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# Some common fixed point theorem for two, three and four mappings in Menger spaces

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**Abstract** The purpose of this article is to prove some common fixed point for a pair of self mappings under strict contractive and owc condition under the concept of menger space. Also we have proved, some common fixed point theorem for quadruplet of self mappings in menger space.

**Key Words** Common fixed point, Weak compatibility, Menger space, Occasionally weakly compatibility
**MSC 2010** 47H10, 54H25

## 1 Introduction and Preliminaries

There have been lots of generalizations of metric space. One such generalization is Menger space in which, used distribution functions instead of nonnegative real numbers as value of metric. A Menger space is a space in which the concept of distance is considered to be a probabilistic, rather than deterministic. For detail discussion of Menger spaces and their applications we refer to Schweizer and Sklar [91]. The theory of Menger space is fundamental importance in probabilistic functional analysis.

A probabilistic metric space shortly PM-Space, is an ordered pair (X, F) consisting of a non empty set X and a mapping F from $X \times X$ to $L$, where $L$ is the collection of all distribution functions (a distribution function $F$ is non decreasing and left continuous mapping of reals in to [0,1] with properties, $\inf F(x) = 0$ and $\sup F(x) = 1$). The value of $F$ at $(u, v) \in X \times X$ is represented by $F_{u,v}$. The function $F_{u,v}$ are assumed satisfy the following conditions;

(a) $F_{u,v}(x) = 1$, for all $x > 0$, iff $u = v$;
(b) $F_{u,v}(0) = 0$, if $x = 0$;
(c) $F_{u,v}(x) = F_{v,u}(x)$;
(d) $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x + y) = 1$.

A mapping $t : 0, 1 \times 0, 1 \to [0, 1]$ is a $t$-norm, if it satisfies the following conditions;

(e) $t(a, 1) = a$ for every $a \in [0, 1]$;
(f) $t(0, 0) = 0$,

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(g) $t(a, b) = t(b, a)$ for every $a, b \in [0, 1]$;
(h) $t(c, d) \geqslant ta, b$ for $c \geqslant a$ and $d \geqslant b$
(i) $t(t(a, b), c) = t(a, t(b, c))$ where $a, b, c, d \in [0, 1]$.

A Menger space is a triplet $(X, F, t)$, where $(X, F)$ is a PM-Space, $X$ is a non-empty set and a $t$-norm satisfying instead of 6.1(i) a stronger requirement.

(j) $F_{u,w}x + y \geqslant tF_{u,v}x, F_{v,w}y$ for all $x \geqslant 0, y \geqslant 0$.

For a given metric space $(X, d)$ with usual metric $d$, one can put $F_{u,v}(x) = H(x - d(u, v))$ for all $x, y \in X$ and $t > 0$. where H is defined as: $H(x) = 1$ if $s > 0$, 0 if $s \leqslant 0$. and t-norm is defined as $ta, b = \min\{a, b\}$.

Examples of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$. Our aim of this paper, we have tried to present a common fixed point for a pair of self mapping under strict contractive and OWC condition under the concept of Menger space.

## 2 Menger Spaces

To proof of our result we need some known definitions which are follows.

**Definition 2.1.** A probabilistic metric space (PM- space) is an ordered pair $(X, F)$ consisting of a non empty set $X$ and a mapping $F$ from $X \times X$ into the collections of all distribution $F \in R$. For $x, y \in X$ we denote the distribution function $F(x, y)$ by $F_{x,y}$ and $F_{x,y}(u)$ is the value of $F_{x,y}$ at $u$ in $R$.

**Definition 2.2.** Self maps A and B of a Menger space $(X, F, *)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

**Definition 2.3.** Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible if $FABx_m, BAx_n$, $(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $Ax_n \to x$, $Bx_n \to x$ for some $x$ in $X$ as $n \to \infty$.

**Definition 2.4.** A sequence $\{x_n\}$ in $(X, F, t)$ is said to be convergent to a point $x$ in $X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\epsilon, \lambda)$ such that $x_n \in Ux\epsilon, \lambda$ for all $n \geqslant N$ or equivalently $Fx_n, x; \epsilon > 1 - \lambda$ for all $n \geqslant N$.

**Definition 2.5.** A sequence $x_n$ in $(X, F, t)$ is said to be Cauchy sequence if for every $\epsilon > 0$ and $\lambda > 0$, there is an integer $N = N(\epsilon, \lambda)$ such that $F(x_n, x_m, \epsilon) > 1 - \lambda$ for all $n, m \geqslant N$.

**Definition 2.6.** A Menger space $(X, F, t)$ with the continuous t-norm is said to be complete if every cauchy sequence in $X$ converges to a point in $X$.

**Definition 2.7.** Let $(X, F, t)$ be a Menger space, two mappings $f, g : X \to X$ are said to be weakly.

**Lemma 2.8.** Let $\{x_n\}$ be a sequence in a Menger space $(X, F, t)$, where $t$ is continuous and $tp, p \geqslant p$ for all $p \in (0, 1)$ and $n \in NF(x_n, x_n + 1, kp) \geqslant F(x_n - 1, x_n, p)$, then $\{x_n\}$ is Cauchy sequence. compatible if they commute at coincidence point.

**Lemma 2.9.** Let $X$ be a set $f, g$ OWC self maps of $X$. If $f$ and $g$ have a unique point of coincidence, $w = fx = gx$, then $w$ is the unique common fixed point of $f$ and $g$.

**Lemma 2.10.** If $(X, d)$ is a metric space, then the metric $d$ induces a mapping $F : X \times X \to L$ defined by $F(p, q) = Hx - dp, q, p, q \in R$. Further if $t : [0, 1] \times [0, 1] \to [0, 1]$ is defined by $t(a, b) = \min\{a, b\}$, then $(X, F, t)$ is a Menger space. It is complete if $(X, d)$ is complete.

**Lemma 2.11.** If for all $x, y \in X, t > 0$ with positive number $q \in (0, 1)$ and $Mx, y(qt) \geqslant Mx, y(t)$ then $x = y$.

**Definition 2.12.** Two self mappings $S$ and $T$ of a menger space are said to be commuting if $MSTx, TSxt = 1$ for all $t > 0$ and for all $x \in X$.

**Definition 2.13.** Two self mappings $S$ and $T$ of a menger space are said to be commuting if $MSTx, TSxt \geqslant MSx, Txt$ for all $t > 0$ and for all $x \in X$.

## 3 Main Result

We now state our main common fixed point result of two, three and four mappings in menger spaces, as follows. First we consider the common fixed point for two mappings.

**Theorem 3.1.** *Let $A$ and $S$ be two self owc mappings of a menger space $(X, M, *)$ with $t * t \geqslant t$ such that for each $x \neq y$ in $X, t > 0$ and for $0 < q < 1$.*

$$MAx, Ay(qt) \geqslant \min\{MSx, Sy(t), MSx, Ay(t), MSy, Ay(t), MAx, Sx(t), MAx, Sy(t)\} \qquad 3.1(a)$$

*Then $A$ and $S$ have a unique common fixed point.*

**Proof:** Since A and S are owc so there exit $a \in X$ such that $Aa = Sa$ implies $ASa = SAa$ That is there exist $a \in X$ such that $MAa, Sa(t) = 1$ implies $MASa, SAa(t) = 1$ for $t > 0$. And since $Sa = Aa, SSa = SAa$ and $ASa = AAa$, we have $SSa = SAa = ASa = AAa$.

Now we show that $Aa = Sa$ is common fixed point of $A$ and $S$. Suppose that $Aa \neq AAa$. Then by 3.1(a)

$$MAa, AAa(qt) \geqslant \min\{MSa, SAa(t), MSa, AAa(T), MSAa, AAa(t), MAa, Sa(t), MAa, SAa(t)\}$$

$$= \min\{MAa, AAa(t), MAa, AAa(t), MAAa, AAa(t), MAa, Aa(t), MAa, AAa(t)\}$$

$$= \min\{MAa, AAa(t), MAa, AAa(t), 1, 1, MAa, AAa(t)\}$$

$$MAa, AAa(qt) \geqslant MAa, AAa(t).$$

Then by lemma 2.11 it follows that $Aa = AAa$ and thus $AAa = SAa = Aa$. Hence $Aa = Sa$ is common fixed point of $A$ and $S$. Finally we show that the fixed point is unique. Let $x_0$ and $y_0$ be two common fixed points of $A$ and $S$. Then $Ax_0 = Sx_0 = x_0$ and $Ay_0 = Sy_0 = y_0$ and by 3.1(a)

$$MAx_0, Ay_0(qt) \geqslant \min\{MSx_0, Sy_0(t), MSx_0, Ay_0(t), MSy_0, Ay_0(t), MAx_0, Sx_0(t), MAx_0, Sy_0(t)\}$$

$$= \min\{MAx_0, Ay_0(t), MAx_0, Ay_0(t), MAy_0, Ay_0(t), MAx_0, Ax_0(t), MAx_0, Ay_0(t)\}$$

$$= \min\{MAx_0, Ay_0(t), MAx_0, Ay_0(t), 1, 1, MAx_0, Ay_0(t)\}$$

$$MAx_0, Ay_0(qt) \geqslant MAx_0, Ay_0(t).$$

Then by lemma (2.11) we have $Ax_0 = Ay_0$, i.e. $x_0 = y_0$. □

Next we consider the common fixed point for three mappings.

**Theorem 3.2.** *Let $A, B$ and $S$ be three self mappings for a menger space $(X, M, *)$ with $t * t \geqslant t$ such that for each $x \neq y$ in $X, t > 0$ and for $0 < q < 1$.*

$$MAx, By(qt) \geqslant \min\{MSx, Sy(t), MSx, By(t), MSy, By(t), MAx, Sx(t), MSy, Ax(t)\} \qquad 3.2(a)$$

*and pair $(A, S)$ or $(B, S)$ is owc pair. 3.2(b). Then $A, B$ and $S$ have a unique common fixed point.*

**Proof:** Since $(A, S)$ is owc pair [from 3.2(b)] Then there is an element $u \in X$ such that $Au = Su$ and $ASu = SAu$. First, we prove that $Au = Bu = Su$. By 3.2(a) we get

$$MAu, Bu(qt) \geqslant \min\{MSu, Su(t), MSu, Bu(t), MSu, Bu(t), ), MAu, Su(t), MSu, Au(t)\}$$

$$= \min\{1, MAu, Bu(t), MAu, Bu(t), MAu, Au(t), MAu, Au(t)\}$$

$$= \min\{1, MAu, Bu(t), MAu, Bu(t), 1, 1\}$$

$$MAu, Bu(qt) \geqslant MAu, Bu(t).$$

Then by lemma 2.11 we hae $Au = Bu$ i.e. $Au = Bu = Su$. Thus $ASu = SAu = ABu = SBu = AAu$. Now suppose that $BAu \neq AAu$. Then from 3.2(a) we get

$$MAAu, BAu(qt) \geqslant \min\{MSAu, SAu(t), MSAu, BAu(t), MSAu, BAu(t), MAAu, SAu(t), MSAu, AAu(t)\}$$

$$= \min\{MAAu, AAu(t), MAAu, BAu(t), MAAu, BAu(t), MSAu, BAu(t), MAAu, AAu(t), MAAu, AAu(t)\}$$

$$= \min\{1, MAAu, AAu(t), MAAu, BAu(t)1, 1\}$$

$$MAAu, BAu(qt) \geqslant MAAu, BAu(t).$$

Hence by lemma 2.11 it follows that $AAu = BAu$, and so $AAu = BAu = SAu$. If $AAu \neq Bu$, we have from 3.2(a)

$$MAAu, Bu(qt) \geqslant \min\{MSAu, Su, (t), MSAu, Bu(t), MSAu, Bu(t), MAAu, SAu(t), MSu, AAu(t)\}$$

$$= \min\{MAAu, Bu(t), MAAu, Bu(t), MAAu, But, MAAu, AAu(t), MBu, AAu(t)\}$$

$$= \min\{MAAu, Bu(t), MAAu, Bu(t), MAAu, Bu(t), 1, MAAu, Bu(t)\}$$

$$MAAu, Bu(qt) \geqslant MAAu, Bu(t).$$

Then by lemma 2.11 we have $AAu = Bu$, i.e $AAu = Au = Bu = Su$. or $AAu = BAu = SAu = Au = a$ (Let) So, $a = Au$ is common fixed point of mappings $A, B$ and $S$.

**Uniqueness:** Now Let $x_0, y_0$ be two distinct common fixed points of mappings $A, B$ and $S$. i.e $Ax_0 = Bx_0 = Sx_0 = x_0$ and $Ay_0 = By_0 = Sy_0 = y_0$. So by condition 3.2(a)

$$MAx_0, By_0(qt) \geqslant \min\{MSx_0, Sy_0(t), MSx_0, By_0(t), MSy_0, By_0(t), MAx_0, Sx_0(t), MSy_0, Ax_0(t)\}$$

$$= \min\{MAx_0, By_0(t), MAx_0, By_0(t), MBy_0, By_0(t), MAx_0, Ax_0(t), MBy_0, Ax_0(t)\}$$

$$= \min\{MAx_0, By_0(t), MAx_0, By_0(t), 1, 1, MAx_0, By_0(t)\}$$

$$MAx_0, By_0(qt) \geqslant MAx_0, By_0(t).$$

Then by lemma 2.11 we have $x_0 = y_0$. □

Finally, we consider common fixed point for four mappings.

**Theorem 3.3.** *Let A,B,S and $T$ be four self mappings of a menger space $(X, M, *)$ with $t * t \geqslant t$ such that for each $x \neq y$ in $X, t > 0$ and for $0 < q < 1$.*

$$MAx, By(qt) \geqslant \min\{MSx, Ty(t), MSx, By(t), MTy, By(t), MSx, Ax(t), MTy, Ax(t)\} \qquad 3.3(a)$$

*and pairs $(A, S), (B, T)$ are owc. 3.3(b). Then $A, B, S$ and $T$ have a unique common fixed point.*

**Proof:** Since $(A, S), (B, T)$ is owc pair [from 3.3(b)]. Then there is an element $u, v \in X$ such that $Au = Su$ and $ASu = SAu, Bv = TvandBTv = TBv$. First, we prove that Au=Bv By 3.3(a) we get

$$MAu, Bv(qt) \geqslant \min\{MSu, Tv(t), MSu, Bv(t), MTv, Bv(t), MSu, Au(t), MTv, Au(t)\}$$

$$= \min\{MAu, Bv(t), MAu, Bv(t), MBv, Bv(t), MAu, Au(t), MBv, Au(t)\}$$

$$= \min\{MAu, Bv(t), MAu, Bv(t), 1, 1, MAu, Bv(t)\}$$

$$MAu, Bv(qt) \geqslant MAu, Bv(t).$$

Then from lemma 2.11, we have $Au = Su = Bv = Tv$. New suppose that $AAu \neq Au$. By using 3.3(a) we obtain $Au = Bv$,

$$MAAu, Bv(qt) \geqslant \min\{MSAu, Tv(t), MSAu, Bv(t), MTv, Bv(t), MSAu, AAu(t), MTv, AAu(t)\}$$

$$= \min\{MASu, Bv(t), MASu, Bv(t), MBv, Bv(t), MASu, AAu(t), MBv, AAu(t)\}$$

$$= \min\{MAAu, Bv(t), MAAu, Bv(t), MBv, Bv(t), MAAu, AAu(t), MBv, AAu(t)\}$$

$$= \min\{MAAu, Bv(t), MAAu, Bv(t), 1, 1, MAAu, Bv(t)\}$$

$$MAAu, Bv(qt) \geqslant MAAu, Bv(t).$$

So by lemma (2.11) we have $AAu = Bv$. Since Au=Bv, it follows that $AAu = Au = ASu = SAu$. Similarly $BAu = TAu = Au$. Therefore $Au = Su = Bv = Tv$ is a common fixed point of mapping $A, B, S$ and $T$. Put $Au = Su = Bv = Tv = x$, then $Ax = Sx = Bx = Tx = x$.

**Uniqueness:** Let $x_0$ and $y_0$ are two common fixed points of mapping $A, B, S$ and T such that $x_0 \neq y_0$ then $x_0 = Ax_0 = Sx_0 = Bx_0 = Tx_0$ and $y_0 = Ay_0 = Sy_0 = By_0 = Ty_0$. From condition 3.3(a) we have

$$MAx_0, By_0(qt) \geqslant \min\{MSx_0, Ty_0(t), MSx_0, By_0(t), MTy_0, By_0(t), MSx_0, Ax_0(t), MTy_0, Ax_0(t)\}$$

$$= \min\{MAx_0, By_0(t), MAx_0, By_0(t), MBy_0, By_0(t), MAx_0, Ax_0(t), MBy_0, Ax_0(t)\}$$

$$= \min\{MAx_0, By_0(t), MAx_0, By_0(t), 1, MAx_0, By_0(t)\}$$

$$MAx_0, By_0qt \geqslant \min\{MAx_0, By_0t\}$$

Then by lemma 2.11. we have $Ax_0 = By_0$ Thus $A, B, S$ and $T$ have unique common fixed point. □

*Remark 1:* If we put $S = T$ in the statement of theorem of 3.3 then we can get statement of theorem 3.2.

*Remark 2:* If we put $A = B$ and $S = T$ in the statement of theorem of 3.4 then we can get statement of theorem 3.1.
*Remark 3:* If we put $A = B$ in the statement of theorem of 3.2 then we can get statement of theorem 3.1.

Thus theorem 3.2 and theorem 3.3 are generalizations of theorem 3.1. The next theorem involves a function $F : [0, 1] \to [0, 1]$ satisfying the following conditions:

(i) $F$ is increasing on $[0, 1]$,
(ii) $F(t) > t$, for any $t \in (0, 1)$ and $F(1) = 1$.

Then we can easily prove these theorems depending upon above three theorems.

**Theorem 3.4.** *Let $A$ and $S$ be two self owc mappings of a menger symmetric space $(X, M, *)$ with $t * t \geqslant t$ such that for each $x \neq, y$ in $X, t > 0$*

*(i) A function $F : [0, 1] \to [0, 1]$ satisfying the following conditions:*
*(a) $F$ is increasing on $[0, 1]$*
*(b) $F(t) > t$, for any $t \in [0, 1]$ and $F(1) = 1$.*

*and $MAx, Ay(t) > F[\min\{MSx, Sy(t), MSx, Ay(t), MSy, Ay(t), MAx, Sx(t), MAx, Sy(t)\}]$, then Aand $S$ have a unique common fixed point.*

**Theorem 3.5.** *Let A,B and S be three self mappings of a menger symmetric space $(X, M, *)$ with $t*t \geqslant t$ such that for each $x \neq, y$ in $X, t > 0$ and function $F : [0, 1] \to [0, 1]$ satisfying the following conditions:*

*(a) $F$ is increasing on $[0, 1]$*
*(b) $F(t) > t$, for any $t \in [0, 1]$ and $F(1) = 1$.*

$$MAx, By(t) > F[\min\{MSx, Sy(t), MSx, By(t), MSy, By(t), MAx, Sx(t), MSy, Ax(t)\}] \qquad 3.5(a)$$

*and pair $(A, S)$ or $(B, S)$ is owc pair. Then $A, B$ and $S$ have a unique common fixed point.*

**Theorem 3.6.** *Let $A, B, S$ and $T$ be four self mappings of a menger space $(X, M, *)$ with $t * t \geqslant t$ such that for each $x \neq, y$ in $X, t > 0$ and function $F : [0, 1]] \to [0, 1]$ satisfying the following conditions:*

*(a) $F$ is increasing on $[0, 1]$*
*(b) $F(t) > t$, for any $t \in [0, 1]$ and $F(1) = 1$.*
*$MAx, By(t) > F[\min\{MSx, Ty(t), MSx, By(t), MTy, By(t), MSx, Ax(t), MTy, Ax(t)\}]$ and pairs $(A, S)$ and $(B, T)$ are owc. Then $A, B, S$ and $T$ have a unique common fixed point.*

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