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RESEARCH ARTICLE

On special concircular K-Lie-recurrence in special Finsler spaces

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Abstract The concept of a Lie-recurrence was given by P. N. Pandey in 1982. Since then several mathematicians contributed significantly towards Lie-recurrence. Some of them used the term curvature inheriting symmetry in place of Lie-recurrence. Recently the present authors (P. N. Pandey and Vaishali Pandey) introduced and studied K-Lie-recurrence in a Finsler space. The aim of the present paper is to discuss a special concircular K-Lie-recurrence in special Finsler spaces such as K-recurrent, K-symmetric, K-birecurrent and K-bisymmetric. Apart from other theorems, it is being proved that a K-recurrent Finsler space $F_n(n > 2)$ can not admit a special concircular K-Lie-recurrence while a non-flat K-symmetric Finsler space $F_n(n > 2)$ is necessarily of constant Riemannian curvature.

Key Words Lie-recurrence, special concircular K-Lie-recurrence, K-recurrent Finsler space

MSC 2010 53B40, 54A05

1 Introduction

In 1982, P. N. Pandey [5] defined the concept of Lie-recurrence in a Finsler space. In 1992, K. L. Duggal [3] studied the Lie-recurrence in a Riemannian space with its application to fluid space time but he used the term curvature inheriting symmetry in place of Lie-recurrence. He also applied the theory to the study of fluid space time. Since then both the terms (Lie-recurrence and curvature inheriting symmetry) are in use [1, 11, 12, 13, 14, 15]. The present authors [12, 15] studied a $\widetilde{K}$-curvature inheritance, $\widetilde{K}$ projective Lie-recurrence and special concircular Lie-recurrence in a Finsler space. Recently present authors (P. N. Pandey and Vaishali Pandey) [13] discussed a K-Lie-recurrence in a Finsler space. Shivalika Saxena and P. N. Pandey [14] studied a Lie-recurrence and Projective Lie-recurrence in a Finsler space generated by contra and concurrent vector fields. P. N. Pandey and Shivalika Saxena [11] also studied a projective N-curvature inheritance. Pandey and Saxena established that an infinitesimal transformation in a Finsler space is Lie-recurrence if and only if the normal projective curvature tensor is Lie-recurrent. C. K. Mishra and Gautam Lodhi [1] discussed curvature inheriting symmetry and Ricci-inheriting symmetry in a Finsler space and investigated some results. The aim of the present paper is to discuss a special concircular K-Lie-recurrence in special Finsler spaces such as K-recurrent, K-symmetric, K-birecurrent and K-bisymmetric.

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Let $F_n = (M^n, L)$ be an n-dimensional Finsler space whose underlying manifold is M^n and the fundamental metric function is $F(x,y)$, where $x = (x^i)$ is a point of M^n and $y = (y^i)$ is an element of support [2]. The metric tensor g_{ij} and the fundamental function F are connected by

$$a) \;\; g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2, \quad b) \;\; g_{ij}y^iy^j = F^2, \tag{1}$$

where $\dot{\partial}_i \equiv \frac{\partial}{\partial y^i}$. The covariant derivative of an arbitrary vector field X^i with respect to connection coefficients Γ^{*i}_{jk} is given by

$$X^i_{|k} = \partial_k X^i - \left(\dot{\partial}_r X^i\right)\Gamma^{*r}_{hk}y^h + X^r\Gamma^{*i}_{rk}, \tag{2}$$

where $\partial_k \equiv \frac{\partial}{\partial x^k}$. This type of covariant derivative introduced by Cartan is called as h-covariant derivative. Ricci-commutation formula for such covariant derivative is given by

$$X^i_{|h|k} - X^i_{|k|h} = X^r K^i_{rhk} - \left(\dot{\partial}_r X^i\right) K^r_{shk}y^s, \tag{3}$$

where $K^i_{rhk} = \left(\partial_k\Gamma^{*i}_{rh}\right) - \left(\partial_h\Gamma^{*i}_{rk}\right) + \left(\dot{\partial}_m\Gamma^{*i}_{rk}\right)\Gamma^{*m}_{ph}y^p - \left(\dot{\partial}_m\Gamma^{*i}_{rh}\right)\Gamma^{*m}_{pk}y^p + \Gamma^{*s}_{rh}\Gamma^{*i}_{sk} - \Gamma^{*s}_{rk}\Gamma^{*i}_{sh}$. The tensor K^i_{jkh} is called Cartan curvature tensor or h-curvature tensor.This tensor is skew-symmetric in last two lower indices and positively homogeneous of degree zero in y^i. Cartan curvature tensor K^i_{jkh}, Berwald curvature tensor H^i_{jkh} and the tensor H^i_{kh} are related by

$$H^i_{jkh} = K^i_{jkh} + y^r\dot{\partial}_j K^i_{rkh}, \tag{4}$$

and

$$K^i_{jkh}y^j = H^i_{kh}. \tag{5}$$

The tensor H^i_{kh} is connected with Berwald deviation tensor H^i_h by

$$a) \;\; y^k H^i_{kh} = H^i_h, \quad b) \;\; \dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k = 3H^i_{kh}. \tag{6}$$

Berwald deviation tensor satisfies the following:

$$a) \;\; g_{ik}H^i_h = g_{ih}H^i_k, \;\; b) \;\; y_iH^i_h = 0, \quad c) \;\; H^i_i = (n-1)H, \tag{7}$$

where $y_i = g_{ij}y^j$ and H is scalar curvature. The commutation formula for the operators of partial differentiation with respect to y^k and h-covariant differentiation is given by

$$\dot{\partial}_k\left(X^i_{|h}\right) - \left(\dot{\partial}_k X^i\right)_{|h} = X^r\dot{\partial}_k\Gamma^{*i}_{rh} - \left(\dot{\partial}_r X^i\right)\left(\dot{\partial}_k\Gamma^{*r}_{sh}\right)y^s. \tag{8}$$

Let us consider an infinitesimal transformation

$$\overline{x}^i = x^i + \epsilon v^i\left(x^j\right), \tag{9}$$

generated by a contravariant vector field $v^i\left(x^j\right)$ which depends upon position co-ordinates only where ϵ is an infinitesimal constant. The Lie-derivative of an arbitrary tensor T^i_j with respect to the infinitesimal transformation(9) is given by [4]

$$\pounds T^i_j = T^i_{j|r}v^r - T^r_j v^i_{|r} + T^i_r v^r_{|j} + (\dot{\partial}_r T^i_j)v^r_{|s}y^s\,. \tag{10}$$

The commutation formula for the operators $\pounds$ and $\dot{\partial}_h$ is given by

$$\dot{\partial}_h\pounds\Omega - \pounds\dot{\partial}_h\Omega = 0\,, \tag{11}$$

where Ω is any geometrical object.
An infinitesimal transformation (9) is called Lie-recurrence or H-Lie-recurrence (curvature inheritance) if the Lie-derivative of Berwald curvature tensor H^i_{jkh} of the Finsler space satisfies

$$\pounds H^i_{jkh} = \Phi H^i_{jkh}, \tag{12}$$

where ϕ is a non-zero scalar field [5]
In view of this terminology, the infinitesimal transformation (9) is called K-Lie-recurrence if the Lie-derivative of Cartan's curvature tensor K^i_{jkh} satisfies [13]

$$\pounds K^i_{jkh} = \Phi K^i_{jkh}, \ \ \phi \neq 0, \tag{13}$$

2 Special Concircular K-Lie-recurrence

Let us consider a Finsler space admitting the infinitesimal transformation (9) generated by a special concircular vector field $v^i(x^j)$ characterized by

$$v^i_{|k} = \rho \delta^i_k, \tag{14}$$

where ρ is not a constant [13, 15].
Differentiating (14) covariantly with respect to x^h, we get

$$v^i_{|k|h} = \rho_h \delta^i_k, \tag{15}$$

where $\rho_h = \rho_{|h}$.
Taking skew-symmetric part of (15) and utilizing the commutation formula (3), we have

$$v^r K^i_{rkh} = \rho_h \delta^i_k - \rho_k \delta^i_h. \tag{16}$$

Contraction of the indices i and h in (16) gives

$$v^r K_{rk} = -(n-1)\rho_k, \tag{17}$$

where K_{rk} is Ricci tensor defined as $K_{rk} = K^h_{rkh}$. From equations (16) and (17), we may write

$$v^r K^i_{rkh} = \frac{1}{n-1} v^r (K_{rk}\delta^i_h - K_{rh}\delta^i_k), \tag{18}$$

which implies

$$\left\{ K^i_{rkh|m} - \frac{1}{n-1}(K_{rk|m}\delta^i_h - K_{rh|m}\delta^i_k) \right\} v^r + \rho \left\{ K^i_{mkh} - \frac{1}{n-1}(K_{mk}\delta^i_h - K_{mh}\delta^i_k) \right\} = 0. \tag{19}$$

Let the Finsler space F_n be K-recurrent characterized by

$$K^i_{jkh|m} = \lambda_m K^i_{jkh}, \tag{20}$$

where λ_m are components of a non-zero covariant vector field [7]. Contracting the indices i and h in (20), we have

$$K_{jk|m} = \lambda_m K_{jk}. \tag{21}$$

From (18), (19), (20) and (21), we get

$$K^i_{mkh} = \frac{1}{n-1}(K_{mk}\delta^i_h - K_{mh}\delta^i_k). \tag{22}$$

Transvecting (22) by y^m and using equation (5) and $K_{mk}y^m = H_k$, we have

$$H^i_{kh} = \frac{1}{n-1}(H_k\delta^i_h - H_h\delta^i_k), \tag{23}$$

Transvecting (23) by y_i and using $y_iH^i_{kh} = 0$ [8], we get $H_ky_h = H_hy_k$, which implies

$$H_k = \frac{n-1}{F^2}Hy_k, \tag{24}$$

for $H_hy^h = (n-1)H$ and $y_hy^h = F^2$. In view of (24), (23) may be rewritten as

$$H^i_{kh} = R(y_k\delta^i_h - y_h\delta^i_k), \tag{25}$$

where $R = \frac{H}{F^2}$. In view of Berwald theorem [2], equation (25) implies that R is a constant and the space $F_n(n > 2)$ is of constant Riemannian curvature. Differentiating (25) covariantly, we find

$$H^i_{kh|m} = 0, \tag{26}$$

for $y_{k|m} = 0$. Transvecting (20) by y^j and using equation (5), we get $H^i_{kh|m} = \lambda_m H^i_{kh}$, which in view of (26), implies $\lambda_m = 0$, a contradiction. Therefore, a K-recurrent Finsler space $F_n(n > 2)$ can not admit a special concircular infinitesimal transformation. This leads to:

Theorem 2.1. *A K-recurrent Finsler space $F_n(n > 2)$ can not admit a special concircular K-Lie-recurrence.*

Let a Finsler space $F_n(n > 2)$ be K-symmetric characterized by [6]

$$K^i_{jkh|m} = 0. \tag{27}$$

Then equation (19) implies equation (22). Adapting the above proceedure, we may show that equation (22) implies that the space $F_n(n > 2)$ is of constant Riemannian curvature if it is non-flat. Suppose that the special concircular transformation (9) is a Lie-recurrence in the K-Symmetric Finsler space $F_n(n > 2)$. Then we have equation (13).
In view of equation (10), equation (13) may be written as

$$K^i_{jkh|r}v^r + (\dot{\partial}_r K^i_{jkh})v^r_{|s}y^s - K^r_{jkh}v^i_{|r} + K^i_{rkh}v^r_{|j} + K^i_{jrh}v^r_{|k} + K^i_{jkr}v^r_{|h} = \Phi K^i_{hjk},$$

Using equation (14), (27) and the fact that the curvature tensor K^i_{hjk} is positively homogeneous of degree zero in y^j, we get $\phi = 2\rho$ if the space is non-flat.
Since ρ is independent of y^i and $\phi = 2\rho$, ϕ is also independent of y^i. In view of Theorem 2.1 established in the paper [13], the special concircular K-Lie-recurrence is an H-Lie-recurrence. Thus we have:

Theorem 2.2. *A special concircular K-Lie-recurrence in a non-flat K-Symmetric Finsler space $F_n(n > 2)$ is an H-Lie-recurrence and the K-Symmetric Finsler space is necessarily of constant Riemannian curvature.*

3 Special Concircular K-Lie-Recurrence in a Birecurrent Finsler Space

Let us consider a birecurrent Finsler space F_n characterized by

$$K^i_{jkh|l|m} = a_{lm} K^i_{jkh}, \tag{28}$$

where a_{lm} are components of a non-zero covariant tensor of type $(0,2)$ and $K^i_{jkh} \neq 0$ [8, 10].
Suppose that this space admits a special concircular K-Lie-recurrence characterised by equations (14) and (13). In view of (10), equation (13) may be written as

$$K^i_{jkh|r} v^r = (\phi - 2\rho) K^i_{jkh} \tag{29}$$

Differentiating equation (29) covariantly with respect to x^m, we get

$$v^r_{|m} K^i_{jkh|r} + v^r K^i_{jkh|r|m} = (\phi_m - 2\rho_m) K^i_{jkh} + (\phi - 2\rho) K^i_{jkh|m}, \tag{30}$$

where $\phi_m = \phi_{|m}$.
Using equation (14) and equation (28) in equation (30), we have

$$(v^r a_{rm} - \phi_m + 2\rho_m) K^i_{jkh} = (\phi - 3\rho) K^i_{jkh|m}. \tag{31}$$

In view of the definition for a birecurrent Finsler space, $K^i_{jkh} \neq 0$. In equation (31), $K^i_{jkh|m} \neq 0$, for implies $a_{lm} = 0$, a contradiction. Therefore, equation (31) implies either of the following conditions:

$$(i)\ \ \phi - 3\rho = 0, \quad v^r a_{mr} - \phi_m + 2\rho_m = 0,$$

$$(ii)\ \ \phi - 3\rho \neq 0, \quad v^r a_{mr} - \phi_m + 2\rho_m \neq 0.$$

We can write the condition (i) as $\phi = 3\rho$, $v^r a_{mr} = \rho_m$.
Let us consider the condition (ii). In this case equation (31) may be written as

$$K^i_{jkh|m} = \frac{(v^r a_{mr} - \phi_m + 2\rho_m)}{\phi - 3\rho} K^i_{jkh}, \tag{32}$$

which shows that the space is recurrent. In view of Theorem 2.1, a K-recurrent Finsler space $F_n (n > 2)$ does not admit a special concircular K-Lie-recurrence. Therefore, the pair of conditions (ii) is not possible. Hence, we may conclude:

Theorem 3.1. *A birecurrent Finsler space $F_n (n > 2)$ admitting a special concircular K-Lie-recurrence necessarily satisfies the conditions $\phi = 3\rho$ and $v^r a_{mr} = \rho_m$.*

Taking skew-symmetric part of equation (28) and using the Ricci-commutation formula exhibited by (3), we have

$$K^r_{jkh} K^i_{rml} - K^i_{rkh} K^r_{jml} - K^i_{jrh} K^r_{kml} - K^i_{jkr} K^r_{hml} - (\dot{\partial}_r K^i_{jkh}) K^r_{ml} = A_{lm} K^i_{jkh}, \tag{33}$$

where $A_{lm} = a_{lm} - a_{ml}$. Operating both sides of equation (33) by the operator $\pounds$ and using equation (13), we get

$$\pounds A_{lm} = \phi A_{lm}.$$

This leads to:

Theorem 3.2. *The skew-symmetric part of the recurrence tensor a_{lm} of a birecurrent Finsler space admitting a special concircular K-Lie-recurrence is Lie-recurrent with respect to the Lie-recurrence.*

4 Special Concircular K-Lie-Recurrence in a Bisymmetric Finsler space

Let us consider a bisymmetric Finsler space F_n characterized by [9]

$$K^i_{jkh|l|m} = 0. \tag{34}$$

admitting a special concircular K-Lie-recurrence.
Differentiating equation (16) covariantly with respect to x^m, we have

$$v^r_{|m} K^i_{jkh|r} + v^r K^i_{jkh|r|m} = (\phi_m - 2\rho_m) K^i_{jkh} + (\phi - 2\rho) K^i_{jkh|m}, \tag{35}$$

Using equations (14) and (34) in equation (35), we get

$$(\phi - 3\rho) K^i_{jkh|m} = (2\rho_m - \phi_m) K^i_{jkh}, \tag{36}$$

If $\phi = 3\rho$, equation (36) reduces to $\rho_m K^i_{jkh} = 0$ which implies $K^i_{jkh} = 0$ for $\rho_m \neq 0$. Thus, we conclude:

Theorem 4.1. *A K-bisymmetric Finsler space F_n admitting a special concircular K-Lie-recurrence with condition $\phi = 3\rho$ is flat.*

If $\phi = 2\rho$ then $\phi_m = 2\rho_m$. Therefore, equation (36) may be written as

$$K^i_{jkh|m} = 0. \tag{37}$$

This shows that the space is symmetric. Thus, we see that a K-bisymmetric Finsler space admitting a special concircular K-Lie-recurrence with $\phi = 2\rho$ is a symmetric space admitting a special concircular K-Lie-recurrence. Thus we conclude:

Theorem 4.2. *A K-bisymmetric Finsler space $F_n(n > 2)$ admitting a special concircular K-Lie-recurrence with $\phi = 2\rho$ is a K-symmetric Finsler space.*

From Theorem 2.2 and Theorem 4.2, we may conclude:

Theorem 4.3. *A special concircular K-Lie-recurrence in a non-flat K-bisymmetric Finsler space $F_n(n > 2)$ with $\phi = 2\rho$ is an H-Lie-recurrence and the K-bisymmetric Finsler space $F_n(n > 2)$ is necessarily of constant Riemannian curvature.*

If $\phi \neq 2\rho$ and $\phi \neq 3\rho$, then equation (36) may be written as

$$K^i_{jkh|m} = \frac{2\rho_m - \phi_m}{\phi - 3\rho} K^i_{jkh}. \tag{38}$$

This shows that the space is recurrent, but in view of Theorem 2.1, a recurrent space does not admit a special concircular K-Lie-recurrence .
Therefore, we may conclude:

Theorem 4.4. *A K-bisymmetric Finsler space $F_n(n > 2)$ can not admit a special concircular K-Lie-recurrence if ϕ is neither 2ρ nor 3ρ.*

From Theorems 4.1, 4.2, 4.3 and 4.4 we may conclude:

Theorem 4.5. *A K-bisymmetric Finsler space $F_n(n > 2)$ admitting a special concircular K-Lie-recurrence is either flat or a Finsler space of constant Riemannian curvature.*

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