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On C-spaces and connective spaces

K.P. Ratheesh ①*, N.M. Madhavan Namboothiri ②

① Department of Mathematics, P. T. M. Govt. College Perithalmanna, Malappuram, Kerala, India
② Department of Mathematics, Govt. College, Kasaragod, Vidyanagar, Kasaragod, Kerala, India
E-mail: ratheeshmath@gmail.com

Received: Dec-9-2013; Accepted: Jan-5-2014 *Corresponding author

Abstract The attempt in this paper is to synthesise the different concepts of c-spaces and connective spaces held by authors [11] and [12] in a systematic manner. The concept of α-generated C-space is introduced and characterizes finite topological C-spaces and connective spaces in connection with 2-generated C-spaces. It also gives a characterization for finite c-spaces and connective spaces with a compatible Čech closure operator. The order properties of the collection of C-spaces and connective spaces on a set are investigated.

Key Words c-space, α-generated c-space, Connective space, Topological c-space, Compatible relation

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1 Introduction

The topological concept of connectedness is originated from C. Jordan's Cours d'Analyse of 1893 [9]. The evolution of connectedness involves the contributions of Bolzano, Schoenflies, Cantor, W. H.Young, G. C. Young, Hausdorff, Lennes and Riesz. Among these W. H.Young, G. C. Young gave a definition of connectedness in terms of regions. Another remarkable contribution is by N.Lennes [10]. He defined connectedness as *"A set of points is connected if in every pair of complementary subsets at least one subset contains a limit point of points in the other set"*. When we go through the developments of the concept of topological connectedness it can be realized that a systematic study on this concept was carried out by Hausdorff, B. Knaster and K. Kuratowski [6].

The purpose of this paper is to explore the concepts of connectedness and to develop a new point of view towards connectedness. The theory is still young and no doubt many concepts are yet to be formulated. In this context the aim of this paper is to focuss on the study of the behaviour and structure of the collection of connected sets in a generalized sense. The digital image processing mainly concerned with defining neighbourhoods of node in the digital array. It can be noted that neighbourhoods of a node in the digital array is defined using a topology on $\mathbb{Z}^2$ or $\mathbb{Z}^n$ [8, 5]. But it is a fact that the topology of $\mathbb{Z}^2$ and $\mathbb{Z}^n$ is not a matter of this theory, but the collection of connected sets is the only concept under consideration. This is the motivation for a detailed study of collection of connected sets on a set considered as a structure on that set. Thus the theory of connectedness has wider scope of applicability particularly in the fields of Digital topology, Image processing and Network theory [7, 4].

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2 Preliminaries

Let (X, τ) be any topological space. Then a subset A of X is said to be *Connected* in (X, τ) if A cannot be written as the union of two non-empty separated subsets of X. ie $A = A_1 \cup A_2$ with $\overline{A}_1 \cap A_2 = \overline{A}_2 \cap A_1 = \emptyset$, then $A_1 = \emptyset$ or $A_2 = \emptyset$ (Where $\overline{A}_1$ and $\overline{A}_2$ denotes the closure of A_1 and A_2 respectively).

In [1] Susan J. Andima and W. J. Thron associated each topology T on a set X with a preorder relation R_T or $\rho(T)$, on X defined by $(a, b) \in R_T$ if and only if every open set containing b contains a. It can be seen that, this correspondence is many-to-one and for a given preorder R on X, there is always a least topology $\mu(R)$ called point closure topology of R and a greatest topology $\nu(R)$ called kernel topology of R associated with R. From [1], a topology T on X has preorder R if and only if $\mu(R) \subseteq T \subseteq \nu(R)$. In particular, $\rho(\mu(R)) = \rho(\nu(R)) = R$.

Let $G = (X,\ E)$ be a graph. Then G is said to be connected if, for every $x, y \in X$ there is a path from x to y. Now we can say that a subset U of X is *Connected* if the underlying vertex spanning subgraph of G with respect to U is connected.

A Čech closure operator on a set X is a function $V : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ such that

1. $V(\phi) = \phi$,
2. $A \subseteq V(A)$ for every $A \in \mathcal{P}(X)$,
3. $V(A \bigcup B) = V(A) \bigcup V(B)$ for every $A, B \in \mathcal{P}(X)$

where $\mathcal{P}(X)$ denotes the power set of X. For brevity we call V a closure operator on X and $(X,\ V)$ is called a closure space.

In [1], P.T. Ramachandran [13] associated each closure operator V on a set X with a reflexive relation ρV, on X defined by $(a, b) \in \rho V$ if and only if $b \in V(\{a\})$. It can be seen that, this correspondence is many-to-one and for a given reflexive R on X, there is always a least closure operator $\mu(R)$ and a greatest closure operator $\nu(R)$ associated with R. From [13], if R is a reflexive relation on a set X, then $R = \rho V$ for some closure operator V on X if and only if $\mu R \leqslant V \leqslant \nu R$. In particular $\rho\mu R = \rho\nu R = R$.

Let $(X,\ V)$ be any closure space. Then a subset A of X is said to be *Connected* in $(X,\ V)$ if A cannot be written as the union of two non-empty semi-separated subsets of $(X,\ V)$. That is $A = A_1 \cup A_2$ with $V(A_1) \cap A_2 = V(A_2) \cap A_1 = \emptyset$, then $A_1 = \emptyset$ or $A_2 = \emptyset$.

3 c-spaces and Connective Spaces

In this section we deal with c-space, connective space as in [11] and 2-generated space and give a characterization about finite connective spaces.

Definition 3.1. *[12], [11] Let X be a set, a $c-structure$ in X is a collection $\mathcal{C}$ of subsets of X satisfying the conditions*

C_1. $\forall C \subset X,\ |C| \leqslant 1 \Rightarrow C \in \mathcal{C}$

C_2. *Let $\{C_i\}_{i \in I}$ be a collection in $\mathcal{C}$ and $\bigcap\limits_{i \in I} C_i \neq \emptyset$, then $\bigcup\limits_{i \in I} C_i \in \mathcal{C}$.*

The set X together with a c-structure $\mathcal{C}$ on X is called a ***c-space*** *and elements of $\mathcal{C}$ are called connected sets in X with respect to $\mathcal{C}$.*

Here onwards $\mathcal{S}$ denote the set of all singleton sets in the concerned space.

Example 3.2. *1. Let* $X = \{1,2,3,4,5\}$ *and* $\mathcal{C} = \{\emptyset,\ \{1,2\},\{3,4\},\{1,2,3,4\}\} \cup \mathcal{S}$. *Then* $(X,\mathcal{C})$ *is a c-space*
2. Let X *be any infinite set and* A, B *subsets of* X *with* $|A| \geqslant 2, |B| \geqslant 2$. *Now* $\mathcal{C} = \emptyset \cup \mathcal{S} \cup \{A,B\} \cup \{C \subset X : A \cup B \subset C\}$. *Then* $(X,\mathcal{C})$ *is a c-space*
Let X *be any set, then*

3. $\mathcal{D} = \{\emptyset\} \cup \mathcal{S}$ *is a c-structure on* X *and* $(X,\mathcal{D})$ *is called* ***discrete*** *c-space.*

4. $\mathcal{I} = \mathcal{P}(X)$ *is a c-structure on* X *and* $(X,\mathcal{I})$ *is called* ***indiscrete*** *c-space.*

5. $\mathcal{C} = \mathcal{D} \cup \{A \subseteq X : A\ is\ infinite\}$ *is a c-structure on* X *and* $(X,\mathcal{C})$ *is called* ***co-finite*** *c-space.*

6. *For* $A \subseteq X$, $\mathcal{C}_A = \{B \subseteq X : A \subseteq B\} \cup \mathcal{D}$ *is a c-structure on* X *and* $(X,\mathcal{C}_A)$ *is c-space.*

Definition 3.3. *Let* $(X,\mathcal{C})$ *be a c-space and* $Y \subset X$, *we define* $\mathcal{C}_Y = \{C \in \mathcal{C} : C \subset Y\}$, *then* $(Y,\mathcal{C}_Y)$ *is a c-space and is called* ***Sub c-space*** *on* Y.

Definition 3.4. *Let* X *be a set and* $\mathcal{B} \subseteq \mathcal{P}(X)$, *then the intersection of all c-structures on* X *containing* $\mathcal{B}$ *is a c-structure and is called the c-structure generated by* $\mathcal{B}$ *and is denoted by* $\langle \mathcal{B} \rangle$.

Note that $\langle \mathcal{B} \rangle$ is the smallest c-structure on X containing $\mathcal{B}$

Definition 3.5. *Let* X *be any set and* α *be any cardinal with* $\alpha \leqslant |X|$, *then a c-structure* $\mathcal{C}$ *on* X *is said to be* α***-generated*** *if there is a sub collection* $\mathcal{B} \subseteq \{A \in \mathcal{C} : |A| \leqslant \alpha\}$ *such that* $\mathcal{C} = \langle \mathcal{B} \rangle$. *An element* A *in a c-structure* $\mathcal{C}$ *on* X *is said to be* α***-generated*** *if there is a sub collection* $\mathcal{B} \subseteq \{A \in \mathcal{C} : |A| \leqslant \alpha\}$ *such that* $A \in \langle \mathcal{B} \rangle$.

It can be noted that the c-space given in Example 3.2, (3) and (4) are 2-generated but (5) is 2-generated if and only if X is finite and (6) is 2-generated if and only if $|X| = 2$.

Proposition 3.6. *Let* A *be any 2-generated connected set in a c-space* $(X,\mathcal{C})$, *then there exists an* $a \in A$ *such that* $A - \{a\}$ *is a 2-generated element in* $\mathcal{C}$

Proof. Since A is 2-generated , then there is a sub collection $\mathcal{B}$ of 2- element sets in $\mathcal{C}$ such that $A \in \langle \mathcal{B} \rangle$. Without loss of generality we can assume that $\mathcal{B}$ is a minimal subfamily of 2-element sets in $\mathcal{C}$ such that $A \in \langle \mathcal{B} \rangle$. That is $A = \bigcup\limits_{C_i \in \mathcal{B}} C_i$. Then there exist $a \in A$ such that $a \in C_j$ for only one $C_j \in \mathcal{B}$. Let $C_j = \{a,b\}$. Therefore $(\bigcup\limits_{C_i \in \mathcal{B}, i \neq j} C_i) \cap C_j = \{b\}$. Hence $A - \{a\} = (\bigcup\limits_{C_i \in \mathcal{B}, i \neq j} C_i) \in \mathcal{C}$ and is 2-generated. $\square$

Definition 3.7. *[12], [11] Let* X *be a set. A* ***Connective Structure*** *or* ***Connectology*** *on* X *is a c-structure* $\mathcal{C}$ *on* X *satisfying the following conditions*

C_3 *: Given any nonempty sets* $A,\ B \in \mathcal{C}$ *with* $A \cup B \in \mathcal{C}$, *then* $\exists\ x \in A \cup B$ *such that* $\{x\} \cup A \in \mathcal{C}$ *and* $\{x\} \cup B \in \mathcal{C}$

C_4 *: If* $A, B, C_i \in \mathcal{C}$ $(i \in I)$ *are disjoint and* $A \cup B \cup \bigcup_{i \in I} C_i \in \mathcal{C}$, *then* $\exists\ J \subseteq I, A \cup \bigcup_{j \in J} C_j \in \mathcal{C}$ *and* $B \cup \bigcup_{i \in I - J} C_i \in \mathcal{C}$.

The set X *together with a connectology* $\mathcal{C}$ *on* X *is called a* ***Connective space*** *and elements of* $\mathcal{C}$ *are called the Connected sets in* X *with respect to* $\mathcal{C}$.

It can be noted that the c-space given in Example 3.2,(3), (4), (5)and (6) are connective spaces but (1) and (2) are not.

Theorem 3.8. *Finite connective spaces are precisely the finite 2- generated c-spaces.*

Proof. We prove this by induction on the number of elements in the connected set $C \in \mathcal{C}$. If $|C| = 2$, it is clear that C is generated by a two element set in $\mathcal{C}$. Suppose that $|C| = 3$, say $C = \{a_1, a_2, a_3\}$. Take $A = \{a_1\},\ B = \{a_2\}$ *and* $D = \{a_3\}$. Then by condition C_4 we get,

1. either $\{a_1, a_3\} \in C$ or $\{a_2, a_3\} \in C$,
2. either $\{a_1, a_2\} \in C$ or $\{a_2, a_3\} \in C$ and
3. either $\{a_1, a_2\} \in C$ or $\{a_1, a_3\} \in C$.

Then it follows that C is generated by two element sets in $\mathcal{C}$. Suppose this result is true for any $m < n, (n > 3)$. Let $C \in \mathcal{C}$ with $|C| = n$ say $C = \{a_1, a_2, \cdots, a_n\}$, Take $A\ ' = \{a_1\}, B\ ' = \{a_2\}$ and $C_i\ ' = \{a_{i+2}\} (i = 1, 2, \cdots, n-2)$. Since $A \cup B \cup \bigcup_{i=1}^{n-2} C_i = C \in \mathcal{C}$. By condition C_4 of Definition 3.7, there exist $J \subseteq \{1, 2, 3, \cdots, n-2\}$ such that $A\ ' \cup (\bigcup_{j\in J} C_j\ ') \in C$ and $B\ ' \cup (\bigcup_{j\in I-J} C_j\ ')$. Let $D = A\ ' \cup (\bigcup_{j\in J} C_j\ ')$ and $E = B\ ' \cup (\bigcup_{j\in I-J} C_j\ ')$ Then by induction hypothesis, D and E are 2-generated elements in $\mathcal{C}$. By condition C_3 of Definition 3.7, there exists $a_j \in D \cup E$ such that $D \cup \{a_j\} \in \mathcal{C}$ and $E \cup \{a_j\} \in \mathcal{C}$, with out loss of generality we may take $a_j \in D$. If $|E \cup \{a_j\}| < n$, then $E \cup \{a_j\}$ is 2-generated element in $\mathcal{C}$. Thus there exists $a_k \in E$ such that $\{a_j,\ a_k\} \in \mathcal{C}$. Now since D is 2-generated, there exist an element $a_i \in A$ such that $\{a_i,\ a_j\} \in \mathcal{C}$. Hence it is clear that $C = D \cup E$ is 2-generated. If $|E \cup \{a_j\}| = n$, that is $E \cup \{a_j\} = C$. Now since E is 2-generated, there exist an element $a_k \in E$ such that $E - \{a_k\}$ is a 2-generated element in $\mathcal{C}$. Let $A\ '' = \{a_k\}$, $B\ '' = E - \{a_k\}$ and $C\ '' = \{a_j\}$, then $\{a_j,\ a_k\} \in \mathcal{C}$ or $C - \{a_k\} = B\ '' \cup \{a_j\} \in \mathcal{C}$. If $\{a_j,\ a_k\} \in \mathcal{C}$ and since E is 2-generated, it is clear that C is 2-generated. If $C - \{a_k\} \in \mathcal{C}$, then by induction hypothesis it is 2-generated and hence by the condition C_3, there exist $a_j \in C$ such that $\{a_j,\ a_k\} \in \mathcal{C}$. Therefore C is 2-generated. Conversely assume that $(X, \mathcal{C})$ is a 2-generated c-space.To show that $\mathcal{C}$ is a connectology on X. Let $A, B \in \mathcal{C}$ and $A \cup B \in \mathcal{C}$ If $A \cap B \neq \emptyset$, take any $x \in A \cap B$ we get $\{x\} \cup A = A \in \mathcal{C}$ and $\{x\} \cup B = B \in \mathcal{C}$ If $A \cap B = \emptyset$. Since $A \cup B$ is 2-generated connected set, by Proposition 3.6 there is an $a_1 \in A \cup B$ such that $(A \cup B) - \{a_1\} \in \mathcal{C}$. Again $(A \cup B) - \{a_1\}$ is 2-generated, delete one more and repeat the process we reach a stage such that $(A \cup B) - \{a_1, a_2, \cdots, a_k\}$ is 2-generated and

1. $|A \cap ((A \cup B) - \{a_1, a_2, \cdots, a_k\})| = 1$ or
2. $|B \cap ((A \cup B) - \{a_1, a_2, \cdots, a_k\})| = 1$

Without loss of generality, we assume (1) happens. Let a_j be the element in $A \cap ((A \cup B) - \{a_1, a_2, \cdots, a_k\})$, since $(A \cup B) - \{a_1, a_2, \cdots, a_k\}$ is 2-generated , there exist $a_i \in B$ such that $\{a_j, a_i\} \in \mathcal{C}$. Take $a_j = x$, therefore $\{x\} \cup A = A \in \mathcal{C}$ and $\{x\} \cup B \in \mathcal{C}$. Hence $\mathcal{C}$ satisfies condition C_3 of Definition 3.7. Now let $A, B, C \in \mathcal{C}$ and are disjoint, $A \cup B \cup C \in \mathcal{C}$. To show that $A \cup C \in \mathcal{C}$ or $B \cup C \in \mathcal{C}$. Let $A = \bigcup_{i=1}^{n} A_i$, $B = \bigcup_{j=1}^{m} B_i$ and $C = \bigcup_{k=1}^{r} C_i$, where $\{A_i, i = 1, 2, \cdots, n\}$, $\{B_j, j = 1, 2, \cdots, m\}$ and $\{C_k, i = 1, 2, \cdots, r\}$ are 2-element generating sets of A, B and C respectively. Now $C \cup (A \cup B) \in \mathcal{C}$, then there exists $a \in C$ and $b \in (A \cup B)$ such that $\{a, b\} \in \mathcal{C}$. If $b \in A$, then $A \cup C = \bigcup_{i=1}^{n} A_i \cup \bigcup_{k=1}^{r} C_i$ where $\{A_i, i = 1, 2, \cdots, n\} \cup \{B_j, j = 1, 2, \cdots, m\}$ is a 2-element generating set of $A \cup B$ and hence $A \cup C \in \mathcal{C}$ If $b \in B$, as above argument we have $B \cup C \in \mathcal{C}$. Now we take $A, B, C_1, C_2 \in \mathcal{C}$ and are disjoint,

$A\cup B\cup\bigcup_{i=1}^{2} C_i\in\mathcal{C}$, then $C_2\cup(A\cup B\cup C_1)\in\mathcal{C}$, which implies there exists $x\in C_2$ and $y\in(A\cup B\cup C_1)$ such that $\{x,y\}\in\mathcal{C}$ If $y\in A$, then $A\cup C_2\in\mathcal{C}$. By applying the proof in last paragraph to $A\cup C_2,\ B$ and C_1, We get ($A\cup C_2\cup C_1$ and B are connected) or ($A\cup C_2$ and $B\cup C_1$ are connected). Similarly we can prove ($B\cup C_2\cup C_1$ and A are connected) or ($B\cup C_2$ and $A\cup C_1$ are connected) if $y\in B$, If $y\in C_1$, the results follows by considering the sets $C_1\cup C_2, A, B$ and apply the argument given for $A,\ B,\ C$ in previous paragraph. Then $\mathcal{C}$ satisfies condition C_4 of Definition 3.7. Hence the result. □

The example 3.2 (5) shows that there is an infinite non 2-generated connective space and example 3.2 (1) shows that a finite c-space need not be 2-generated.

Remark 3.9. *It is clear that a finite 2-generated c-space $(X,\ \mathcal{C})$ uniquely corresponds to a simple graph $G=(X,E)$, where $E=\{A : A\in\mathcal{C}$ and $|A|=2\}$. Also note that the set of all subsets of X connected in G is $\mathcal{C}$. Thus from Theorem 3.8 finite connective spaces precisely simple graphs.*

4 Topological c-spaces and Connective Spaces

Definition 4.1. *A c-space $(X,\mathcal{C})$ is said to be* ***topological c-space*** *if there exist a topology τ on X such that the collection of all connected subsets of $(X,\ \tau)$ is $\mathcal{C}$.*

Example 4.2. *1. The indiscrete c-space $(X,\mathcal{I})$ is a topological c-spaces, since the set of connected sets of X with indiscrete topology is $\mathcal{I}=\mathcal{P}(X)$.*

2. The discrete c-space $(X,\mathcal{D})$ is a topological c-spaces, since the set of connected sets of X with discrete topology is $\mathcal{D}$.

3. The Co-finite c-space $(X,\mathcal{C})$ is topological, since the set of connected sets of X with Co-finite topology is $\mathcal{C}$.

4. Let X be any finite set with $|\ X\ |\geqslant 3$, consider a c-structure $\mathcal{C}=\mathcal{D}\cup\{X\}$ on X, then $(X,\mathcal{C})$ is a c-space. It is not topological since finite T_1 spaces are discrete.

Theorem 4.3. *A finite topological c-space is 2-generated*

Proof. Let $(X,\ \mathcal{C})$ be a finite topological c-space and $\mathcal{B}=\{A\in\mathcal{C} : |A|=2\}$. Now we claim $\mathcal{C}=\langle\mathcal{B}\rangle$. Suppose not, then $\langle\mathcal{B}\rangle\subsetneqq\mathcal{C}$, that is there exists $C\in\mathcal{C}$ such that $C\notin\langle\mathcal{B}\rangle$. Clearly $|C|>2$, then there exist at least two elements $a,b\in C$ such that there is no 2-element sequence will connect them. Let C_a be the set of all elements in C which are connected to a and $C_b=C-C_a$. Fix $y\in C_b$, then for $\forall x\in C_a,\ \{x,y\}\notin\langle\mathcal{B}\rangle$ (by definition of C_a and C_b). Now by definition of $\langle\mathcal{B}\rangle$, $\{x,y\}\notin\mathcal{C}$. Since $(X,\ \mathcal{C})$ is topological, $\{x,y\}$ is disconnected implies that there exists an open set $O_{x,\ y}$ containing x which does not contains y and $O_{y,\ x}$ containing y which does not contains X. Define $U_y=\cup_{x\in C_a}O_{x,\ y}$ and $V_y=\cap_{x\in C_a}O_{y,\ x}$, then $C_a\subset U_y$ and $y\in V_y$. But $y\notin U_y$ and $C_a\cap C_b=\emptyset$. Now take $V_a=\cup_{y\in C_b}V_y$, then $C_b\subset V_a$ and $C_a\cap V_a=\cup_{y\in C_b}(C_a\cap V_y)=\emptyset$. Thus V_a is an open set containing C_b which does not meets C_a. Similarly there exists U_a containing C_a which does not meets C_b. Thus C_a and C_b are separated in C and $C_a\cup C_b=C$. Therefore C is disconnected, a contradiction. Hence $\mathcal{C}=\langle\mathcal{B}\rangle$. Thus any finite topological c-space is 2-generated. □

Remark 4.4. *In general the converse of Theorem 4.3 is not true, it follows from Example 4.5. From Example 4.2 (3) we can note that Theorem 4.3 is not true for infinite space.*

Example 4.5. *Let $X = \{a, b, c, d, e, \}$, $\mathcal{B} = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{e, a\}\}$ and $\mathcal{C} = \langle\mathcal{B}\rangle$. Then $(X, \mathcal{C})$ is a 2-generated c-space.*

suppose $(X, \mathcal{C})$ is topological, then there exist a topology τ on X such that the collection of all connected subsets of $(X, \mathcal{C})$ is $\mathcal{C}$.

Let R be the associated preorder of τ [1]. Since $\{a, b\} \in \mathcal{C}$ if and only if $(a, b) \in R$ or $(b, a) \in R$. Similarly $\{b, c\} \in \mathcal{C}$ if and only if $(b, c) \in R$ or $(c, b) \in R$ $\{c, d\} \in \mathcal{C}$ if and only if $(c, d) \in R$ or $(d, c) \in R$ $\{d, e\} \in \mathcal{C}$ if and only if $(d, e) \in R$ or $(e, d) \in R$ $\{e, a\} \in \mathcal{C}$ if and only if $(e, a) \in R$ or $(a, e) \in R$.

Case I: $(a, b) \in R$ and $\{a, c\} \notin \mathcal{C}$ implies $(b, c) \notin R$ [Since R is transitive]. Therefore $(c, b) \in R$. Continuing the above argument the collection $\{(a, b), (c, b), (c, d), (e, d), (e, a)\}$ is subset of R and R is transitive. Therefore $(a, b) \in R$ and $(e, a) \in R$ implies $(e, b) \in R$. Thus $\{e, b\}$ is connected, a contradiction. Case II: Starting from $(b, a) \in R$ and proceed as in Case I, we get a collection $\{(b, a), (b, c), (d, c), (d, e), (a, e)\}$ subset of R and R is transitive. Then $(b, a) \in R$ and $(a, e) \in R$ implies $(b, e) \in R$. Thus $\{b, e\}$ is connected, a contradiction. Hence $(X, \mathcal{C})$ is not a topological c-space, but it is 2-generated.

Definition 4.6. *Let R be a relation on X and $(X, \mathcal{C})$ be a c-space, then R is said to be* ***Compatible*** *with $(X, \mathcal{C})$ if $\{x, y\} \in \mathcal{C}$ if and only if $(x, y) \in R \cup R^{-1}$*

Theorem 4.7. *A c-space $(X, \mathcal{C})$ is topological if it is 2-generated and there is a compatible transitive relation R on X.*

Proof. Suppose that $(X, \mathcal{C})$ is a 2-generated c-space with a compatible transitive relation R. Then $\{x\} \in \mathcal{C}$ implies $(x, x) \in R$, thus R is a pre-order on X. Let τ be any topology on X such that $\mu(R) \leqslant \tau \leqslant \nu(R)$, then $\rho(\tau) = R$. Now let $\mathcal{C}_\tau$ be the collection of all connected subsets of (X, τ). Suppose $\{x, y\} \in \mathcal{C}_\tau$ if and only if $(x, y) \in R$ or $(y, x) \in R$ if and only if $(x, y) \in R \cup R^{-1}$ if and only if $\{x, y\} \in \mathcal{C}$. Therefore $\mathcal{C}_\tau = \mathcal{C}$, since $\mathcal{C}$ are 2-generated.Thus $(X, \mathcal{C})$ is topological. □

From Theorem 4.3 and 4.7, we have the following

Theorem 4.8. *A finite c-space $(X, \mathcal{C})$ is topological if and only if it is 2-generated and there is a compatible transitive relation R on X.*

Theorem 4.9. *A finite connective space $(X, \mathcal{C})$ is topological if and only if there exists a compatible transitive relation R on X.*

Proof. It follows from Theorem 3.8 and 4.7. □

5 Closure Operator

Definition 5.1. *A c-space $(X, \mathcal{C})$ is said to be induced by a closure operator V on X , if the collection of all connected subsets of (X, V) is $\mathcal{C}$.*

Theorem 5.2. *A finite c-space is induced by a closure operator is 2-generated.*

Proof. Let $(X, \mathcal{C})$ be a finite c-space is induced by a closure operator V and $\mathcal{B} = \{A \in \mathcal{C} : |A| = 2\}$. Now we claim $\mathcal{C} = \langle\mathcal{B}\rangle$. Suppose not , then $\langle\mathcal{B}\rangle \subsetneqq \mathcal{C}$, there exists $C \in \mathcal{C}$ such that $C \notin \langle\mathcal{B}\rangle$. Clearly $|C| > 2$, then there exists at least two elements $a, b \in C$ such that there is no 2-element sequence will connect them. Let C_a be the set of all elements in C which are connected to a and $C_b = C - C_a$. Fix $y \in C_b$, then for $\forall x \in C_a$, $\{x, y\} \notin \langle\mathcal{B}\rangle$. Now by definition of $\langle\mathcal{B}\rangle$, $\{x, y\} \notin \mathcal{C}$. Since the closure operator V induces the c-space $(X, \mathcal{C})$, $\{x, y\} \notin \mathcal{C}$ implies that $\{x, y\}$ is disconnected with respect to V and hence $y \notin V(\{x\})$ and $x \notin V(\{y\})$. Therefore $y \notin V(\{x\}), \forall x \in C_a$. Thus $y \notin V(C_a)$ since $V(C_a) = \bigcup_{x \in C_a} V(\{x\})$. That is $y \in C_b$ implies that $y \notin V(C_a)$, and hence $V(C_a) \cap C_b = \emptyset$. Similarly $C_a \cap V(C_b) = \emptyset$. Thus C_a and C_b are semi-separated in C and $C_a \cup C_b = C$. Therefore C is disconnected, a contradiction to our assumption. Hence $\mathcal{C} = \langle\mathcal{B}\rangle$. Thus any finite c-space is induced by a closure operator is 2-generated. □

Theorem 5.3. *Every finite 2-generated c-space is induced by a closure operator.*

Proof. Suppose $(X, \mathcal{C})$ is a finite 2-generated c-space. Now define a relation R on X by $(x,\ y) \in R$ if and only if $\{x,\ y\} \in \mathcal{C}$. Clearly R is a reflexive relation on X. Consider any closure operator V on X such that $\mu(R) \leqslant V \leqslant \nu(R)$, then $\rho(V) = R$ (where $\rho(V)$ is the relation on X defined by $(x,\ y) \in \rho(V)$ if and only if $x \in V(\{y\})$)[13]. Let $\mathcal{C}_V$ be the c-structure with respect to V on X.Then $\{x, y\} \in \mathcal{C}_V$ if and only if $(x, y) \in R$ or $(y, x) \in R$ if and only if $(x, y) \in R \cup R^{-1}$ if and only if $\{x, y\} \in \mathcal{C}$. Therefore $\mathcal{C}_V = \mathcal{C}$, since $\mathcal{C}$ are 2-generated. Thus $(X, \mathcal{C})$ is induced by a closure operator. □

6 Lattice Properties of c-spaces

The set of all c-structures on a set X is a partially ordered set under usual set inclusion. This poset is denoted by $LCS(X)$. Also note that the discrete c-structure $\mathcal{D}$ is the least element and the indiscrete c-structure $\mathcal{I}$ is the greatest element in $LCS(X)$.

Theorem 6.1. *$LCS(X)$ is a complete lattice.*

Proof. Let $\{C_i : i \in I\}$ be any subset of $LCS(X)$ and let $C = \cap C_i$. Then C is a lower bound of $\{C_i : i \in I\}$ Now $C\ ' \in LCS(X)$ and $C\ ' \subseteq C_i$ for $i \in I$ implies that $C\ ' \subseteq C$. Thus C is the g.l.b of $\{C_i : i \in I\}$. Thus any subset of $LCS(X)$ has a meet. Hence $LCS(X)$ is a complete lattice. □

Remark 6.2. *It can be note that $LCS(X)$ is an atomic lattice, whose atoms are the c-structures of the form $\{A\} \cup \mathcal{D}$ where $A \subseteq X$, $|A| \geqslant 2$. If X is a finite set with n elements, then $LCS(X)$ has $2^n - (n+1)$ atoms. And if X an infinite set, then $LCS(X)$ has $2^{|X|}$ atoms.*

Theorem 6.3. *The dual atoms in $LCS(X)$ are of the form $\mathcal{P}(X) - \{\{a, b\}\},\ a, b \in X$and $a \neq b$. If $|X| = n$, then $LCS(X)$ contains nC_2 dual atoms.*

Proof. Let $\mathcal{C}$ be any element in $(LCS(X), \subseteq)$. If $\mathcal{C}$ contains all 2-element sets ($ie.\{\{a, b\}\}$), then $\mathcal{C} = \mathcal{P}(X)$ If $\mathcal{C}$ does not contains $\{a, b\}$, and $\{c, d\}$ with $\{a, b\} \neq \{c, d\}$, then $\mathcal{C}' =< \mathcal{C} \cup \{a, b\} >$ is an element in $LCS(X)$ such that $\mathcal{C}' \neq \mathcal{P}(X)$ and $\mathcal{C} \subsetneq \mathcal{C}'$ Therefore $\mathcal{C}$ is not a dual atom in $LCS(X)$. Thus any dual atom contains all but one 2- element set. Hence it is the form $\mathcal{P}(X) - \{\{a, b\}\}, \quad a, b \in X$ and $a \neq b$. □

Remark 6.4. *If $|X| > 2$, then $LCS(X)$ is non-modular and hence non-distributive.*

Example 6.5. *Let $X = \{a, b, c\}$, Consider $\mathcal{C}_1 = \{\{a, b\}\} \cup \mathcal{D}$, $\mathcal{C}_2 = \{\{b, c\}\} \cup \mathcal{D}$ and $\mathcal{C}_3 = \{\{a, b\}, \{a, b, c\}\} \cup \mathcal{D}$ Note that $\mathcal{C}_1 \subseteq \mathcal{C}_3$, $\mathcal{C}_1 \vee (\mathcal{C}_2 \wedge \mathcal{C}_3) = \mathcal{C}_1$ and $(\mathcal{C}_1 \vee \mathcal{C}_2) \wedge \mathcal{C}_3) = \mathcal{C}_3$ $\therefore \mathcal{C}_1 \vee (\mathcal{C}_2 \wedge \mathcal{C}_3) \neq (\mathcal{C}_1 \vee \mathcal{C}_2) \wedge \mathcal{C}_3)$. Hence $LCS(X)$ is non- modular.*

Remark 6.6. *Let X be any set with $|X| \geqslant 3$. Then $LCS(X)$ is not dually atomic, since the meet of all dual atoms in $LSC(X)$ is the c-structure $\mathcal{C} = \{A \subseteq X : |A| \geqslant 3\} \cup \mathcal{D}$; and there are c-structures which are properly contained in $\mathcal{C}$*

Remark 6.7. *For $|X| > 2$ the lattice $LCS(X)$ is not complemented.*

Example 6.8. *Let $X = \{a, b, c,\}$ and $\mathcal{C}_1 = \{\{a, b\}, \{a, b, c\}\} \cup \mathcal{D}$ Suppose $\mathcal{C}_1$ is complemented, that is there exists $\mathcal{C}_2$ such that $\mathcal{C}_1 \vee \mathcal{C}_2 = \mathcal{P}(X)$. Then $\mathcal{C}_2 \supseteq \mathcal{P}(X) - \{\{a, b\}\}$ and hence $\mathcal{C}_2 = \mathcal{P}(X)$ or $\mathcal{C}_2 = \mathcal{P}(X) - \{\{a, b\}\}$. In either case $\mathcal{C}_1 \wedge \mathcal{C}_2 \supseteq \{\{a, b, c\}\} \cup \mathcal{D}$. Therefore $\mathcal{C}_1 \wedge \mathcal{C}_2 \neq \mathcal{D}$ Hence $LCS(X)$ is not complemented*

Theorem 6.9. *Let X be any set. Then an atom $\mathcal{C} = \{A\} \cup \mathcal{D}$ of $LCS(X)$ is topological if $A \subseteq X$ and $|A| = 2$.*

Proof. If $|A| = 2$, consider the topology $\tau = \{A, X\} \cup \mathcal{P}(X - A)$ on X, then the connected sets of X w.r.t. τ is $\mathcal{C}$. Therefore $\mathcal{C}$ is topological. □

Theorem 6.10. *Let X be any finite set. Then an atom $\mathcal{C} = \{A\} \cup \mathcal{D}$ of $LCS(X)$ is topological if and only if $|A| = 2$*

Proof. From Theorem 6.9 it is clear that if $|A| = 2$, then $\mathcal{C}$ is topological. Now suppose that $|A| \geqslant 3$. If $(X, \mathcal{C})$ is topological c-space, then there exists a topology τ on X such that the connected sets of X are precisely $\mathcal{C}$. Therefore any 2-elements set $\{a, b\}$ of X is disconnected, then there exists an open set in X which containing a, not contains b and vice-versa.That is (X, τ) is a finite T_1 space. So it is a discrete space. Therefore A is disconnected, a contradiction. Hence $(X, \mathcal{C})$ is not topological. □

Remark 6.11. *Let X be any set with $|X| \geqslant 5$. Then the set of all topological c-structures on X is not a sublattice of $LCS(X)$.*

Proof. Let $\{a, b, c, d, e\} \subseteq X$ and $\mathcal{C}_1 = \langle\{\{a, b\}, \{b, c\}, \{c, d\}\}\rangle$ and $\mathcal{C}_2 = \langle\{\{d, e\},\ \{e, a\}\}\rangle$. Then $\mathcal{C}_1 \vee \mathcal{C}_2 = \langle\{\{a, b\},\ \{b, c\},\ \{c, d\},\ \{d, e\},\ \{e, a\}\}\rangle$. Then by Example 4.5, $\mathcal{C}_1 \vee \mathcal{C}_2$ is not topological. □

Theorem 6.12. *The dual atoms in $LCS(X)$ are topological*

Proof. Let us consider any dual atom $\mathcal{C} = \mathcal{P}(X) - \{\{a, b\}\}$, $a, b \in X$ and $a \neq b$. Consider the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ on X, then the connected sets of X w.r.t. τ is $\mathcal{C}$. Hence $\mathcal{C}$ is topological. □

Remark 6.13. *The collection of connectologies on a set X is a partially ordered set with usual inclusion relation. This poset is denoted by $CNS(X)$. But Example 6.14 shows that it is not a lattice*

Example 6.14. *Let $X = \mathbb{R}$ and let $A = [-10, \infty)$, $B = (-\infty, 10], C = [-10, 0), D = [0, 10], C' = [-10, 0]$ and $D' = [0, 10] \cup \{-1\}$ are subset of X Define $\mathcal{C}_1 = \{A, B, C, A \cap B\} \cup \mathcal{D}$, $\mathcal{C}_2 = \{A, B, D, A \cap B\} \cup \mathcal{D}$, $\mathcal{C}_3 = \{A, B, C, D, C', A \cap B\} \cup \mathcal{D}$, $\mathcal{C}_4 = \{A, B, C, D, D', A \cap B\} \cup \mathcal{D}$. It is clear that $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ are connectologies on X and $\mathcal{C}_1 \cup \mathcal{C}_2 = \{A, B, C, D, A \cap B\} \cup \mathcal{D}$. Also note that $C \cup D = A \cap B \in \mathcal{C}_1 \cup \mathcal{C}_2$, but no $x \in C \cup D$ such that $C \cup \{x\}$ and $D \cup \{x\}$ is in $\mathcal{C}_1 \cup \mathcal{C}_2$. Therefore $\mathcal{C}_1 \cup \mathcal{C}_2$ is not a Connectology on X. Now $\mathcal{C}_1 \cup \mathcal{C}_2 \subsetneq \mathcal{C}_3$ and $\mathcal{C}_1 \cup \mathcal{C}_2 \subsetneq \mathcal{C}_4$. Also there is no connectology $\mathcal{C}$ on X such that $\mathcal{C}_1 \cup \mathcal{C}_2 \subsetneq \mathcal{C} \subsetneq \mathcal{C}_3$ or $\mathcal{C}_1 \cup \mathcal{C}_2 \subsetneq \mathcal{C} \subsetneq \mathcal{C}_4$. Thus $\mathcal{C}_1 \vee \mathcal{C}_2$ does not exists in $CNS(X)$.*

Theorem 6.15. *Let $L2GCS(X)$ be the collection of all 2-generated c-structures on a set X. Then it is a complete, atomic Boolean lattice with respect to inclusion relation.*

Proof. Clearly $L2GCS(X)$ is a poset under set inclusion relation. Consider a subfamily $\{C_i : i \in I\}$ of $L2GCS(X)$. Then for each $i \in I$, define $\mathcal{B}_i = \{A \in C_i : |A| = 2\}$. then $C_i = \langle \mathcal{B}_i \rangle$, for all $i \in I$. Define $C = \langle \cup_{i \in I} \mathcal{B}_i \rangle$, then $C \in \mathcal{G}$ and $C_i \subseteq C$ for each $i \in I$. Let us suppose that there exist $C\,' \in \mathcal{G}$ with $C_i \subseteq C\,', \forall i \in I$ and $C\,' \subsetneq C$. Then there exists $\{a,\ b\} \in C$ such that $\{a,\ b\} \notin C\,'$. Now $\{a,\ b\} \in C$ implies there exists $i \in I$ such that $\{a,\ b\} \in \mathcal{B}_i$ and hence there exists $i \in I$ such that $\{a,\ b\} \in C_i$. Therefore $\{a,\ b\} \notin C\,'$ implies $C_i \subsetneq C\,'$. It is a contradiction. Therefore C is the l.u.b of $\{C_i : i \in I\}$, ie $L2GCS(X)$ is a partially ordered set with every subset has a join . Hence $L2GCS(X)$ is a complete lattice. The atoms of $L2GCS(X)$ are precisely $\{\mathcal{C}_{a,\ b} : a, b \in X\}$ where $\mathcal{C}_{a,\ b} = \{\emptyset\} \cup \mathcal{S} \cup \{\{a,\ b\}\}$. Thus for any $\mathcal{C} \in \mathcal{G}(X)$ it is clear that $\mathcal{C} = \bigcup_{\{a,\ b\} \in \mathcal{C}} \mathcal{C}_{a,\ b}$. Therefore $L2GCS(X)$ is atomic. Also note that any $\mathcal{C} \in \mathcal{G}(X)$ has the complement $\mathcal{C}\,'$ given by $\mathcal{C}\,' = \bigcup_{\{a,\ b\} \notin \mathcal{C}} \mathcal{C}_{a,\ b}$. Therefore $L2GCS(X)$ is complemented and dually atomic. Also it is clear that $L2GCS(X)$ is a distributive lattice. Thus $L2GCS(X)$ is a Boolean lattice. $\square$

Remark 6.16. *From Theorem 6.15 and Theorem 3.8, it is clear that when X is finite, the family of Connectologies on X is a complete lattice under usual inclusion relation.*

The following Example 6.17 shows that the family of 2-generated c-spaces on a set X is not a sublattice of the complete lattice $LCS(X)$

Example 6.17. *Let X be any set with $|X| \geqslant 3$ and let $\{a, b, c\} \subseteq X$. Then $\mathcal{P}(X) - \{a,\ b\}$ and $\mathcal{P}(X) - \{b,\ c\}$ are 2-generated c-spaces. Now the meet of $\mathcal{P}(X) - \{a,\ b\}$ and $\mathcal{P}(X) - \{b,\ c\}$ in $LCS(X)$ is $\mathcal{P}(X) - \{\{a,\ b\},\ \{b, c\}\}$. But it is not 2-generated since $\{a, b, c\}$ is not a 2-generated set in $\mathcal{P}(X) - \{\{a,\ b\},\ \{b, c\}\}$.*

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