costability) are unrelated to S-compactness (resp. semi-cocompactness), as shown in the following examples.

**Examples 4.1.** (1) Let \(X = (0,1], \; L = \{(0,r) : r \in X\}, \; \tau = L, \; K = \{X,\phi\}\). Then the ditopology \((\tau,K)\) is not semi-compact as shown in examples 3.1. Since the only semi-closed sets are \(X, \phi\), we see that \((\tau,K)\) is semi-stable.

(2) Dually let \(X = (0,1], \; L = \{(0,r) : r \in X\}, \; \tau = \{X,\phi\}, \; K = L\). Then the ditopology \((\tau,K)\) is semi-costable but not semi-cocompact.

**Remark 4.2.** Note that the notions of semi-stability and semi-costability are independent, as shown in the following examples.

**Examples 4.2.** (1) Let \(X = (0,1], \; L = \{(0,r) : r \in X\}, \; \tau = L, \; K = \{X,\phi,(0,\frac{1}{2}]\}\). Then it is clearly that \(SO(X) = L\) and \(SC(X) = K\). Since \(C = \{(0,\frac{1}{2} - \frac{1}{n}) : n = 3, 4, 5, ...\}\) is a semi-open cover of the semi-closed set \([0,\frac{1}{2}]\) with no finite subcover, we see that \((\tau,K)\) is not semi-stable. On the other hand it is semi-costable because \(SC(X) = K\) is finite.

(2) Dually let \(X = (0,1], \; L = \{(0,r) : r \in X\}, \; \tau = \{X,\phi,(0,\frac{1}{2}]\}, \; K = L\). Then the ditopology \((\tau,K)\) is semi-stable but not semi-costable.

On the other hand, for complemented ditopological texture spaces these two properties are equivalent, as shown in the following theorem.
Theorem 4.1. Let \((X, L, \tau, K, \sigma)\) be a complemented ditopological texture space. Then \((X, L, \tau, K, \sigma)\) is semi-stable if and only if it is semi-costable.

**Proof.** Suppose that \((X, L, \tau, K, \sigma)\) is semi-stable, let \(A \in SO(X)\), \(A \neq \phi\) and \(C\) be a semi-closed cocover of \(A\). Then \(B = \sigma(A)\) is semi closed set and \(B \neq X\). Let \(D = \{\sigma(F) : F \in C\}\). Since \(\bigcap C \subseteq A\), then \(\sigma(A) \subseteq \sigma(\bigcap C)\), i.e. \(B \subseteq \sqrt{ \mathcal{D}}\). This means that \(D\) is a semi-open cover of \(B\). Hence there exists \(F_1, F_2, ..., F_n \in D\) s.t \(B \subseteq \sigma(F_1) \cup \sigma(F_2) \cup \sigma(F_3) \cup ... \sigma(F_n) = \sigma(F_1 \cap F_2 \cap F_3 \cap ... F_n)\). Thus \(F_1 \cap F_2 \cap F_3 \cap ... F_n \subseteq \sigma(B) = A\). It follows \(A\) is semi-cocompact. Hence \((\tau, K)\) is semi-costable. Conversely, if \((X, L, \tau, K, \sigma)\) is semi-costable, then by a similar way we can proof that \((X, L, \tau, K, \sigma)\) is semi-stable.

Theorem 4.2. Let \((f, F) : (X_1, L_1, \tau_1, K_1) \rightarrow (X_2, L_2, \tau_2, K_2)\) be a surjective MS-bicontinuous difunction. Then if \(X_1\) is semi-stable, then \(X_2\) is semi-stable.

**Proof.** Let \(F \in SC(X_2)\) s.t \(F \neq X_2\). Since \((f, F)\) is semi-cocontinuous, \(f^{-1}(F) \in SO(X_1)\). Now we want prove that \(f^{-1}(F) \neq X_1\). Suppose that \(f^{-1}(F) = X_1\). Since \(f^{-1}(X_2) = X_1\) from Proposition 2.1(1(c)), then \(f^{-1}(F) = f^{-1}(X_2)\). It follows \(F = X_2\) from Corollary 2.2(1(b)), which is a contradiction with \(F \neq X_2\), so \(f^{-1}(F) \neq X_1\). Thus \(f^{-1}(F)\) is semi-compact in \(X_1\) by semi-stability, also \(f^{-1}(f^{-1}(F))\) is semi-compact in \(X_2\) from Theorem 3.4 and \(f^{-1}(f^{-1}(F)) = F\) from Corollary 2.2(1), i.e. \(F\) is semi-compact in \(X_2\). Hence \(X_2\) is semi-stable.

Theorem 4.3. Let \((f, F) : (X_1, L_1, \tau_1, K_1) \rightarrow (X_2, L_2, \tau_2, K_2)\) be a surjective MS-bicontinuous difunction. Then if \(X_1\) is semi-costable, then \(X_2\) is semi-costable.

**Proof.** The proof is similar to Theorem 4.2.

Definition 4.2. A ditopology \((\tau, K)\) on \((X, L)\) is said to be semi-dicompact if it is semi-compact, semi-cocompact, semi-stable and semi-costable.

Theorem 4.4. Semi-dicompactness is preserved under a surjective MS-bicontinuous difunction.

**Proof.** Immediate from Proposition 3.1, Proposition 3.2, Theorem 4.2 and Theorem 4.3.

5 Semi compactness in ditopological texture spaces via idealization

Definition 5.1. Let \(I\) be an ideal on a ditopological texture space \((X, L, \tau, K)\). A cover \(\{G_\alpha : \alpha \in \Lambda\}\) of \(X\) is said to be an \(I\)-cover if there exists a finite subset \(\Lambda_o\) of \(\Lambda\) such that \(\{G_\alpha : \alpha \in \Lambda_o\}\) covers \(X\) except, perhaps, for some subset which belongs to the ideal \(I\), i.e. \(X \setminus \bigcup_{\alpha \in \Lambda_o}G_\alpha \in I\).

Dually we may speak of an \(I\)-cocover namely, a family \(\{F_j : j \in J\}\) is said to be an \(I\)-cocover if whenever the family \(\{F_j : j \in J\}\) is a cocover of \(\phi\), there exists a finite subset \(J_o\) of \(J\) such that \(X \setminus (\bigcap_{j \in J_o}F_j)' \in I\).

Definition 5.2. A ditopology \((\tau, K)\) on \((X, L)\) is said to be:

1. \(I\)-compact if every \(\tau\)-open cover of \(X\) is \(I\)-cover.
2. \(I\)-cocompact if every \(K\)-closed cocover of \(\phi\) is \(I\)-cocover.

**Proposition 5.1.** (1) Every compact ditopological space \((\tau, K)\) on a texture space \((X, L)\) is \(I\)-compact for any ideal \(I\) on \(X\).
(2) Every cocompact ditopological space \((\tau, K)\) on a texture space \((X, L)\) is \(I\)-cocompact for any ideal \(I\) on \(X\).

(3) If \(I = \phi\), then \((\tau, K)\) is compact \(\iff\) it is \(I\)-compact.

(4) If \(I = \phi\), then \((\tau, K)\) is cocompact \(\iff\) it is \(I\)-cocompact.

(5) If the \(*\)-ditopology \((X, L^*, \tau^*, K^*)\) is \(I\)-compact, then the ditopology \((X, L, \tau, K)\) is \(I\)-compact.

(6) If the \(*\)-ditopology \((X, L^*, \tau^*, K^*)\) is \(I\)-cocompact, then the ditopology \((X, L, \tau, K)\) is \(I\)-cocompact.

**Proof.** Immediate.

**Theorem 5.1.** Let \((X, L, \tau, K)\) be a ditopological texture space, \(I\) be an ideal on \(X\) and \((X, L^*, \tau^*, K^*)\) be a \(*\)-ditopological texture space. Then \((X, L, \tau, K)\) is \(I\)-compact if and only if \((X, L^*, \tau^*, K^*)\) is \(I\)-compact.

**Proof.** Let \((X, L, \tau, K)\) be \(I\)-compact and \(\{G_{\alpha} : \alpha \in \Lambda\}\) be \(\tau^*\)-open cover of \(X\). Then \(G_{\alpha} = (V_{\alpha} - A_{\alpha})\forall \alpha \in \Lambda\), where \(V_{\alpha} \in \tau\) and \(A_{\alpha} \in I\). It follows that \(\{V_{\alpha} : \alpha \in \Lambda\}\) is a \(\tau^*\)-open cover of \(X\). Thus by \(I\)-compactness of \((X, L, \tau, K)\), \(\exists \Lambda_0 \subseteq \Lambda\) finite s.t \(X - \bigcup_{\alpha \in \Lambda_0} V_{\alpha} \in I\). So \(X - \bigcup_{\alpha \in \Lambda_0} G_{\alpha} = X - \bigcup_{\alpha \in \Lambda_0} (V_{\alpha} - A_{\alpha}) \subseteq (X - \bigcup_{\alpha \in \Lambda_0} (V_{\alpha})) \cup (\bigcup_{\alpha \in \Lambda_0} (A_{\alpha})) \in I\), where \(A_{\alpha} \in I\forall \alpha \in \Lambda\). Hence \((X, L^*, \tau^*, K^*)\) is \(I\)-compact. Conversely, if \((X, L^*, \tau^*, K^*)\) is \(I\)-compact. It follows that \((X, L, \tau, K)\) be \(I\)-compact from Proposition 5.2.

**Theorem 5.2.** Let \((X, L, \tau, K)\) be a ditopological texture space, \(I\) be an ideal on \(X\) and \((X, L^*, \tau^*, K^*)\) be a \(*\)-ditopological texture space. Then \((X, L, \tau, K)\) is \(I\)-cocompact if and only if \((X, L^*, \tau^*, K^*)\) is \(I\)-cocompact.

**Proof.** The proof is similar to Theorem 5.1.

**Theorem 5.3.** Let \((X, L, \tau, K)\) be a ditopological texture space, \(I\) be an ideal on \(X\) and \((X, L^*, \tau^*, K^*)\) be a \(*\)-ditopological texture space. Then the following implications hold.

\[\begin{align*}
(X, L^*, \tau^*, K^*) \text{ is compact} & \quad \Rightarrow \quad (X, L, \tau, K) \text{ is compact} \\
\downarrow & \\
(X, L^*, \tau^*, K^*) \text{ is } I\text{-compact} & \quad \iff \quad (X, L, \tau, K) \text{ is } I\text{-compact}
\end{align*}\]

**Proof.** Immediate from Proposition 5.2 and Theorem 5.1.

**Theorem 5.4.** Let \((X, L, \tau, K)\) be a ditopological texture space, \(I\) be an ideal on \(X\) and \((X, L^*, \tau^*, K^*)\) be a \(*\)-ditopological texture space. Then the following implications hold.

\[\begin{align*}
(X, L^*, \tau^*, K^*) \text{ is cocompact} & \quad \Rightarrow \quad (X, L, \tau, K) \text{ is cocompact} \\
\downarrow & \\
(X, L^*, \tau^*, K^*) \text{ is } I\text{-cocompact} & \quad \iff \quad (X, L, \tau, K) \text{ is } I\text{-cocompact}
\end{align*}\]

**Proof.** Immediate from Proposition 5.2 and Theorem 5.2.

**Definition 5.3.** A ditopological texture space \((X, L, \tau, K, I)\) with ideal is said to be semi-\(I\)-compact if for every semi-open cover \(\{G_{\alpha} : \alpha \in \Lambda\}\) of \(X\) there exists a finite subset \(\Lambda_0\) of \(\Lambda\) such that \(\{G_{\alpha} : \alpha \in \Lambda_0\}\) covers \(X\) except, perhaps, for some subset which belongs to the ideal \(I\), i.e. \(X - \bigcup_{\alpha \in \Lambda_0} G_{\alpha} \in I\).

Dually A ditopological texture space \((X, L, \tau, K, I)\) with ideal is said to be semi-\(I\)-cocompact if for every semi-closed co-cover \(\{F_{j} : j \in J\}\) of \(\phi\) there exists a finite subset \(J_0\) of \(J\) such that \(X - (\bigcap_{j \in J_0} F_{j})' \in I\).
Proposition 5.2. (1) Every semi-I-compact ditopological space with ideal \((\tau, K, I)\) on a texture space \((X, L)\) is I-compact for any ideal \(I\) on \(X\).

(2) Every semi-I-cocompact ditopological space \((\tau, K, I)\) on a texture space \((X, L)\) is I-cocompact for any ideal \(I\) on \(X\).

(3) If the \(\ast\)-ditopology \((X, L^\ast, \tau^\ast, K^\ast)\) is semi-I-compact, then the ditopology \((X, L, \tau, K)\) is semi-compact.

(4) If the \(\ast\)-ditopology \((X, L^\ast, \tau^\ast, K^\ast)\) is semi-I-cocompact, then the ditopology \((X, L, \tau, K)\) is semi-I-cocompact.

Proof. Immediate.

Theorem 5.5. A ditopological texture space \((X, L, \tau, K)\) is semi-compact if and only if \((X, L, \tau, K, I_f)\) is semi-I-compact, where \(I_f = \{A \subseteq X : A\text{ is finite}\}\).

Proof. Let \((X, L, \tau, K)\) be a semi-compact ditopological texture space and let \(\{G_\alpha : \alpha \in \Lambda\}\) be a semi-open cover of \(X\). Then there exists a finite subset \(\Lambda_o\) of \(\Lambda\) such that \(\{G_\alpha : \alpha \in \Lambda_o\}\) covers \(X\), i.e. \(X = \bigcup_{\alpha \in \Lambda_o} G_\alpha\). It follows that \(X - \bigcup_{\alpha \in \Lambda_o} G_\alpha = \phi \in I_f\). Hence \((X, L, \tau, K, I_f)\) is SI-f-compact. Conversely, let \((X, L, \tau, K, I_f)\) is semi-I-compact and let \(\{G_\alpha : \alpha \in \Lambda\}\) be a semi-open cover of \(X\). Then there exists a finite subset \(\Lambda_o\) of \(\Lambda\) such that \(X - \bigcup_{\alpha \in \Lambda_o} G_\alpha \in I_f\). Thus \(X = \bigcup_{\alpha \in \Lambda_o} G_\alpha\). Hence \((X, L, \tau, K)\) is semi-compact.

Theorem 5.6. A ditopological texture space \((X, L, \tau, K)\) is semi-cocompact if and only if \((X, L, \tau, K, I)\) is semi-\(\{\phi\}\)-cocompact.

Proof. The proof is similar to Theorem 5.5.

Theorem 5.7. Let \((X, L, \tau, K, I)\) be a semi-I-compact ditopological texture space and \(J\) be an ideal on \(X\) s.t \(I \subseteq J\). Then \((X, L, \tau, K, J)\) is a semi-J-compact.

Proof. Immediate.

Theorem 5.8. Let \((X, L, \tau, K, I)\) be a semi-I-cocompact ditopological texture space and \(J\) be an ideal on \(X\) s.t \(I \subseteq J\). Then \((X, L, \tau, K, J)\) is a semi-J-cocompact.

Proof. Immediate.

Theorem 5.9. Let \((X, L, \tau, K, I)\) be a ditopological texture space. Then the following are equivalent:

(1) \((X, L, \tau, K)\) is semi-I-cocompact.

(2) For every family \(\{F_j : j \in J\}\) of semi-closed sets of \(X\) for which \(\bigcap\{F_j : j \in J\} = \phi\), there exists \(J_o \subseteq J\) finite s.t \(\bigcap\{F_j : j \in J_o\} \in I\).

Proof.

(1) ⇒ (2) \(\{F_j : j \in J\}\) be a family of semi-closed sets of \(X\) s.t \(\bigcap\{F_j : j \in J\} = \phi\). Then \(\bigcup\{X - F_j : j \in J\}\) is a family of semi-open sets of \(X\) s.t \(X = \bigcup\{X - F_j : j \in J\}\). By (1), there exists \(J_o \subseteq J\) finite s.t \(X - (\bigcup\{X - F_j : j \in J_o\}) \in I\), i.e. \(X - (X \cup (\bigcap_{j \in J_o} F'_j)) = X - (X \cap (\bigcup_{j \in J_o} F'_j)) = X - (X \cap (\bigcap_{j \in J_o} F_j)) = \bigcap_{j \in J_o} F_j \in I\).
(2) $\Rightarrow$ (1) Suppose that \( \{G_j : j \in J\} \) be a family of semi-open cover of \( X \). Then \( \{X - G_j : j \in J\} \) is a family of semi-closed sets of \( X \) with \( \bigcap_{j \in J}(X - G_j) = \phi \). By (2), there exists \( J_o \subseteq J \) finite s.t \( \bigcap_{j \in J_o}(X - G_j) \in I \). Thus \( \bigcap_{j \in J_o}(X - G_j) = X \cap (\bigcup_{j \in J_o}G_j)' = X - \bigcup_{j \in J_o}(G_j) = X - X = \phi \). Then by (2) \( \bigcup_{j \in J_o}(X - G_j) = X \cap (\bigcup_{j \in J_o}G_j) \in I \). Hence \( (X, L, \tau, K) \) is semi-I-cocompact.

**Theorem 5.10.** Let \( (X, L, \tau, K, I) \) be a ditopological texture space. Then the following are equivalent:

1. \((X, L, \tau, K)\) is semi-I-cocompact.
2. For every family \( \{G_\alpha : \alpha \in \Lambda\} \) of semi-open sets of \( X \) for which \( \bigcup_{\alpha \in \Lambda}G_\alpha = X \), there exists \( \alpha_o \in \Lambda \) finite s.t \( \bigcup_{\alpha \in \Lambda}G_\alpha \in I \).

**Proof.** The proof is similar to Theorem 5.9.

6 Conclusion

Topology is an important and major area of mathematics and it can give many relationships between other scientific areas and mathematical models. The notion of a texture space, under the name of fuzzy structure, was introduced by L. M. Brown in [1], as a means of representing a lattice of fuzzy sets as a lattice of crisp subsets of some base set. The notion of a texture space, under the name of fuzzy structure, was introduced by L. M. Brown in [1], as a means of representing a lattice of fuzzy sets as a lattice of crisp subsets of some base set. The notion of semi-open sets in ditopological texture spaces was initiated by S. Dost in [11], which is being extended to a ditopological texture space modulo an ideal in [14]. The purpose of this paper is to generalize the notion of Semi-continuity and MS-continuity and go on to study Semi-compactness, Semi-cocompactness, Semi-stability and Semi-costability in a ditopological texture space. We also extends the notion of Semi-compactness and Semi-cocompactness to a ditopological texture space modulo an ideal [13].

**References**