South Asian Journal of Mathematics 2013 , Vol. 4 (1) : 23 ∼ 27
www.sajm-online.com ISSN 2251-1512 RESEARCH ARTICLE

On generalized common fixed point theorem in complete fuzzy metric spaces

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Received: July-18-2013; Accepted: September-5-2013 *Corresponding author

Abstract In this paper we prove common fixed point theorem for six mappings in fuzzy metric space.

Key Words fuzzy contractive mapping , complete fuzzy metric space

MSC 2010 20M17, 20N15

1 Introduction and preliminaries

Zadesh was introduced the concept of fuzzy sets in 1965.analysis many authors have expansively developed the theory of fuzzy sets and application. Michalek [6] have introduced the concept of fuzzy topological spaces induced by fuzzy metric , which have very important application in quantum particle physics .many authors have proved fixed point theorem in fuzzy metric spaces.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \to [0;1]$ is a continuous t-norm if it satisfies the following conditions

(1) $*$ is associative and commutative,

(2) $*$ is continuous,

(3) $a * 1 = a$ for all $a \in [0,1]$

(4) $a * b\cdot \leqslant c * d$ whenever $a\cdot \leqslant candb\cdot \leqslant d$; for each $a, b, c, d \in [0,1]$. Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = min(a;b)$.

Definition 1.2. A 3-tuple $(X; M.*)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0,\infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(1) $M(x,y,t) > 0$,

(2) $M(x,y,t) = 1$ if and lonely if $x = y$,

(3) $M(x,y,t) = M(y,x,t)$,

(4) $M(x,y,t) * M(y,z,s) \leqslant M(x,z,t+s)$,

(5) $M(x,y,t) : (0,\infty) \to [0,1]$ is continuous,

Citation: Manda L.L. Panikath, Vengopalam, Vijaya Kumar, On generalized common fixed point theorem in complete fuzzy metric spaces, South Asian J Math, 2013, 4(1), 23-27.

(6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Remark 1.3. Let (X, M, T) be fuzzy metric space. for $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$ Let $(X, M, *)$ be a fuzzy metric space. (ii) Let be the set of all AX with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t)A$. then τ is a topology on X, This topology is Hausdorff and first countable.

Definition 1.4. A sequence $\{x_n\}$ in X

(1) converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$.

(2) It is called a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exist $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geqslant n_0$.

(3) The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent.

Definition 1.5. A subset A of X is said to be bounded if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all x, y belong to A.

Example 1.6. Let $X = R$. Denote $a * b = ab$ for all $a, b \in [0, 1]$. for each $t \in (0, \infty)$, define $M(x, y, t) = t/(t + |x - y|)$ for all $x, y \in X$.

Definition 1.7. Let $(X, M, *)$ be a metric space, M is said to be continuous of $X^2 \times (0, \infty)$ i.e $\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$ Whenever a sequence $(x_n, y_n, t_n) \in X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times ((0, \infty)$ i.e

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1, \ \lim_{n\to\infty} M(x, x, t_n) = \lim_{n\to\infty} M(x, y, t) = 1.$$

Lemma 1.9. *Let $(X, M, *)$ be a fuzzy metric space. Then M is continuous function of $X^2 \times (0, \infty)$.*

Definition 1.10. Let A and P be mappings from a fuzzy metric space $(X; M; *)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point.

Definition 1.11. Let A and P be mappings from a fuzzy metric space $(X; M; *)$ into itself. Then the mappings are said to be compatible if $APx_n, PAx_n, t) = 1, t > 0$ Whenever $\{x_n\}$ is d a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = x \in X$.

Lemma 1.12 [10]. *Self-mappings A and P of a fuzzy metric space $(x, M, *)$ are compatible. Then they are weak compatible.*

Lemma 1.13. *Let $(x, M, *)$ be a fuzzy metric space.*

(i) If we define $E_\mu M(x_1, x_n) \leqslant E_\mu M(x_1 x_2) + E_\mu M(x_2 x_3) + \cdots + E_\mu M(x_{n-1} x_n)$ for any $x_1, x_2, \cdots, x_n \in X$.

(ii) the sequence $\{x_n\}_{n\in N}$ is convergent in fuzzy metric space $(x, M, *)$ if and only if $E_\mu M(x_1, x) \to 0$. Also the sequence $\{x_n\}_{n\in N}$ is a cauchy sequence if and only it is cauchy with $E_\mu M$.

(iii) If there is a sequence $\{x_n\}$ in X, such that for every $n \in N$ $\lim_{n\to\infty} M(x_n, x_{n+1}, t) \geqslant M(x_0, x_1, k^n t)$ for every $k > 1$, then the sequence $\{x_n\}$ is a Cauchy sequence.

2 The main results

Theorem 2.1. *Let A, T, P, and Q be self mappings of a complete fuzzy metric space $(X, M, *)$ satisfying: $P(X)T(X), Q(X)A(X)$ and $P(X)orQ(X)$ is a closed subset of X, and*

$[F(Pu, Qv, (kx)]^2 \geqslant [F(Au, Tv, (x)]^2, F(Au, Pu, (x).\ F(Tv, Qv, (x).\ F(Au, Tv, (x).F(Au, Pu, (x).$
$F(Au, Tv, (x).\ F(Tv, Qv, (x).F(Au, Tv, (x).\ F(Av, Qu, (x).\ F(Au, Tv, (x).\ F(Tv, Pu, (x).$
$F(Au, Qu, (x).F(Tv, Pu, (x).\ F(Au, Qv(2x), F(Tv, Pu, (x)[F(Au, Qv, (2x)]^2 F(Tv, Qv(x),$

for every x, y in $X, k > 1$. The pairs (A, P) and (Q, T) are weak compatible. Then A, T, P, Q have a unique common fixed point in X.

Proof: for any point x_0 in X, there exists a point $x_1 \in X$, such that $Px_0 = Tx_1$. For this point x_1, we can choose a point x_2 in X, such $Qx_1 = Ax_2$ and so on, in this manner we can define a sequence $\{y_n\}$ in X such that $y_{2n} = Px_{2n} = Tx_{2n+1} = Ax_{2n+2}$ for $n = 0, 1, 2, \cdots$. Now we shall prove $F(y_{2n}, y_{2n+1}, (kx) \geqslant F(y_{2n-1}, y_{2n}, (x)$ for $x > 0$, where $k \in (0, 1)$. Suppose that $F(y_{2n}, y_{2n+1}, (kx) < F(y_{2n-1}, y_{2n}, (x)$ then by using

(ii) $F(y_{2n}, y_{2n+1}, (kx) \leqslant F(y_{2n}, y_{2n+1}, (x))$ we have $[F(y_{2n}, y_{2n+1}, (kx)]^2$
$= [F(Px_{2n}, Qx_{2n+1}), , (kx)]^2$
$\geqslant \min\{[F(y_{2n-1}, y_{2n}, (kx)]^2 F(y_{2n-1}, y_{2n}, (x)F(y_{2n}, y_{2n+1}, (x))$
$F(y_{2n-1}, y_{2n}, (x)F(y_{2n-1}, y_{2n}, (x)F(y_{2n-1}, y_{2n}, (x))F(y_{2n}, y_{2n+1}, (x))F(y_{2n-1}, y_{2n}, (x))$
$F(y_{2n-1}, y_{2n+1}, (2x))F(y_{2n-1}, y_{2n}, (x))F(y_{2n}, y_{2n}, (x))F(y_{2n-1}, y_{2n+1}, (2x))$
$F(y_{2n}, y_{2n}, (x))F(y_{2n-1}, y_{2n}, (x))F(y_{2n}, y_{2n}, (x))F(y_{2n-1}, y_(2n, +1)(2x))F(y_{2n}, y_{2n+1}, (x)\}$
$\geqslant \min\{[F(y_{2n-1}, y_{2n}, (kx)]^2 F(y_{2n-1}, y_{2n}, (x)F(y_{2n}, y_{2n+1}, (x))[F(y_{2n-1}, y_{2n}, (kx)]^2 F(y_{2n-1}, y_{2n}, (x)$
$F(y_{2n} y_{2n+1}, (x)F(y_{2n-1}, y_{2n}, (x)), tF(y_{2n-1}, y_{2n}, (x))F(y_{2n}, y_{2n+1}, (x))$
$F(y_{2n-1}, y_{2n}, (x)), tF(y_{2n-1}, y_{2n}, (x)F(y_{2n}, y_{2n+1}, (x)$
$F(y_{2n-1}, y_{2n}, (x)tF(y_{2n-1}, y_{2n}, (x))F(y_{2n}, y_{2n+1}(x)F(F(y_{2n}, y_{2n+1}, (x)\}x_{2n+1},$
$\geqslant \min\{[F(y_{2n}, y_{2n+1}, (kx)]^2[F(y_{2n}, y_{2n+1}, (kx)]^2[F(y_{2n}, y_{2n+1}, (kx)]^2$
$[F(y_{2n}, y_{2n+1}, (kx)]^2[F(y_{2n}, y_{2n+1}, (kx)]^2[F(y_{2n}, y_{2n+1}, (kx)]^[F(y_{2n}, y_{2n+1}, (kx)]^2$
$[F(y_{2n}, y_{2n+1}, (kx)]^[F(y_{2n}, y_{2n+1}, (kx)]^2[F(y_{2n}, y_{2n+1}, (kx)]^2\}$

which is a contradiction. Thus we have $F(y_{2n}, y_{2n+1}, (kx) \geqslant F(y_{2n-1}, y_{2n}, (x)$ similarly we can have $F(y_{2n+1}, y_(2n+2,), (kx) \geqslant F(y_{2n}, y_{2n+1}, (x)$. Therefore, for every $n \in N, F(y_(n,)y_(n+1,), (kx) \geqslant F(y_(n-1,)y_(n,)(x)$. There fore it is a Cauchy sequence in X. since space $(X, M, *)$ is complete $\{y_n\}$ converges to a point z in X. and the subsequences $\{Px_{2n}\}, \{Qx_{2n+1}\}\{Ax_{2n}\}, \{Tx_{2n+1}\}$ of $\{y_{2n}\}$ also converges to Z. Now suppose that P is continuous, since P and A are weak compatible, it follow from $(APx_{2n}) \to Pz$, and $PPx_{2n}, \to Pz$ as $n \to \infty$. Now $u = Px_{2n}$, and $v = x_{2n+1}$, in the equation (ii)we have

$[F(PPx_{2n}, Qx_{2n+1}), , (kx)]^2$
$\geqslant min[F(APx_{2n}, Tx_{2n+1}), , (x)]^2[F(APx_{2n}, PPx_{2n}, (x)][F(Tx_{2n+1}), Qx_{2n+1}), , (x)]$
$[F(APx_{2n}, Tx_{2n+1}), (x)][F(APx_{2n}, PPx_{2n}, (x)[F(APx_{2n}, Tx_(2n, +1), (x)]$
$[F(TPx_{2n+1}), Qx_{2n+1}), , (x)]\ [F(APx_{2n}, Tx_{2n+1}), , (x)]\ [F(APx_{2n}, Qx_{2n}, (2x)]$
$[F(APx_{2n}, Tx_{2n+1}), , (x)]\ [F(Tx_{2n+1}), PPx_{2n}, (x)]\ [F(APx_{2n}, Qx_(2n, +1), (x)]$
$[F(Tx_{2n+1}), PPx_{2n}, (x)][F(APx_{2n}, PPx_{2n}, (x)][F(APx_{2n+1}), Qx_{2n}, (x)][F(APx_{2n}, Qx_{2n+1}), , (2x)]$

$[F(Tx_{2n+1}), Qx_(2n, +1), (x)]$.

Taking the limit $n \to \infty$, we have $[F(Pz, z, (kx)]^2 \geqslant \min\{[F(Pz, z, (x)]^2$

$[F(Pz, Pz, (x)]^2$ $[F(z, z, (x)][F(Pz, z, (x)][F(Pz, Pz, (x)]$ $[F(Pz, z, (x)][F(z, z, (x)][F(Pz, z, (x)]$

$[F(Pz, z, (2x)][F(Pz, z, (x)]$ $[F(z, Pz, (x)][F(z, z, (2x)][F(z, Pz, (x)][F(Pz, Pz, (x)][F(z, Pz, (x)]$

$[F(Pz, z, (2x)][F(z, z, (x)]\} = [F(Pz, z, (x)]^2$, which is a contradiction. Thus we have $Pz = z$, since $P(x)T(X)$, there exist appoint u belong to X such that $z = Pz = Tp$. Again putting $u = Px_{2n}, v = p$ in (ii) we have

$[F(PPx_{2n}, Qpx_{2n+1}), (kx)]^2$

$\geqslant \min[F(APx_{2n}, TP, (x)]^2[F(APx_{2n}, PPx_{2n}, (x)][F(Tp, QP_,(x)]$ $[F(APx_{2n}, Tp_,(x)]$

$[F(TPx_{2n}, PPx_{2n}, (x)[F(APx_{2n}, TPx_(2n, +1), (x)]$

$[F(Tp, Qp, (x)][F(Apx_{2n}, Tp, (x)][F(APx_{2n}, Tp_,(x)][F(APx_{2n}, Qp_,(2x)][F(APx_{2n}, TP, (x)]$

$[F(TP, PPx_{2n}, (x)][F(Apx_{2n}QPx_{2n}, (2x)]$

$[F(TP, PPx_{2n}, (x)][F(APx_{2n}, PPx_{2n}, (x)][F(APx_{2n}, pPx_{2n}, (2x)][F(Tp, PPx_{2n}, (x)]$.

Taking the limit $n \to \infty$, we have $[F(z, Qp, (kx)]^2 \geqslant [F(z, Qp, (x)]^2$ which is a contradiction , there fore $z = Qp$. Since Q and T are weak compatible and $Tp = Qp = Z, TQp = QTp$ and hence $Tz = TQp = QTp = Qz$. Again by putting $u = x_{2n}$ and $v = z$, we have $[F(Px_{2n}, Qz, (kx)]^2 \geqslant \min\{[F(Ax_{2n}, Tz, (x)]^2 F(Ax_{2n}, Px_{2n}, (X)) F(Tz, Qz, (X)), F(Ax_{2n}, Tz, (x)),$

$F(Ax_{2n}, Px_{2n}, (x)) F(Ax_{2n}, Tz, (x)) F(Tz, Qz, (x), F(Ax_{2n}, Tz(x))$

$F(Ax_{2n}, Qz, (2x)), F(Ax_{2n}, Tz, (x)) F(Tz, Px_{2n}, (x) F(Ax_{2n}, Qz, (2x),$

$F(Tz, Px_{2n}, (x), F(Ax_{2n}, Px_{2n}, (x)) F(Tz, Px_{2n}, (x)) F(Ax_{2n}, Qz, (2x), F(Tz, Qz, (x))\}$. Taking the limit $n \to \infty$, we have $[F(z, Qz, (kx)]^2 \geqslant [F(z, Q, (x)]^2$ Which is a contraction therefore we have $Qz = z$. Thus $Qz = Tz = z$, similarly since P and A are weak compatible and we have $Az = Pz = z$. Now We prove $Az = z$. Suppose that $Az \neq z$ then by putting $u = Az$ and $v = z$ in (iii) we have

$[F(PAz, Qz, (kx)]^2 \geqslant \min\{F(AAz, Tz, (x)]^2 F(AAz, PAz, (x), F(Tz, Qz, (x),) F(AAz, Tz, (x),$

$F(AAz, PAz, (x), F(AAz, Tz, (x), F(Tz, Qz, (2x), F(AAz, Tz, (x), F(AAz, Qz, (2x),$

$F(Tz, PAz, (x)), F(AAz, Qz, (x), F(AAz, Qz, (2x)) F(Tz, PAz, (x), F(AAz, Qz, (2x)) F(Tz, PAz, (x),$

$F(AAz, Qz, (2x), F(Tz, Az, (x),)\}$ which yields $[F(Az, z, (kx)]^2 \geqslant [F(Az, z, (x)]^2$ which is a contradiction there fore we have $Az = z$, similarly if we put $u = z$ and $y = z$ we have

$[F(Pz, Qz, (kx)]^2 \geqslant \min\{F(Az, Tz, (x)]^2, F(Az, Pz, (x)) F(Tz, Qz, (x)) F(Az, Tz, (x))$

$F(Az, Pz, (x)) F(Az, Tz, (x)) F(Tz, Qz, (x)) F(Az, Tz, (x)) F(Az, Qz, (2x)) F(Az, Tz, (x)) F(Tz, Pz, (x))$

$F(Az, Qz, (2x)) F(Tz, Pz, (x)), F(Az, Pz, (x)) F(Tz, Pz, (x)) F(Az, Qz, (2x)) F(Tz, Qz, (x))\},$

which yields $[F(z, z, (kx)]^2 \geqslant [F(z, z, (x)]^2$ which is contradiction , therefore we we have $Pz = Qz = Az = Tz = z$. Thus combining the results . thus z is a common fixed point AT.P.Q. For uniqueness let $w(zw)$ be another common fixed point A,B,P,Q, then we have

$[F(z, w, (kx)]^2 \geqslant [F(Pz, QW, (x)]^2 \geqslant min\{[[F(z, w, (x)]^2 F(z, z, (x)) F(w, w, (x))$

$F(z, w, (x)) F(z, z, (x)) F(z, w, (x)) F(w, w, (x)) F(z, w, (x)) F(z, w, (2x)) F(z, w, (x))$

$F(w, z, (x)) F(z, w, (2x)) F(w, z, (x)) F(z, z, (x)) F(w, z, (x)) F(z, w, (2x)) F(w, w, (x))\} = [F(z, w, (x)]^2$

which is a contradiction, therefore $z = w$. hence z is unique common fixed point A, T, P, and Q. If we put $T = I (I$ is identity mapping on $X)$. $\square$

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