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New hypergeometric summation formulae arising from the summation formulae of Prudnikov

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Abstract In this paper, we aim to establish the explicit expression of

$$ {}_2F_1\left[\begin{matrix} a, & n-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] $$

for $n = 7, 8, 9, 10, 11$ & 12. For $n = 0, 1, 2, 3, 4, 5, 6$, the results were established by Prudnikov et al[2,p.414].The results are derived with the help of Contiguous relation[1,p.558]and the result from Prudinov et al[2,p. 414].The results established in this research paper are simple,interesting and may be potentially useful.

Key Words Contiguous relation, Summation formulae
MSC 2010 33C05, 33C20

1 Introduction and Results Required

Special functions and their applications are now awe-inspiring in their scope, variety and depth. Not only in their rapid growth in pure Mathematics and its applications to the traditional fields of Physics, Engineering and Statistics but in new fields of applications like Behavioral Science, Optimization, Biology, Environmental Science and Economics, etc. they are emerging. Summation formulae for hypergeometric function has an important role in applied mathematics.

Prudnikov et al[2,p.414] derived the following seven summation formulae

$$ {}_2F_1\left[\begin{matrix} a, & -a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = \frac{\sqrt{\pi}\,\Gamma(c)}{2^c}\Big[\frac{1}{\Gamma(\frac{c+a+1}{2})\,\Gamma(\frac{c-a}{2})} + \frac{1}{\Gamma(\frac{c+a}{2})\,\Gamma(\frac{c-a+1}{2})}\Big] \tag{1} $$

$$ {}_2F_1\left[\begin{matrix} a, & 1-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-1}}\Big[\frac{1}{\Gamma(\frac{c+a}{2})\,\Gamma(\frac{c-a+1}{2})}\Big] \tag{2} $$

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$$ {}_2F_1\left[\begin{matrix} a, & 2-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = \frac{\sqrt{\pi}\,\Gamma(c)}{(a-1)\,2^{c-2}}\Big[\frac{1}{\Gamma(\frac{c+a-2}{2})\,\Gamma(\frac{c-a+1}{2})} - \frac{1}{\Gamma(\frac{c+a-1}{2})\,\Gamma(\frac{c-a}{2})}\Big] \tag{3} $$

$$ {}_2F_1\left[\begin{matrix} a, & 3-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = \frac{\sqrt{\pi}\,\Gamma(c)}{(a-1)(a-2)\,2^{c-3}}\Big[\frac{(c-2)}{\Gamma(\frac{c+a-2}{2})\,\Gamma(\frac{c-a+1}{2})} - \frac{2}{\Gamma(\frac{c+a-3}{2})\,\Gamma(\frac{c-a}{2})}\Big] \tag{4} $$

$$ {}_2F_1\left[\begin{matrix} a, & 4-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = \frac{\sqrt{\pi}\,\Gamma(c)}{(1-a)(2-a)(3-a)\,2^{c-4}}\Big[\frac{(a-2c+3)}{\Gamma(\frac{c+a-4}{2})\,\Gamma(\frac{c-a+1}{2})} + \frac{(a+2c-7)}{\Gamma(\frac{c+a-3}{2})\,\Gamma(\frac{c-a}{2})}\Big] \tag{5} $$

$$ {}_2F_1\left[\begin{matrix} a, & 5-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-5}\{\prod\limits_{\gamma=1}^{4}(\gamma-a)\}}\Big[\frac{\{2(c-2)(c-4)-(a-1)(a-4)\}}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-4}{2})} + \frac{(12-4c)}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-5}{2})}\Big] \tag{6} $$

$$ {}_2F_1\left[\begin{matrix} a, & 6-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = $$

$$ = \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-6}\{\prod\limits_{\delta=1}^{5}(\delta-a)\}}\Big[\frac{(4c^2+2ac-a^2-a-34c+62)}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-5}{2})} - \frac{(4c^2-2ac-a^2+13a-22c+20)}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-6}{2})}\Big] \tag{7} $$

The contiguous relation is defined as Abramowitz et al[1,p.558]

$$ b\;{}_2F_1\left[\begin{matrix} a, & b+1 & ; & z \\ c & & ; & \end{matrix}\right] = (b-c+1)\;{}_2F_1\left[\begin{matrix} a, & b & ; & z \\ c & & ; & \end{matrix}\right] + (c-1)\;{}_2F_1\left[\begin{matrix} a, & b & ; & z \\ c-1 & & ; & \end{matrix}\right] \tag{8} $$

2 Main Summation Formulae

$$ {}_2F_1\left[\begin{matrix} a, & 7-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = $$

$$ = \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-7}\{\prod\limits_{\varsigma=1}^{6}(\varsigma-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-6}{2})}(-3a^2c+12a^2+21ac-84a+4c^3-48c^2+158c-120)+ $$

$$ +\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-7}{2})}(2a^2-14a-8c^2+64c-108)\Big] \tag{9} $$

$$ {}_2F_1\left[\begin{matrix} a, & 8-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = $$

$$ = \frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-8}\{\prod\limits_{\xi=1}^{7}(\xi-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-7}{2})}(-a^3-4a^2c+30a^2+4ac^2-4ac-107a+8c^3-124c^2+576c-762)+ $$

$$ +\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-8}{2})}(-a^3+4a^2c-6a^2+4ac^2-68ac+181a-8c^3+92c^2-288c+210)\Big] \tag{10} $$

$$
{}_2F_1\left[\begin{array}{lll} a, & 9-a & ; & \frac{1}{2} \\ c & & ; & \end{array}\right]=
$$

$$
=\frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-9}\{\prod_{\varpi=1}^{8}(\varpi-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-8}{2})}(a^4-18a^3-8a^2c^2+80a^2c-85a^2+72ac^2-720ac+1494a+8c^4-
$$

$$
-160c^3+1056c^2-2560c+1680)+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-9}{2})}(8a^2c-40a^2-72ac+360a-16c^3+240c^2-1072c+1360)\Big] \tag{11}
$$

$$
{}_2F_1\left[\begin{array}{lll} a, & 10-a & ; & \frac{1}{2} \\ c & & ; & \end{array}\right]=
$$

$$
=\frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-10}\{\prod_{\upsilon=1}^{9}(\upsilon-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-10}{2})}(-a^4-4a^3c+42a^3+12a^2c^2-72a^2c-107a^2+8ac^3-252ac^2+
$$

$$
+1772ac-3054a-16c^4+312c^3-2000c^2+4704c-3024)+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-9}{2})}(a^4-4a^3c+2a^3-12a^2c^2+192a^2c-
$$

$$
-553a^2+8ac^3-12ac^2-868ac+3406a+16c^4-392c^3+3320c^2-11224c+12264)\Big] \tag{12}
$$

$$
{}_2F_1\left[\begin{array}{lll} a, & 11-a & ; & \frac{1}{2} \\ c & & ; & \end{array}\right]=
$$

$$
=\frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-11}\{\prod_{\varphi=1}^{10}(\varphi-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-10}{2})}(5a^4c-30a^4-110a^3c+660a^3-20a^2c^3+360a^2c^2-1305a^2c-
$$

$$
-810a^2+220ac^3-3960ac^2+21010ac-31020a+16c^5-480c^4+5240c^3-25200c^2+50544c-30240)+
$$

$$
+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-11}{2})}(-2a^4+44a^3+24a^2c^2-288a^2c+530a^2-264ac^2+3168ac-8492a-32c^4+768c^3-6352c^2+
$$

$$
+20928c-22320)\Big] \tag{13}
$$

$$
{}_2F_1\left[\begin{array}{lll} a, & 12-a & ; & \frac{1}{2} \\ c & & ; & \end{array}\right]=
$$

$$
=\frac{\sqrt{\pi}\,\Gamma(c)}{2^{c-12}\{\prod_{\chi=1}^{11}(\chi-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\,\Gamma(\frac{c+a-12}{2})}(a^5-6a^4c+9a^4-12a^3c^2+300a^3c-1103a^3+32a^2c^3-
$$

$$
-408a^2c^2+46a^2c+6351a^2+16ac^4-800ac^3+10364ac^2-46852ac+62182a-32c^5+944c^4-10112c^3+47656c^2-
$$

$$
-93776c+55440)+\frac{1}{\Gamma(\frac{c-a}{2})\,\Gamma(\frac{c+a-11}{2})}(a^5+6a^4c-69a^4-12a^3c^2+12a^3c+769a^3-32a^2c^3+840a^2c^2-
$$

$$
-5662a^2c+8301a^2+16ac^4-32ac^3-4612ac^2+42380ac-96002a+32c^5-1136c^4+15104c^3-
$$

$$
-92536c^2+255392c-245640)\Big] \tag{14}
$$

3 Derivation of the Main Forumlae

Derivation of formula (9). Putting $b = (6-a)$ and $z = \frac{1}{2}$ in (8), we have

$$(6-a)\ {}_2F_1\left[\begin{matrix} a, & 7-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] = (7-c-a)\ {}_2F_1\left[\begin{matrix} a, & 6-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] + (c-1)\ {}_2F_1\left[\begin{matrix} a, & 6-a & ; & \frac{1}{2} \\ c-1 & & ; & \end{matrix}\right]$$

Now using(9), we have

$$(6-a)\ {}_2F_1\left[\begin{matrix} a, & 7-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] =$$

$$= (7-c-a)\frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-6}\{\prod_{\delta=1}^{5}(\delta-a)\}}\Big[\frac{(4c^2+2ac-a^2-a-34c+62)}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a-5}{2})} - \frac{(4c^2-2ac-a^2+13a-22c+20)}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a-6}{2})}\Big]+$$

$$+\frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-7}\{\prod_{\delta=1}^{5}(\delta-a)\}}\Big[\frac{(a^2+2ac-15a-4c^2+30c-46)}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a-7}{2})} + \frac{(-a^2+2ac-3a+4c^2-42c+100)}{\Gamma(\frac{c-a-1}{2})\ \Gamma(\frac{c+a-6}{2})}\Big] =$$

$$= \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-7}\{\prod_{\delta=1}^{5}(\delta-a)\}}\Big[\frac{-(4c^2-34c+2ac-a^2-a+62)}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a-7}{2})} + \frac{(-a^3-3a^2c+20a^2+2ac^2+5ac-71a)}{2\ \Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a-6}{2})}+$$

$$+\frac{(4c^3-50c^2+174c-140)}{2\ \Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a-6}{2})}\Big] + \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-7}\{\prod_{\delta=1}^{5}(\delta-a)\}}\Big[\frac{(a^2+2ac-15a-4c^2+30c-46)}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a-7}{2})}+$$

$$+\frac{(a^3-3a^2c+4a^2-2ac^2+37ac-97a+4c^3-46c^2+142c-100)}{2\ \Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a-6}{2})}\Big]$$

$$= \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-7}\{\prod_{\delta=1}^{5}(\delta-a)\}}\Big[\frac{(2a^2-14a-8c^2+64c-108)}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a-7}{2})} + \frac{(-3a^2c+12a^2+21ac-84a+4c^3)}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a-6}{2})}+$$

$$+\frac{(-48c^2+158c-120)}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a-6}{2})}\Big]$$

Now dividing both sides by $(6-a)$, we get

$${}_2F_1\left[\begin{matrix} a, & 7-a & ; & \frac{1}{2} \\ c & & ; & \end{matrix}\right] =$$

$$= \frac{\sqrt{\pi}\ \Gamma(c)}{2^{c-7}\{\prod_{\varsigma=1}^{6}(\varsigma-a)\}}\Big[\frac{1}{\Gamma(\frac{c-a+1}{2})\ \Gamma(\frac{c+a-6}{2})}(-3a^2c+12a^2+21ac-84a+4c^3-48c^2+158c-120)+$$

$$+\frac{1}{\Gamma(\frac{c-a}{2})\ \Gamma(\frac{c+a-7}{2})}(2a^2-14a-8c^2+64c-108)\Big].$$

On the similar way , other formulae can be derived.

References

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