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Finite non-solvable groups whose monolithic characters vanish on at most three conjugacy classes

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Abstract The aim of this note is to classify the finite non-solvable groups whose monolithic characters vanish on at most three conjugacy classes in the character table.

Key Words finite groups, characters, zeros of characters

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1 Introduction

A group G is said to be a monolith if it contains only one minimal normal subgroup. We consider the identity group as a monolith. A character χ of a group G is said to be a monolithic character if $\chi \in \mathrm{Irr}(G)$ and the factor group $G/ker(\chi)$ is a monolith, where $\mathrm{Irr}(G)$ denotes the set of irreducible complex characters of G.

J. Zhang, J. Shi and Z. Shen [8] investigated the finite groups in which every irreducible character vanishes on at most three conjugacy classes of G. In this paper, we study the finite non-solvable groups G in which every monolithic character vanishes on at most three conjugacy classes.

Remark and notation: Denote $\mathrm{Irr}_m(G)$ the set of all monolithic characters of G, $\mathrm{cd}_m(G)$ the set of monolithic character degrees of G. Note that all irreducible characters of p-groups are monolithic, and that $\mathrm{cd}_m(G) = 1$ if and only if G is abelian. Since $\bigcap_{\chi\in \mathrm{Irr}_m(G)} ker(\chi) = 1$ (see [1, Lemma 2(a)]), G is a subgroup of a direct product of monoliths. Hence the set $\mathrm{Irr}_m(G)$ is sufficiently large to have a strong influence on the structure of G. On the other hand, in many cases, the set $\mathrm{Irr}_m(G)$ is a rather small subset of $\mathrm{Irr}(G)$ (see [1] for examples). Now our result can be stated as follows.

Theorem. *Let G be a finite non-solvable group. If every monolithic character of G vanishes on at most three conjugacy classes, then $G \cong A_5$, $L_2(7)$, or A_6.*

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In this paper, G always denotes a finite group, p always denotes a prime. Notation is standard and taken from [4]. In particular, for $\chi \in \mathrm{Irr}(G)$, set $\upsilon(\chi) := \{g \in G \mid \chi(g) = 0\}$, denote $\mathrm{cd}(G)$ the set of irreducible character degrees of G, and $k_G(N)$ the number of conjugacy classes of G contained in N, where N is a normal subset of G. For $N \lhd G$, set $\mathrm{Irr}(G|N) = \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$.

We shall freely use the following facts: Let $N \lhd G$ and write $\overline{G} = G/N$.

(1) If N is contained in $ker(\chi)$ where $\chi \in \mathrm{Irr}(G)$, then χ is monolithic as a character of G/N if and only if it is monolithic as a character of G.

(2) For any $x \in G$, $\overline{x}^{\overline{G}}$ (when viewed as a subset of G, that is, the set $\bigcup_{g\in G} x^g N$) is a union of conjugacy classes of G; furthermore, $k_G(\overline{x}^{\overline{G}}) = 1$ if and only if $\chi(x) = 0$ for any $\chi \in \mathrm{Irr}(G|N)$.

2 Proof of Theorem

The following result, which appears as Theorem 3.9 in [8], will turn out to be useful in proof of Theorem.

Lemma 2.1. *Let G be a non-abelian simple group. If every irreducible character vanish on at most three conjugacy classes of G, then G is isomorphic to A_5, $L_2(7)$, or A_6.*

We will use the following lemma (see [7, Theorem 2.1]).

Lemma 2.2. *Let G be non-abelian, and let $\chi \in \mathrm{Irr}_1(G)$. Assume that N is a normal subgroup of G such that $G' \leqslant N < G$. If χ_N is not irreducible, then the following two statements hold:*

(1) There exists a normal subgroup H of G such that $N \leqslant H < G$ and $G \backslash H \subseteq \upsilon(\chi)$.

(2) If $(G\backslash G') \cap \upsilon(\chi)$ consists of n conjugacy classes of G, then $[H : G']\,([G : H] - 1) \leqslant n$.

Next, we study the p-groups satisfying the hypothesis. Recall that all irreducible characters of p-groups are monolithic. We get the following easy result.

Lemma 2.3. *Suppose that G is a non-abelian p-group. If every monolithic character of G vanishes on at most three conjugacy classes, then $G \cong D_8$ or Q_8.*

Proof. Take $\varphi \in \mathrm{Irr}_1(G)$ such that $\varphi_{G'}$ is not irreducible. It follows from the hypothesis and Lemma 2.2 that G has a proper subgroup H such that $G' \leqslant H < G$, $G - H \subseteq \upsilon(\varphi)$ and $[H : G']\,([G : H] - 1) \leqslant 3$.

Since G is nilpotent, we easily conclude that G is a non-abelian p-group. It implies that $|G/G'| \geqslant p^2$. Note that $[H : G']\,([G : H] - 1) \leqslant 3$; then we obtain that $p = 2$ and $|G/G'| = 4$, and so G is of maximal class (see [3, P.375]). Suppose that $|G| \geqslant 16$. As G is of maximal class, one of the upper central series members must have index 16. Now every group of order 16 has a non-linear irreducible character which vanishes on at least 4 conjugacy classes (see [5, P.300]). Hence $|G| = 8$, and thus $G \cong \mathrm{D}_8$ or Q_8. The proof is complete. □

Proof of Theorem. We need only prove necessity in the Theorem. Clearly our hypothesis is inherited by any factor group. Let L be a normal subgroup of G maximal with respect to G/L being non-abelian. Then $(G/L)'$ is the unique minimal normal subgroup of G/L, and thus all non-linear irreducible character of G/L is monolithic.

We claim that $G \cong \mathrm{A}_5$, $\mathrm{L}_2(7)$, or A_6. By Lemma 2.1, it suffices to show that G is a non-abelian simple group. Assume that G is a minimal counter-example.

Now we show that G/L is non-solvable. Otherwise, G/L is solvable. Then by [4, Corollary 12.3], we have to discuss the following two cases.

Assume that G/L is a p-group, for some prime p. Then by Lemma 2.3, $G/L \cong \mathrm{D}_8$ or Q_8. Set $N/L = Z(G/L)$. Let λ be a non-principal character of N/L. Then $\lambda^G = \chi \in \mathrm{Irr}_1(G)$, and so χ vanishes on $G \backslash N$. Recall that all non-linear irreducible characters of G/L are monolithic. It follows from the hypothesis that $k_G(G \backslash N) = 3$. Then it follows by [6, Theorem 3.5] that G has a normal subgroup E such that $G/E \cong \mathrm{S}_5$ or M_{10}, then we obtain a contradiction from [2].

Assume that G/L is a Frobenius group with kernel N/L. Let λ be a non-principal character of N/L. Then $\lambda^G = \chi \in \mathrm{Irr}_1(G)$, and so χ vanishes on $G \backslash N$. It follows from the hypothesis that $k_G(G \backslash N) \leqslant 3$. If $k_G(G \backslash N) = 1$, then G is a Frobenius group with abelian kernel G' and complement of order 2, a contradiction. Suppose that $k_G(G \backslash N) = 2$. Then by [6, Theorem 2.2], we obtain that G is solvable, a contradiction. If $k_G(G \backslash N) = 3$, then arguing as the above paragraph, we also obtain a contradiction. Therefore G/L is non-solvable.

Next we show that $L = 1$. Assume that that $L > 1$. To reach a contradiction, we may assume that L is a minimal normal subgroup of G. Recall that G/L is non-solvable, then by induction, G/L is a non-abelian simple group. Applying Lemma 2.1, we obtain that $G/L \cong \mathrm{A}_5$, $\mathrm{L}_2(7)$, or A_6. since $\bigcap_{\chi \in \mathrm{Irr_m(G)}} ker(\chi) = 1$, the set $\mathrm{Irr}(G|L)$ contain at least a non-linear monolithic character of G. Then Arguing as in Theorem B of [8], we obtain a contradiction. Hence $L = 1$.

Since $L = 1$, all non-linear irreducible characters of G are monolithic. The hypothesis implies that every irreducible character χ of G vanishes on at most three conjugacy classes of G. Hence, by [8, Theorem B], we have $G \cong \mathrm{A}_5$, $\mathrm{L}_2(7)$, or A_6. The proof is completed. □

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