Positive solutions for boundary value problem of impulsive fractional differential equations

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Received: Sep-18-2014; Accepted: Nov-20-2014 *Corresponding author

Abstract In this paper, we discuss the existence of positive solutions for the boundary value problem of impulsive fractional differential equations

\[
\begin{align*}
^{c}D^{q}u(t) &= f(t, u(t)), \quad 0 < t < 1, t \neq t_{k}, k = 1, 2, \ldots, p. \\
\Delta u|_{t=t_{k}} &= I_{k}(u(t_{k})), \quad \Delta u'|_{t=t_{k}} = T_{k}(u(t_{k})), k = 1, 2, \ldots, p, \\
u(0) &= 0, \quad u'(0) = \xi u'(1).
\end{align*}
\]

where \(^{c}D^{q}\) is Caputo fractional derivative, \(1 < q \leq 2, 0 < \xi < 1\). By means of fixed point theorem of cone expansion and compression, some new results for the boundary value problem are obtained.

Key Words boundary value problem; impulsive fractional differential equations; positive solution

MSC 2010 26A33, 34B25

1 Introduction

Fractional differential equations serve as an excellent apparatus for simulation of process and phenomena observed in the fields of control theory, physics, chemistry, aerodynamics, electrodynamics of a complex medium, engineering, etc. For details, see [8, 10, 15] and references therein. Recently, there are some papers investigate the existence of solution for initial value or boundary value problem of fractional differential equations [1, 5, 17, 18, 23].

The theory of integer order impulsive differential equations is attracting much attention in recent years. This is mostly because they efficiently describe many phenomena arising in engineering, physics, and science [2, 16]. Recently, the boundary value problem (BVP) for integer order differential equations with impulse have been studied extensively in the literatures (see [6, 7, 9, 13, 19, 20, 21]).

However, the research of fractional differential equations with impulse is still in the initial stages. In [14], R. P. Agarwal and M. Benchohra establish sufficient conditions for the existence of solutions of initial problem of impulsive fractional differential equations involving the Caputo fractional derivative of order
0 < q ≤ 1 and 1 < q ≤ 2. In [3, 4], B. Ahmad and S. Sivasundaram give some existence results for two-point BVP and integral boundary value problems involving nonlinear impulsive differential equations of fractional 1 < q ≤ 2. In [22], Y. Tian and Z. Bai used Schauder’s fixed point theorem to study the existence solutions of the three-point impulsive boundary value problem of fractional differential equations.

In this paper, we are concerned with positive solution for BVP of impulsive fractional differential equations

\[
\begin{cases}
  ^cD^q u(t) = f(t, u(t)), & 0 < t < 1, t \neq t_k, k = 1, 2, \ldots, p. \\
  \triangle u|_{t=t_k} = I_k(u(t_k)), \triangle u'|_{t=t_k} = T_k(u(t_k)), k = 1, 2, \ldots, p, \\
  u(0) = 0, u'(0) = \xi u'(1).
\end{cases}
\]

where \(^cD^q\) is the Caputo fractional derivative, 1 < q ≤ 2, 0 < ξ < 1 and f : [0, 1] × [0, ∞) → [0, ∞) is a continuous function. \(I_k, T_k : [0, +\infty) \rightarrow [0, +\infty)\) is continuous function, \(\triangle u(t_k) = u(t_k^+) - u(t_k^-), \triangle u'(t_k) = u'(t_k^+) - u'(t_k^-)\) with \(u(t_k^+) = \lim_{h \rightarrow 0^+} u(t_k + h), u(t_k^-) = \lim_{h \rightarrow 0^-} u(t_k + h), u'(t_k^+) = \lim_{h \rightarrow 0^+} u'(t_k + h), u'(t_k^-) = \lim_{h \rightarrow 0^-} u'(t_k + h), k = 1, 2, \ldots, p\) for \(0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1\).

To the best of our knowledge, there are few papers that consider the positive solutions for boundary value problem of impulsive fractional differential equations. In the present paper, we firstly reduce BVP (1.1) to an equivalent integral equation. Next, by using fixed-point theorems of cone expansion and compression, the existence results for the BVP (1.1) are obtained. Finally, an example is given to illustrate the effect of these results.

2 Preliminary results

Definition 2.1 ([8, 10, 15]). For a function \(f : (0, \infty) \rightarrow R\), the Caputo derivative of fractional order \(q > 0\) is defined as

\[ ^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q-\xi}} ds, n = [q] + 1, \]

where \([q]\) denotes the integer part of real number \(q\), \(\Gamma(\cdot)\) is the (Euler’s) Gamma function defined by \(\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt, \xi > 0\).

Let \(J = [0, 1], J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_p = (t_p, 1], J^c = J \setminus \{t_1, t_2, \ldots, t_p\} \). We define \(PC(J, R) = \{u : J \rightarrow R; u \in C(J^c), u(t_k^-)\text{ exist, and } u(t_k^+) = u(t_k), 1 \leq k \leq p\}\) and \(PC^1(J, R) = \{u \in PC(J, R), u'(t_k^-)\text{ exist, and } u'\text{ is left continuous at } t_k, \text{ for } 1 \leq k \leq p\}\). It is easy to see that \(PC^1(J, R)\) is a Banach space with the norm \(\|u\| = \sup_{t \in J} |u(t)|, |u'(t)|\).

Lemma 2.2. Let \(g(t) \in C[0, 1]\) and \(1 < q \leq 2, 0 < \xi < 1\), the unique solution of

\[
\begin{cases}
  ^cD^q u(t) = g(t), & 0 < t < 1, t \neq t_k, k = 1, 2, \ldots, p. \\
  \triangle u|_{t=t_k} = I_k(u(t_k)), \triangle u'|_{t=t_k} = T_k(u(t_k)), k = 1, 2, \ldots, p, \\
  u(0) = 0, u'(0) = \xi u'(1).
\end{cases}
\]
is
\[
\begin{cases}
\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}g(s)ds + t\Theta, \ t \in J_0, \\
\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}g(s)ds + t\Theta + \sum_{i=1}^{k} \frac{1}{\Gamma(q-i)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1}g(s)ds \\
+ I_i(u(t_i^-))] + \sum_{i=1}^k (t - t_i)[\frac{1}{\Gamma(q-i)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2}g(s)ds + \mathcal{T}_i(u(t_i^-))],
\end{cases}
\tag{2.2}
\]

where
\[
\Theta = \frac{k}{\Gamma(q-1)(1-\xi)} \int_0^1 (1-s)^{q-2}g(s)ds + \frac{k}{1-\xi} \sum_{i=1}^p \frac{1}{\Gamma(q-i)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2}g(s)ds + \mathcal{T}_i(u(t_i^-)).
\]

Lemma 2.3 ([11], ch 2). Let Q be a cone of a Banach space E, and \( \Omega_1, \Omega_2 \) are open subsets of E with \( 0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \). Suppose that \( A : Q \to Q \) is a completely continuous operator such that one of the following two conditions is satisfied:
(i) \( \|Ax\| \leq \|x\| \) for \( x \in Q \cap \partial \Omega_1 \) and \( \|Ax\| \geq \|x\| \) for \( x \in Q \cap \partial \Omega_2 \).
(ii) \( \|Ax\| \geq \|x\| \) for \( x \in Q \cap \partial \Omega_1 \) and \( \|Ax\| \leq \|x\| \) for \( x \in Q \cap \partial \Omega_2 \).
Then, \( A \) has a fixed point \( x \in Q \cap \overline{\Omega}_2 \setminus \Omega_1 \).

3 Main results

We define an operator \( A : PC(J, R) \to PC(J, R) \) by
\[
(Au)(t) = \begin{cases}
(A_0u)(t), \ t \in J_0, \\
(A_ku)(t), \ t \in J_k, k = 1, 2, \ldots, p.
\end{cases}
\tag{3.1}
\]

where
\[
(A_0u)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, u(s))ds + t\Theta,
\]
and
\[
(A_ku)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}f(s, u(s))ds + t\Theta + \sum_{i=1}^{k} \frac{1}{\Gamma(q-i)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1}f(s, u(s))ds \\
+ I_i(u(t_i^-))] + \sum_{i=1}^k (t - t_i)[\frac{1}{\Gamma(q-i)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2}f(s, u(s))ds + \mathcal{T}_i(u(t_i^-))].
\]

It follows that \( u \in PC(J) \) is a solution of BVP (1.1) if and only if \( u \) is a fixed point of the operator \( A \). Therefore, we need to consider only the existence of fixed points for the operator \( A \).

Let us denote the cone \( P = \{u \in PC(J) : u(t) \geq 0, t \in J \} \), we have the following Lemma.

Lemma 3.1. Let \( A, A_k, k = 0, 1, \ldots, p \) be defined by (3.1), and assume that \( u \in P \). Then for \( \forall t \in J_p, (A_pu)(t) \geq \max_{s \in J_p} \{ (A_lu)(s) \}, l = 0, 1, \ldots, p - 1 \).

Lemma 3.2. The operator \( A \) is defined by (3.1). Then \( A : PC(J, R) \to PC(J, R) \) is completely continuous.

Proof. In view of nonnegativeness and continuity of \( f(t, u) \) with respect to \( u \), we can see that \( A \) is continuous. Let \( \Omega \in PC(J, R) \) is bounded. Then, there exist positive constants \( \lambda_i > 0(i = 1, 2, 3) \) such
that that $|f(t, u)| \leq \lambda_1$, $|I_k(u)| \leq \lambda_2$ and $|T_k(u)| \leq \lambda_3$, $\forall u \in \Omega$. Hence $\forall u \in \Omega$, we have

$$
(A u)(t) \leq \frac{1}{\Gamma(q)} \int_1^t (1-s)^{q-1} |f(s, u(s))| ds + t(1-s)^{q-2} |f(s, u(s))| ds \\
+ \frac{\xi}{1-\xi} \sum_{i=1}^p I_i(t_i) (1-t_i)^{q-2} |f(s, u(s))| ds + \|T_i(u(t_i^-))\|
$$

which implies that there exists $\lambda > 0$ such that

$$
\|(Au)(t)\| \leq \lambda.
$$

Applying the above method, we can show that there exists $\overline{X} > 0$ such that

$$
|(Au)'(t)| \leq \overline{X}.
$$

Therefore, for $t', t'' \in J_p, t' < t''$, we have

$$
|(Au)(t'') - (Au)(t')| \leq \int_{t'}^{t''} |(Au)'(s)| ds \leq \overline{X}(t'' - t').
$$

This implies that $A$ is uniformly bounded and is equi-continuity on all $J_p$. The Arzela-Ascoli theorem implies that $A$ is completely continuous.

**Theorem 3.3.** Suppose that there exist constants $r_2 > r_1 > 0, M, N \geq 0, \alpha_i, \beta_i, \gamma_i, \mu_i \geq 0, i = 1, 2, \ldots, p$ such that the following conditions hold:

$(H_1)$ $f(t, u) \geq M r_1, I_k(u) \geq \alpha_i r_1, T_k(u) \geq \beta_i r_1$ for $(t, u) \in [0, 1] \times [0, r_1], i = 1, 2, \ldots, p$;

$(H_2)$ $f(t, u) \leq N r_2, I_k(u) \leq \gamma_i r_2, T_k(u) \leq \mu_i r_2$ for $(t, u) \in [0, 1] \times [0, r_2], i = 1, 2, \ldots, p$;

$(H_3)$

$$
\frac{M}{\Gamma(q+1)} \sum_{i=1}^{p+1} \Delta^{q-1} + \frac{M}{\Gamma(q)} \sum_{i=1}^p (\frac{1}{1-\xi} - t_i) \Delta^{q-1} + \frac{\xi M}{\Gamma(q)(1-\xi)} \Delta^{q-1} + \sum_{i=1}^p \alpha_i + \frac{1}{1-\xi} - t_i) \beta_i \geq 1;
$$

$(H_4)$

$$
\frac{N}{\Gamma(q+1)} \sum_{i=1}^{p+1} \Delta^{q-1} + \frac{N}{\Gamma(q)} \sum_{i=1}^p (\frac{1}{1-\xi} - t_i) \Delta^{q-1} + \frac{\xi N}{\Gamma(q)(1-\xi)} \Delta^{q-1} + \sum_{i=1}^p \gamma_i + \frac{1}{1-\xi} - t_i) \mu_i \leq 1,
$$

where $\Delta_l = t_{l+1} - t_l, l = 0, 1, \ldots, p$. Then BVP (1.1) has at least one positive solution.

**Proof.** Let $\Omega_{r_1} = \{ u \in PC(J) : \| u \| \leq r_1 \}$, suppose that condition $(H_1)$ holds, for $u \in P \cap \partial \Omega_{r_1}$,
we have \(0 \leq u(t) \leq r_1\) for \(t \in J\). Then from \((H_4)\), we have for \(u \in P \cap \partial \Omega_{r_1}\)

\[
(A_p \ u)(t) = \frac{1}{\Gamma(q)} \int_0^t (1-s)^{q-1} f(s, u(s))ds + \frac{\xi}{\Gamma(q-1)(1-\xi)} \int_{t_p}^1 (1-s)^{q-2} f(s, u(s))ds \\
+ \frac{p}{\Gamma(q-1)} \sum_{i=1}^p \frac{1}{1-i} \int_{t_{i-1}}^{t_{i-1}} (t_i - s)^{q-2} f(s, u(s))ds + I_i(u(t_{i-1})) \\
+ \frac{p}{\Gamma(q)} \sum_{i=1}^p (1-t_i) \frac{1}{1-i} \int_{t_{i-1}}^{t_{i-1}} (t_i - s)^{q-2} g(s)ds + I_i(u(t_{i-1})) \\
\geq \frac{M^p}{\Gamma(q+1)} \int_0^1 (1-s)^{q-1} ds + \frac{\xi M^p}{\Gamma(q-1)(1-\xi)} \int_{t_p}^1 (1-s)^{q-2} ds \\
+ \frac{p}{\Gamma(q)} \sum_{i=1}^p \frac{1}{M^p} \int_{t_{i-1}}^{t_{i-1}} (t_i - s)^{q-2} ds + \beta_i r_1 + \frac{\xi M^p}{\Gamma(q)} \sum_{i=1}^p \frac{1}{1-i} \int_{t_{i-1}}^{t_{i-1}} (t_i - s)^{q-1} ds + \alpha_i r_1 \\
+ \frac{p}{\Gamma(q)} \sum_{i=1}^p (1-t_i) \frac{1}{M^p} \int_{t_{i-1}}^{t_{i-1}} (t_i - s)^{q-2} ds + \beta_i r_1 \\
= r_1 \left\{ \frac{M^p}{\Gamma(q+1)} \sum_{i=1}^p \Delta q_{i-1} + \frac{M^p}{\Gamma(q)} \sum_{i=1}^p \frac{1}{1-i} - t_i \Delta q_{i-1} + \frac{\xi M^p}{\Gamma(q)} \Delta q_{i-1} \\
+ \frac{p}{\Gamma(q)} \sum_{i=1}^p [\alpha_i + (1-t_i) \beta_i] \right\} \geq r_1 = \| u \|.
\]

It follows that for \(u \in P \cap \partial \Omega_{r_1}\)

\[
\| Au \| \geq \| u \|.
\]

On the other hand, let \(\Omega_{r_2} = \{ u \in PC(J) : \| u \| \leq r_2 \}\), suppose that condition \((H_2)\) holds, then for \(u \in P \cap \partial \Omega_{r_2}\), we have \(0 \leq u(t) \leq r_2\) for \(t \in J\). It follows from \((H_4)\) that for \(u \in P \cap \partial \Omega_{r_2}, t \in J_p\)

\[
(A_p \ u)(t) = \frac{1}{\Gamma(q)} \int_0^t (1-s)^{q-1} f(s, u(s))ds + \frac{\xi}{\Gamma(q-1)(1-\xi)} \int_{t_p}^1 (1-s)^{q-2} f(s, u(s))ds \\
+ \frac{p}{\Gamma(q-1)} \sum_{i=1}^p \frac{1}{1-i} \int_{t_{i-1}}^{t_{i-1}} (t_i - s)^{q-2} f(s, u(s))ds + I_i(u(t_{i-1})) \\
+ \frac{p}{\Gamma(q)} \sum_{i=1}^p (t - t_i) \frac{1}{1-i} \int_{t_{i-1}}^{t_{i-1}} (t_i - s)^{q-2} g(s)ds + I_i(u(t_{i-1})) \\
\leq \frac{M^p}{\Gamma(q+1)} \int_0^1 (1-s)^{q-1} ds + \frac{\xi N r_2}{\Gamma(q-1)(1-\xi)} \int_{t_p}^1 (1-s)^{q-2} ds \\
+ \frac{p}{\Gamma(q)} \sum_{i=1}^p \frac{N r_2}{\Gamma(q)} \int_{t_{i-1}}^{t_{i-1}} (t_i - s)^{q-2} ds + \mu_i r_2 + \frac{\xi N}{\Gamma(q)} \int_{t_{i-1}}^{t_{i-1}} (t_i - s)^{q-1} ds + \gamma_i r_2 \\
+ \frac{p}{\Gamma(q)} \sum_{i=1}^p (1-t_i) \frac{N r_2}{\Gamma(q)} \int_{t_{i-1}}^{t_{i-1}} (t_i - s)^{q-2} ds + \mu_i r_2 \\
= r_2 \left\{ \frac{N^p}{\Gamma(q+1)} \sum_{i=1}^p \Delta q_{i-1} + \frac{N^p}{\Gamma(q)} \sum_{i=1}^p \frac{1}{1-i} - t_i \Delta q_{i-1} + \frac{\xi N}{\Gamma(q)} \Delta q_{i-1} \\
+ \frac{p}{\Gamma(q)} \sum_{i=1}^p [\gamma_i + (1-t_i) \mu_i] \right\} \leq r_2 = \| u \|.
\]

In view of Lemma 3.1, we have

\[
\| Au \| \leq \| u \|, u \in P \cap \partial \Omega_{r_2}.
\]

Consequently, by Lemma 2.3, BVP (1.1) has at least one positive solution in \(P \cap (\Omega_{r_2} \setminus \Omega_{r_1})\).

**Theorem 3.4.** Suppose that there exist constants \(r_1 > 0, M > 0, \eta_i > 0, \tau_i > 0, i = 1, 2, \ldots, p\) such that the following conditions hold

\((H_5)\) \(f(t, u) \geq M r_1, I_i(u) \geq \alpha_i r_1, \mathcal{I}_i(u) \geq \beta_i r_1\) for \((t, u) \in [0,1] \times [0, r_1], i = 1, 2, \ldots, p;\)
\((H_6)\) \(\lim_{u \to +\infty} \max_{t \in [0,1]} \frac{f(t, u)}{u} \leq M, \lim_{u \to +\infty} \frac{I_i(u)}{u} \leq \eta_i, \lim_{u \to +\infty} \frac{T_i(u)}{u} \leq \tau_i, i = 1, 2, \ldots, p;\)

\((H_7)\)

\[
\frac{M}{G(q+1)} \sum_{i=1}^{p+1} \Delta q_{i-1} + \frac{M}{G(q)} \sum_{i=1}^{p} \left( \frac{1}{1-\xi} - t_i \right) \Delta q_{i-1} + \frac{\xi M}{G(q)(1-\xi)} \Delta q_{i-1} + \sum_{i=1}^{p} [\alpha_i + \left( \frac{1}{1-\xi} - t_i \right) \beta_i] \geq 1;
\]

\((H_8)\)

\[
\frac{M}{G(q+1)} \sum_{i=1}^{p+1} \Delta q_{i-1} + \frac{M}{G(q)} \sum_{i=1}^{p} \left( \frac{1}{1-\xi} - t_i \right) \Delta q_{i-1} + \frac{\xi M}{G(q)(1-\xi)} \Delta q_{i-1} + \sum_{i=1}^{p} [\eta_i + \left( \frac{1}{1-\xi} - t_i \right) \tau_i] \leq 1.
\]

Then BVP (1.1) has at least one positive solution.

**Proof.** According to the proof of the Theorem 3.1, we have for \(u \in P \cap \partial \Omega_r,\)

\[\|Au\| \geq \|u\|, u \in P \cap \partial \Omega_r.\]

Next, we show that there exists \(R > r_1\) such that for \(u \in P \cap \partial \Omega_R\)

\[\|Au\| \leq \|u\|, u \in P \cap \partial \Omega_R.\]

Since condition \((H_6)\) holds, there exist \(\varepsilon \in (0, \theta_1)\) and \(H > r_1\) such that

\[f(t, u) \leq (M - \varepsilon)u, \text{ for } t \in J, u \geq H\]

and

\[I_i(u) \leq (\eta_i - \varepsilon)u, \quad T_i(u) \leq (\tau_i - \varepsilon)u, \text{ for } u \geq H, i = 1, 2, \ldots, p\]

where \(\theta_1 = \min\{M, \eta_1, \ldots, \eta_p, \tau_1, \ldots, \tau_p\}.\) If \(\max f(t, u), I_i(u), T_i(u), i = 1, 2, \ldots, p\) are bounded for \(u \in [0, +\infty),\) that is to say that there exist \(N_1, N_2, N_3 > 0\) such that for all \(u \in [0, +\infty)\)

\[f(t, u) \leq N_1, I_i(u) \leq N_2, T_i(u) \leq N_3, i = 1, 2, \ldots, p.\]

Let \(R_1 > \max\{H, \Phi\},\) where

\[\Phi = N_1 \frac{N}{(q+1)} \sum_{i=1}^{p+1} \Delta q_{i-1} + \frac{\xi N_1 q}{G(q)(1-\xi)} \Delta q_{i-1} + \frac{N_2 q}{G(q)} \sum_{i=1}^{p} \left( \frac{1}{1-\xi} - t_i \right) \Delta q_{i-1} + pN_2 + N_3 \sum_{i=1}^{p} \left( \frac{1}{1-\xi} - t_i \right).\]

For \(u \in P \cap \partial \Omega_{R_1}, t \in J_p,\) from the above inequality, we have

\[(A_p u)(t) = \frac{1}{G(q)} \int_t^1 (t-s)^{q-1} f(s, u(s))ds + t\frac{\xi}{G(q)(1-\xi)} \int_0^1 (1-s)^{q-2} f(s, u(s))ds + \frac{\xi N}{G(q)(1-\xi)} \int_0^1 (1-s)^{q-2} f(s, u(s))ds + I_i(u(t^*_i)) + T_i(u(t^*_i)) + \sum_{i=1}^{p} (t - t_i) \left[ \frac{N}{G(q-1)} \int_{t_i}^1 (t-s)^{q-2} f(s, u(s))ds + T_i(u(t^-_i)) \right] \leq \frac{N_1}{G(q)} \int_0^1 (1-s)^{q-1} ds + \frac{\xi N_1}{G(q)(1-\xi)} \int_0^1 (1-s)^{q-2} ds
\]

\[+ \frac{\xi N_2}{G(q-1)} \int_{t_i}^1 (t-s)^{q-2} ds + N_3] + \sum_{i=1}^{p} \left[ \frac{N}{G(q)} \int_{t_i}^1 (t-s)^{q-1} ds + N_2 \right]
\]

\[+ \sum_{i=1}^{p} (1 - t_i) \left[ \frac{N}{G(q-1)} \int_{t_i}^1 (t-s)^{q-2} ds + N_3 \right]
\]

\[= \frac{N_1}{G(q+1)} \sum_{i=1}^{p+1} \Delta q_{i-1} + \frac{\xi N_1 q}{G(q)(1-\xi)} \Delta q_{i-1} + \frac{N_2 q}{G(q)} \sum_{i=1}^{p} \left( \frac{1}{1-\xi} - t_i \right) \Delta q_{i-1}
\]

\[+ pN_2 + N_3 \sum_{i=1}^{p} \left( \frac{1}{1-\xi} - t_i \right) \leq R_1 = \|u\|.
\]

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If \( \max_{t \in J} f(t, u), I_i(u), T_i(u), i = 1, 2, \ldots, p \) are unbounded for \( u \in [0, +\infty) \), then there exists \( R_2 > H \) such that \( u \in [0, R_2] \)

\[
f(t, u) \leq \max_{t \in J} I_i(u) \leq I_i(R_2), T_i(u) \leq T_i(R_2), i = 1, 2, \ldots, p.
\] (3.4)

It follows from (3.2) (3.3) and (3.4) that for \( u \in P \cap \partial\Omega_{R_2}, t \in J_p \)

\[
(A_p u)(t) = \frac{1}{\Gamma(\xi)} \int_{t}^{1} (1-s)^{\xi-1} f(s, u(s)) ds + \frac{\epsilon}{\Gamma(\xi)} \sum_{i=1}^{p} \frac{1}{(1-q)^{\xi-i}} \int_{t_i}^{1} (1-s)^{\xi-2} f(s, u(s)) ds + \frac{\epsilon}{\Gamma(\xi)} \sum_{i=1}^{p} \sum_{s \in J} (t-s)^{\xi-2} f(s, u(s)) ds + I_i(u(t^-_i))
\]

\[
+ \frac{\epsilon}{\Gamma(\xi)} \sum_{i=1}^{p} \sum_{s \in J} (t-s)^{\xi-2} f(s, u(s)) ds + I_i(u(t^-_i))
\]

\[
\leq \frac{1}{\Gamma(\xi)} \int_{t}^{1} (1-s)^{\xi-1} f(s, R_2) ds + \frac{\epsilon}{\Gamma(\xi)} \sum_{i=1}^{p} \sum_{s \in J} (t-s)^{\xi-2} f(s, R_2) ds + I_i(R_2)
\]

If \( \max_{t \in J} f(t, u), I_i(u), T_i(u), i = 1, 2, \ldots, p \) are not all bounded or unbounded for \( u \in [0, +\infty) \), similar to the above method, we can show that there exists \( R_3 > r_1 \) such that \( u \in P \cap \partial\Omega_{R_3}, t \in J_p \)

\[
(A_p u)(t) \leq \|u\|.
\]

In view of Lemma 3.1, we see that there exists \( R > r_1 \) such that for \( u \in P \cap \partial\Omega_R \)

\[
\|Au\| \leq \|u\|.
\]

So, by Lemma 2.3, BVP (1.1) has at least one positive solution in \( P \cap \overline{\Omega}_R \setminus \Omega_{r_1} \).

**Remark 3.5.** If we replace condition \((H_6)\) and \((H_5)\) with

\[
(H_6)' \lim_{u \to \infty} \sup_{t \in [0,1]} \frac{f(t, u)}{u} = 0, \lim_{u \to \infty} \frac{I_i(u)}{u} = 0, \lim_{u \to \infty} \frac{T_i(u)}{u} = 0, i = 1, 2, \ldots, p,
\]

the conclusion of the above theorems is valid.

**Theorem 3.6.** Suppose that

\[
(H_9) \text{ There exists a constant } L_1 > 0 \text{ such that } |f(t, x) - f(t, y)| \leq L_1 |x - y|, \text{ for } t \in J \text{ and } x, y \in R.
\]
(H\textsubscript{10}) There exist constants $L_2, L_3 > 0$ such that $|I_k(x) - I_k(y)| \leq L_2|x - y|, |\overline{T}_i(x) - \overline{T}_i(y)| \leq L_3|x - y|$, for $t \in J$ and $x, y \in R, i = 1, 2, \ldots, p$.

If

$$
\frac{L_1}{\Gamma(q+1)} \sum_{i=1}^{p+1} \Delta^{q}_{-1} + \frac{\xi L_1}{\Gamma(q)(1 - \xi)} \Delta^{q-1} + \frac{L_2}{\Gamma(q)} \sum_{i=1}^{p} \left( \frac{1}{1 - \xi} - t_i \right) \Delta^{q-1}_{-1} + pL_2 + L_3 \sum_{i=1}^{p} \left( \frac{1}{1 - \xi} - t_i \right) < 1,
$$

(3.5)

Then BVP (1.1) has a unique positive solution.

**Proof.** Let $x, y \in P$. Then for $t \in J_p$, we get

$$
|\langle A_p \rangle x \rangle(t) - \langle A_p y \rangle(t)| \leq \frac{1}{1 - \xi} \int_{t_p}^{1} (1 - s)^{q-1} |f(s, x(s)) - f(s, y(s))| ds
$$

$$
+ \frac{\xi}{\Gamma(q)(1 - \xi)} \int_{t_p}^{1} (1 - s)^{q-2} |f(s, x(s)) - f(s, y(s))| ds
$$

$$
+ \frac{L_2}{\Gamma(q)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} |f(s, x(s)) - f(s, y(s))| ds + |\overline{T}_i(x(t_i^-)) - \overline{T}_i(y(t_i^-))|)
$$

$$
+ \sum_{i=1}^{p} (t - t_i) \left( \frac{1}{\Gamma(q-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} |f(s, x(s)) - f(s, y(s))| ds + |I_i(x(t_i^-)) - I_i(y(t_i^-))|)
$$

$$
\leq \left\{ \frac{L_1}{\Gamma(q)} \int_{t_p}^{1} (1 - s)^{q-1} ds + \frac{\xi L_1}{\Gamma(q)(1 - \xi)} \int_{t_p}^{1} (1 - s)^{q-2} ds + \frac{L_2}{\Gamma(q)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} ds
$$

$$
+ L_3 \right) + \sum_{i=1}^{p} \int_{t_p}^{t_i} (t_i - s)^{q-3} ds + L_2 + \sum_{i=1}^{p} (1 - t_i) \left( \frac{L_1}{\Gamma(q)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} ds + L_3 \right) \|x - y\|
$$

$$
= \left[ \frac{L_1}{\Gamma(q+1)} \sum_{i=1}^{p+1} \Delta^{q}_{-1} + \frac{\xi L_1}{\Gamma(q)(1 - \xi)} \Delta^{q-1} + \frac{L_2}{\Gamma(q)} \sum_{i=1}^{p} \left( \frac{1}{1 - \xi} - t_i \right) \Delta^{q-1}_{-1}
$$

$$
+ pL_2 + L_3 \sum_{i=1}^{p} \left( \frac{1}{1 - \xi} - t_i \right) \|x - y\| \right) \|x - y\|
$$

In view of Lemma 3.1, we have

$$
\|Ax - Ay\| \leq \left[ \frac{L_1}{\Gamma(q+1)} \sum_{i=1}^{p+1} \Delta^{q}_{-1} + \frac{\xi L_1}{\Gamma(q)(1 - \xi)} \Delta^{q-1} + \frac{L_2}{\Gamma(q)} \sum_{i=1}^{p} \left( \frac{1}{1 - \xi} - t_i \right) \Delta^{q-1}_{-1}
$$

$$
+ pL_2 + L_3 \sum_{i=1}^{p} \left( \frac{1}{1 - \xi} - t_i \right) \|x - y\| \right) \|x - y\|
$$

Since (3.5) holds, consequently $A$ is a contraction. We deduce the conclusion of the theorem by the contraction mapping principle.

**Remark 3.7.** The previous theorems are proved by using the operator theory in space with cones. However, in the proof of Theorem 3.3, the contraction mapping principle is used which is distinct from the previous theorems.

As an application of our results, we consider the following BVP

\[ cD^\frac{1}{2} u(t) = \frac{1}{20} (u^2 + \sin^2 t + 1), \quad 0 < t < 1, t \neq \frac{1}{2}, \]

\[ \Delta u \left( \frac{1}{2} \right) = \frac{1}{10} u \left( \frac{1}{2} \right) + \frac{1}{10}, \quad \Delta u' \left( \frac{1}{2} \right) = \frac{1}{20} u \left( \frac{1}{2} \right) + \frac{1}{20}, \]

\[ u(0) = 0, u'(0) = \frac{1}{2} u'(1). \]

where $f(t, u) = \frac{1}{20} (u^2 + \sin^2 t + 1), q = \frac{1}{2}, \xi = \frac{1}{2}, t_1 = \frac{1}{2}, I_1(u) = \frac{1}{20} u + \frac{1}{10}, \overline{T}_1(u) = \frac{1}{20} u + \frac{1}{10}$. Choosing $r_1 = \frac{1}{10}, r_2 = 1$, we take $M = \frac{1}{2}, \alpha_1 = \beta_1 = 1, N = \frac{1}{20}, \gamma_1 = \frac{1}{2}, \mu_1 = \frac{1}{2}(\beta_1) = 1$ by a simple computation. Then it follows that

\[ f(t, u) \geq M r_1, I_1(u) \geq \alpha_1 r_1, \overline{T}_1(u) \geq \beta_1 r_1, \quad \text{for} \ (t, u) \in J \times \left[ 0, \frac{1}{10} \right], \]

\[ f(t, u) \leq N r_2, I_1(u) \leq \gamma_1 r_2, \overline{T}_1(u) \leq \mu_1 r_2, \quad \text{for} \ (t, u) \in J \times [0, 1]. \]
We see that $(H_2)$ and $(H_3)$ hold. By Theorem 3.1, we conclude that BVP (3.6) has at least one solution.

References