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RESEARCH ARTICLE

Matrices in interval-valued fuzzy soft set theory and their application

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Abstract The purpose of this paper is to introduce the concept of Interval-Valued Fuzzy Soft Matrix ($IVFS$-matrix) together with some different types of matrices in interval-valued fuzzy soft set theory. We have defined here some new operations on these matrices and discussed all these definitions and operations by appropriate examples. In addition we have proven some theorems along with few properties on these matrices. Moreover a new efficient $IVFSM$-algorithm based on these new matrix operations has been developed to solve interval-valued fuzzy soft set based group decision making problems. After that the $IVFSM$-algorithm has been applied to a real life group decision making problem and then we have described the feasibility of this proposed method.

Key Words Interval-valued Fuzzy Soft Set; Interval-Valued Fuzzy Soft Matrix($IVFS$-Matrix); Choice Matrix;
MSC 2010 03E72, 65F30

1 Introduction

In real scenario we need strategies which provide some flexible information processing capacity to deal with uncertainties. Soft set theory is generally used to solve such problems. In the year 1999 Molodtsov [2] introduced soft set as a completely generic mathematical tool for modeling uncertainties. Since there is hardly any limitation in describing the objects, researchers simplify the decision making process by choosing the form of parameters they require and subsequently makes it more efficient in the absence of partial information. Maji et al. have done further research on soft set theory [9, 10] and on fuzzy soft set theory [11]. Then the different researchers Chen et al.[1] in 2005, Zou et al. [16] and Kong et al. [18] in 2008 have worked on parameter reduction of soft set and fuzzy soft set theory.

All of these works are based on the classical soft set theory. The soft set model, however, can also be combined with other mathematical models. For example, by amalgamating the soft set and algebra, Akta et al.[5] proposed the definition of soft groups, Feng et al. [3] proposed the concept of soft semirings, soft subsemirings, soft ideals, idealistic soft semirings and soft semiring homomorphisms. Then Y.B. Jun [17] applied the notion of soft sets to the theory of BCK/BCI-algebras. Maji et al. presented the concept of the fuzzy soft set [11, 12] which is based on a combination of the fuzzy set and soft set models. Yang et al. [13] defined the operations on fuzzy soft sets, which are based on three fuzzy logic operators: negation, triangular norm and triangular conorm. Zou et al.[15] introduced the soft set and fuzzy soft set into the

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incomplete environment. Cagman et al. [7, 8] have proposed the definition of soft matrix which is the representation of a soft set and they also introduced a new soft set based decision making method. Then in the year 2009, Yang et al. [14] have combined the interval-valued fuzzy set [6] and soft set[2], from which a new soft set model: interval-valued fuzzy soft set(IVFSs)[14] is obtained. They have also given an algorithm to solve IVFSs based decision making problems. Then Feng et al[4] have shown that Yang's algorithm has some drawbacks and they have proposed another method for solving IVFSs based decision making problems.

But according to Feng's method [4], the decision maker has to form a reduct fuzzy soft set (of pessimistic or optimistic or neutral type) of the given IVFSs and then can select any level to form the level soft set. There does not exist any unique or uniform criterion for the selection of the level. So by this method the decision maker will be puzzled to decide that which type and which level is most suitable for the selection of the object. Moreover till now researchers [14, 4] have worked on finding solution of the IVFSs based decision making problems involving **only one** decision maker. There does not exist any method for solving a IVFSs-based **group** decision making problem.

Group Decision is the academic and professional field that aims to improve collective decision process by taking into account the needs and opinions of every group assisting groups or individuals within groups, as they interact and collaborate to reach a collective decision.

In this paper, we have introduced the concept of $IVFS$-matrix. Here we have presented the concept of choice matrix which represents the choice parameters of the decision makers and then we have introduced some new operations on $IVFS$-matrix and choice matrix. Moreover we have proven some theorems along with few properties on these matrices. Finally we have proposed the new $IVFSM$-algorithm based on some of these new matrix operations to solve interval-valued fuzzy soft set based decision making problems **involving any number of decision maker.** To realize this newly proposed algorithm we have applied it to a real life group decision making problem and also described the purpose of its introduction.

2 Preliminaries

Definition 2.1([2]) Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U. Let $A \subseteq E$. A pair (F_A, E) is called a **soft set** over U, where F_A is a mapping given by, $F_A : A \longrightarrow P(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U.

Definition 2.2([11]) Let U be an initial universe set and E be a set of parameters(which are fuzzy words or sentences involving fuzzy words). Let $P(U)$ denotes the set of all fuzzy sets of U. Let $A \subset E$. A pair (F_A, E) is called a **Fuzzy Soft Set** (FSS) over U, where F_A is a mapping given by, $F_A : E \longrightarrow P(U)$ such that $F_A(e) = \tilde{\phi}$ if $e \notin A$ where $\tilde{\phi}$ is a null fuzzy set.

Example 2.1. Let U be the set of four cities, given by, $U = \{C_1, C_2, C_3, C_4\}$. Let E be the set of parameters (each parameter is a fuzzy word), given by, $E = \{$ highly, immensely, moderately, average, less$\} = \{e_1, e_2, e_3, e_4, e_5\}$, where e_1 stands for the parameter 'highly', e_2 stands for the parameter 'immensely', e_3 stands for the parameter 'moderately', e_4 stands for the parameter 'average'. e_5 stands for the parameter 'less'. Let $A \subset E$, given by, $A = \{e_1, e_2, e_3, e_5\}$, Now suppose that, $F_A(e_1) = \{C_1/.2, C_2/.9, C_3/.4, C_4/.6\}$, $F_A(e_2) = \{C_2/1, C_3/.3, C_4/.4\}$, $F_A(e_3) = \{C_1/.3, C_2/.4, C_3/.8\}$, $F_A(e_5) = \{C_1/.9, C_2/.1, C_3/.5, C_4/.3\}$.

Then the fuzzy soft set is given by,

$$\begin{aligned}(F_A, E) \quad = \quad & \{\text{highly polluted city } = \{C_1/.2, C_2/.9, C_3/.4, C_4/.6\}, \\ & \text{immensely polluted city} = \{C_2/1, C_3/.3, C_4/.4\}, \\ & \text{moderately polluted city} = \{C_1/.3, C_2/.4, C_3/.8\}, \\ & \text{average polluted city} = \tilde{\phi}, \\ & \text{less polluted city} = \{C_1/.9, C_2/.1, C_3/.5, C_4/.3\}\}\end{aligned}$$

Definition 2.3([14]) Let U be an initial universe, E be a set of parameters and $A \subseteq E$. Then a pair $(\bar{F}_A, E)$ is called an **interval-valued fuzzy soft set** over $P(U)$ where $\bar{F}_A$ is a mapping given by $\bar{F}_A : A \longrightarrow P(U)$.

An interval-valued fuzzy soft set is a parameterized family of interval-valued fuzzy subsets of U. An interval-valued fuzzy soft set is also a special case of a soft set because it is still a mapping from parameters to $P(U)$. $\forall e \in A, \bar{F}_A(e)$ is referred as an interval-valued fuzzy set of U. It can be written as: $\bar{F}_A(e) = \{(x, \mu_{\bar{F}_A}(x)) : x \in U\}$ where $\mu_{\bar{F}_A}(x)$ is the interval-valued fuzzy membership degree that object x holds on parameter e. If $\forall e \in E, \forall x \in U, \mu^-_{\bar{F}_A}(x) = \mu^+_{\bar{F}_A}(x)$, then $\bar{F}_A$ will degenerated to be a standard fuzzy set and then $(\bar{F}_A, E)$ will be degenerated to be a traditional fuzzy soft set.

Example 2.2. Suppose that, U be the set of six houses $h_1, h_2, h_3, h_4, h_5, h_6$ and E be the set of parameters given by $E = \{$ beautiful, wooden, cheap, in the green surroundings$\} = \{e_1, e_2, e_3, e_4\}$ and $A = E$. The tabular representation of the interval-valued fuzzy soft set $(\bar{F}_A, E)$ is shown in Table 1. In Table 1, we can see that the precise evaluation for each object on each parameter is unknown while the lower and upper limits of such an evaluation is given. For example, we cannot present the precise degree of how beautiful the house h_1 is, however, the house h_1 is at least beautiful on the degree of 0.7 and it is at most beautiful on the degree of 0.9.

Table-1: Tabular representation of $(\bar{F}_A, E)$

	e_1	e_2	e_3	e_4
h_1	[0.7,0.9]	[0.6, 0.7]	[0.3, 0.5]	[0.5, 0.8]
h_2	[0.6, 0.8]	[0.8,1.0]	[0.8, 0.9]	[0.9, 1.0]
h_3	[0.5, 0.6]	[0.2,0.4]	[0.5, 0.7]	[0.7, 0.9]
h_4	[0.6, 0.8]	[0.0,0.1]	[0.7, 1.0]	[0.6, 0.8]
h_5	[0.8, 0.9]	[0.1,0.3]	[0.9, 1.0]	[0.2, 0.5]
h_6	[0.8, 1.0]	[0.7, 0.8]	[0.2, 0.5]	[0.7,1.0]

Definition 2.4([7]) Let (F_A, E) be a soft set over U. Then a subset of $U \times E$ is uniquely defined by $R_A = \{(u, e) : e \in A, u \in F_A(e)\}$ which is called a relation form of (F_A, E). Now the **characteristic function** of R_A is written by,

$$\chi_{R_A} : U \times E \longrightarrow \{0, 1\}, \;\; \chi_{R_A} \quad = \quad \begin{cases} 1, (u, e) \in R_A \\ 0, (u, e) \notin R_A \end{cases}$$

Definition 2.5([7]) Let $U = \{u_1, u_2, .u_m\}$, $E = \{e_1, e_2, \cdots, e_n\}$, then R_A can be presented by a table as in the following form

Table-2

	e_1	e_2	$\cdots$	e_n
u_1	$\chi_{R_A}(u_1, e_1)$	$\chi_{R_A}(u_1, e_2)$	$\cdots$	$\chi_{R_A}(u_1, e_n)$
u_2	$\chi_{R_A}(u_2, e_1)$	$\chi_{R_A}(u_2, e_2)$	$\cdots$	$\chi_{R_A}(u_2, e_n)$
$\cdots$	v	$\cdots$	$\cdots$	$\cdots$
u_m	$\chi_{R_A}(u_m, e_1)$	$\chi_{R_A}(u_m, e_2)$	$\cdots$	$\chi_{R_A}(u_m, e_n)$

If $a_{ij} = \chi_{R_A}(u_i, e_j)$, we can define a matrix

$$[a_{ij}]_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

which is called a **soft matrix** of order $m \times n$ corresponding to the soft set (F_A, E) over U. A soft set (F_A, E) is uniquely characterized by the matrix $[a_{ij}]_{m\times n}$. Therefore we shall identify any soft set with its soft matrix and use these two concepts as interchangeable.

3 Some New Concepts of Matrices in Interval-Valued Fuzzy Soft Set Theory

Definition 3.1 (The Concept of $IVFS$-Matrix) Let $(\bar{F}_A, E)$ be an interval-valued fuzzy soft set over U. Then a subset of $U \times E$ is uniquely defined by

$$R_A = \{(u, e) : e \in A, u \in \bar{F}_A(e)\}$$

which is called a relation form of $(\bar{F}_A, E)$. Now the relation R_A is characterized by the membership function $\mu_A : U \times E \longrightarrow \text{Int}([0, 1])$ such that

$$\mu_A(u, e) = \begin{cases} [\mu^-_{\bar{F}_A(e)}(u), \mu^+_{\bar{F}_A(e)}(u)], \;\; if \;\; e \in A \\ [0, 0], \;\; if \;\; e \notin A \end{cases}$$

where $\text{Int}([0, 1])$ stands for the set of all closed subintervals of $[0, 1]$ and $[\mu^-_{\bar{F}_A(e)}(u), \mu^+_{\bar{F}_A(e)}(u)]$ denotes the interval-valued fuzzy membership degree of the object u associated with the parameter e.

Now if the set of universe $U = \{u_1, u_2, \cdots, u_m\}$ and the set of parameters $E = \{e_1, e_2, \cdots, e_n\}$, then R_A can be presented by a table in the following form

Table-3: Tabular representation of R_A

	e_1	e_2	$\cdots$	e_n
u_1	$\mu_A(u_1, e_1)$	$\mu_A(u_1, e_2)$	$\cdots$	$\mu_A(u_1, e_n)$
u_2	$\mu_A(u_2, e_1)$	$\mu_A(u_2, e_2)$	$\cdots$	$\mu_A(u_2, e_n)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
u_m	$\mu_A(u_m, e_1)$	$\mu_A(u_m, e_2)$	$\cdots$	$\mu_A(u_m, e_n)$

where $\mu_A(u_m, e_n) = [\mu^-_{\bar{F}_A(e_n)}(u_m), \mu^+_{\bar{F}_A(e_n)}(u_m)]$. If $a_{ij} = [\mu^-_{\bar{F}_A(e_j)}(u_i), \mu^+_{\bar{F}_A(e_j)}(u_i)]$, then from table-4 we can define a matrix

$$(\bar{a}_{ij})_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

which is called an **interval-valued fuzzy soft matrix or, simply $IVFS$-matrix** of order $m \times n$ corresponding to the interval-valued fuzzy soft set $(\bar{F}_A, E)$ over U. An interval-valued fuzzy soft set $(\bar{F}_A, E)$ is uniquely characterized by the matrix $(\bar{a}_{ij})_{m\times n}$. Therefore we shall identify any interval-valued fuzzy soft set with its $IVFS$-matrix and use these two concepts as interchangeable.

Example 3.1. Let U be the set of five cities, given by, $U = \{C_1, C_2, C_3, C_4, C_5\}$. Let E be the set of parameters (each parameter is an interval-valued fuzzy word), given by,

$E = \{$ highly, immensely, moderately, average, less $\} = \{e_1, e_2, e_3, e_4, e_5\}$(say)

Let $A \subset E$, given by, $A = \{e_1, e_2, e_3, e_5\}$ (say) Now suppose that, $\bar{F}_A : A \longrightarrow P(U)$ describing "the pollution of the cities"is given by,

$\bar{F}_A(e_1) = \{C_1/[.2, .4], C_2/[.8, .9], C_3/[.4, .5], C_4/[.6, .7], C_5/[.7, .8]\}$,
$\bar{F}_A(e_2) = \{C_1/[0, .1], C_2/[.9, 1], C_3/[.3, .4], C_4/[.4, .6], C_5/[.6, .7]\}$,
$\bar{F}_A(e_3) = \{C_1/[.3, .5], C_2/[.4, .5], C_3/[.8, .9], C_4/[.1, .2], C_5/[.3, .5]\}$,
$\bar{F}_A(e_5) = \{C_1/[.9, 1], C_2/[.1, .2], C_3/[.5, .6], C_4/[.3, .5], C_5/[.1, .2]\}$

Therefore the Interval-Valued Fuzzy Soft Set is,

$$\begin{aligned} (\bar{F}_A, E) \quad = \quad & \{\text{highly polluted city } = \{C_1/[.2, .4], C_2/[.8, .9], C_3/[.4, .5], C_4/[.6, .7], C_5/[.6, .8]\}, \\ & \text{immensely polluted city} = \{C_1/[0, .1], C_2/[.9, 1], C_3/[.3, .4], C_4/[.4, .6], C_5/[.6, .7]\}, \\ & \text{moderately polluted city} = \{C_1/[.3, .5], C_2/[.4, .5], C_3/[.8, 1], C_4/[.1, .2], C_5/[.3, .5]\}, \\ & \text{less polluted city} = \{C_1/[.9, 1], C_2/[.1, .2], C_3/[.5, .7], C_4/[.3, .5], C_5/[.1, .2]\}\} \end{aligned}$$

Then the relation form of $(\bar{F}_A, E)$ is written by,

$$\begin{aligned} R_A \quad = \quad & \{(\{C_1/[.2, .4], C_2/[.8, .9], C_3/[.4, .5], C_4/[.6, .7], C_5/[.6, .8]\}, e_1), \\ & (\{C_1/[0, .1], C_2/[.9, 1], C_3/[.3, .4], C_4/[.4, .6], C_5/[.6, .7]\}, e_2), \\ & (\{C_1/[.3, .5], C_2/[.4, .5], C_3/[.8, 1], C_4/[.1, .2], C_5/[.3, .5]\}, e_3), \\ & (\{C_1/[0, 0], C_2/[0, 0], C_3/[0, 0], C_4/[0, 0], C_5/[0, 0]\}, e_4), \\ & (\{C_1/[.9, 1], C_2/[.1, .2], C_3/[.5, .7], C_4/[.3, .5], C_5/[.1, .2]\}, e_5)\} \end{aligned}$$

Hence the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [\ .2,\ .4\ ] & [\ 0,\ .1\ ] & [\ .3,\ .5\ ] & [0,0] & [\ .9,\ 1\ ] \\ [\ .8,\ .9\ ] & [\ .9,\ 1\ ] & [\ .4,\ .5\ ] & [0,0] & [\ .1,\ .2] \\ [\ .4,\ .5] & [\ .3,\ .4] & [\ .8,\ 1\ ] & [0,0] & [\ .5,\ .7\ ] \\ [\ .6,\ .7\ ] & [\ .4,\ .6\ ] & [\ .1,\ .2\ ] & [0,0] & [\ .3,\ .5\ ] \\ [.6,\ .8\ ] & [.6,\ .7] & [.3,\ .5] & [0,0] & [.1,\ .2] \end{pmatrix}.$$

Definition 3.2(Row $IVFS$-Matrix) An $IVFS$-matrix of order $1 \times n$ i.e., with a single row is called a **row $IVFS$-matrix**. Physically, a row $IVFS$-matrix formally corresponds to an interval-valued fuzzy soft set whose universal set contains only one object.

Example 3.2. Suppose the universe set U contains only one dress d_1 and parameter set $E = \{\text{costly, beautiful, cheap, comfortable}\} = \{e_1, e_2, e_3, e_4\}$. Let $A = \{e_2, e_3, e_4\} \subset E$ and $\bar{F}_A : A \longrightarrow P(U)$ s.t., $\bar{F}_A(e_2) = \{d_1/[.8, .9]\}, \bar{F}_A(e_3) = \{d_1/[.2, .4]\}, \bar{F}_A(e_4) = \{d_1/[.5, .6]\}$. Then the interval-valued fuzzy soft set $(\bar{F}_A, E) = \{(e_2, \{d_1/[.8, .9]\}), (e_3, \{d_1/[.2, .4]\}), (e_4, \{d_1/[.5, .6]\})\}$ and then the relation form of $(\bar{F}_A, E)$ is written by, $R_A = \{(\{d_1/[0, 0]\}, e_1), (\{d_1/[.8, .9]\}, e_2), (\{d_1/[.2, .4]\}, e_3), (\{d_1/[.5, .6]\}, e_4)\}$ Hence the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0,\ 0] & [.8,\ .9] & [.2,.4] & [.5,\ .6] \end{pmatrix}$$

which contains a single row and so it is a row $IVFS$-matrix.

Definition 3.3(Column $IVFS$-Matrix) An $IVFS$-matrix of order $m \times 1$ i.e., with a single column is called a **column** $IVFS$**-matrix**. Physically, a column $IVFS$-matrix formally corresponds to an interval-valued fuzzy soft set whose parameter set contains only one parameter.

Example 3.3. Suppose the initial universe set U contains four dresses d_1, d_2, d_3, d_4 and the parameter set E contains only one parameter given by, $E = \{\text{ beautiful }\} = \{e_1\}$. $\bar{F} : E \longrightarrow P(U)$ s.t, $\bar{F}(e_1) = \{d_1/[0.6, 0.8], d_2/[0.1, 0.3], d_3/[0.7, 0.9], d_4/[0.3, 0.5]\}$. Then the interval-valued fuzzy soft set $(\bar{F}, E) = \{(e_1, \{d_1/[0.6, 0.8], d_2/[0.1, 0.3], d_3/[0.7, 0.9], d_4/[0.3, 0.5]\})\}$ and then the relation form of $(\bar{F}, E)$ is written by, $R_E = \{(\{d_1/[0.6, 0.8], d_2/[0.1, 0.3], d_3/[0.7, 0.9], d_4/[0.3, 0.5]\}, e_1)\}$. Hence the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.6,\ 0.8] \\ [0.1,\ 0.3] \\ [0.7,\ 0.9] \\ [0.3,\ 0.5] \end{pmatrix}$$

which contains a single column and so it is an example of column $IVFS$-matrix.

Definition 3.4(Square $IVFS$-Matrix) An $IVFS$-matrix of order $m \times n$ is said to be an **square** $IVFS$**-matrix** if $m = n$ ie., the number of rows and the number of columns are equal. That means a square $IVFS$-matrix is formally equal to an interval-valued fuzzy soft set having the same number of objects and parameters.

Example 3.4. Consider the example 3.1. Here since the $IVFS$-matrix $(\bar{a}_{ij})$ contains five rows and five columns, so it is a square $IVFS$-matrix.

Definition 3.5 (Complement of an $IVFS$-matrix) Let $(\bar{a}_{ij})$ be an $m \times n$ $IVFS$-matrix, where $(\bar{a}_{ij})$ is the matrix representation of the interval-valued fuzzy soft set $(\bar{F}_A, E)$. Then the **complement** of $(\bar{a}_{ij})$ is denoted by $(\bar{a}_{ij})^c$ and is defined by, $(\bar{a}_{ij})^c = (\bar{c}_{ij})$, where $(\bar{c}_{ij})$ is also an $IVFS$-matrix of order $m \times n$ and it is the matrix representation of the interval-valued fuzzy soft set $(\bar{F}^c_{\neg A}, E)$, ie., $c_{ij} = [\mu^-_{c_{ij}}, \mu^+_{c_{ij}}] = [1 - \mu^+_{a_{ij}}, 1 - \mu^-_{a_{ij}}]$.

Example 3.5. Consider the example 3.1. Then the complement of $(\bar{a}_{ij})$ is,

$$(\bar{a}_{ij})^c = \begin{pmatrix} [\ .6,\ .8\ ] & [\ .9,\ 1\ ] & [\ .5,\ .7\ ] & [1,1] & [\ 0,\ .1\ ] \\ [\ .1,\ .2\ ] & [\ 0,\ .1\ ] & [\ .5,\ .6\ ] & [1,1] & [\ .8,\ .9] \\ [\ .5,\ .6] & [\ .6,\ .7] & [\ 0,\ .2\ ] & [1,1] & [\ .3,\ .5\ ] \\ [\ .3,\ .4\ ] & [\ .4,\ .6\ ] & [\ .8,\ .9\ ] & [1,1] & [\ .5,\ .7\ ] \\ [.2,\ .4\ ] & [.3,\ .4] & [.5,\ .7] & [1,1] & [.8,\ .9] \end{pmatrix}.$$

Definition 3.6 (Null $IVFS$-Matrix) An $IVFS$-matrix of order $m \times n$ is said to be a **null $IVFS$-matrix or zero $IVFS$-matrix** if all of its elements are $[0, 0]$. A null $IVFS$-matrix is denoted by, $\bar{\Phi}$. Now the interval-valued fuzzy soft set associated with a null $IVFS$-matrix must be a null interval-valued fuzzy soft set.

Example 3.6. Let there are four dresses in the universe set U given by, $U = \{d_1, d_2, d_3, d_4\}$ and the parameter set $E = \{\text{beautiful, cheap, comfortable, gorgeous}\} = \{e_1, e_2, e_3, e_4\}$.
Let $A = \{e_1, e_2, e_3\} \subset E$. Now let $\bar{F}_A : E \longrightarrow P(U)$ s.t, $\bar{F}_A(e_1) = \{d_1/[0, 0], d_2/[0, 0], d_3/[0, 0], d_4/[0, 0]\} = \bar{\phi}, \bar{F}_A(e_2) = \bar{\phi}, \bar{F}_A(e_3) = \bar{\phi}$ (where $\bar{\phi}$ is a null interval-valued fuzzy set). Then the interval-valued fuzzy soft set $(\bar{F}_A, E) = \{(e_1, \bar{\phi}), (e_2, \bar{\phi}), (e_3, \bar{\phi})\}$ and then the relation form of $(\bar{F}_A, E)$ is written by, $R_A = \{(\bar{\phi}, e_1), (\bar{\phi}, e_2), (\bar{\phi}, e_3)\}$ Hence the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix} = \bar{\Phi}.$$

Definition 3.7 (Complete $IVFS$-Matrix or, Absolute $IVFS$-Matrix) An $IVFS$-matrix of order $m \times n$ is said to be a **complete $IVFS$-matrix or, absolute $IVFS$-matrix** if all of its elements are $[1, 1]$. A complete or absolute $IVFS$-matrix is denoted by, $\bar{C}_A$. Now the interval-valued fuzzy soft set associated with an absolute $IVFS$-matrix must be an absolute interval-valued fuzzy soft set.

Example 3.7. Let there are four dresses in the universe set U given by, $U = \{d_1, d_2, d_3, d_4\}$ and the parameter set $E = \{\text{beautiful, cheap, comfortable, gorgeous}\} = \{e_1, e_2, e_3, e_4\}$. Let $A = \{e_1, e_2, e_3, e_4\} \subseteq E$ and $\bar{F}_A : E \longrightarrow P(U)$ s.t,

$$\bar{F}_A(e_1) = \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\},$$
$$\bar{F}_A(e_2) = \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\},$$
$$\bar{F}_A(e_3) = \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\},$$
$$\bar{F}_A(e_4) = \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}.$$

Then the interval-valued fuzzy soft set

$$\begin{aligned}(\bar{F}_A, E) = \{&(e_1, \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}),\\ &(e_2, \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}),\\ &(e_3, \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}),\\ &(e_4, \{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\})\}\end{aligned}$$

and then the relation form of $(\bar{F}_A, E)$ is written by,

$$\begin{aligned}R_A = \{&(\{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}, e_1), (\{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}, e_2),\\ &(\{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}, e_3), (\{d_1/[1, 1], d_2/[1, 1], d_3/[1, 1], d_4/[1, 1]\}, e_4)\}.\end{aligned}$$

Hence the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [1,1] & [1,1] & [1,1] & [1,1] \\ [1,1] & [1,1] & [1,1] & [1,1] \\ [1,1] & [1,1] & [1,1] & [1,1] \\ [1,1] & [1,1] & [1,1] & [1,1] \end{pmatrix} = \bar{C}_A.$$

Definition 3.8 (Diagonal $IVFS$-Matrix) A square $IVFS$-matrix of order $n \times n$ is said to be a **diagonal**

$IVFS$**-matrix** if all of its non-diagonal elements are $[0,0]$. If the diagonal elements of a diagonal $IVFS$-matrix be all equal, then the matrix is called a **scalar** $IVFS$**-matrix**. If the diagonal elements of a diagonal $IVFS$-matrix be all $[1,1]$, then the matrix is called a **unit or identity** $IVFS$**-matrix**.

Example 3.8. Suppose the initial universe set $U = \{d_1, d_2, d_3, d_4, d_5\}$ and the parameter set $E = \{e_1, e_2, e_3, e_4, e_5\}$. Let $\bar{F} : E \longrightarrow P(U)$ s.t,

$$\bar{F}(e_1) = \{d_1/[0.8,1], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0,0]\},$$
$$\bar{F}(e_2) = \{d_1/[0,0], d_2/[0.3,0.5], d_3/[0,0], d_4/[0,0], d_5/[0,0]\},$$
$$\bar{F}(e_3) = \{d_1/[0,0], d_2/[0,0], d_3/[0.8,1], d_4/[0,0], d_5/[0,0]\},$$
$$\bar{F}(e_4) = \{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0.7,0.9], d_5/[0,0]\},$$
$$\bar{F}(e_5) = \{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0.6,0.8]\}.$$

Then the interval-valued fuzzy soft set

$$\begin{aligned}(\bar{F}, E) \quad = \quad & \{(e_1, \{d_1/[0.8,1], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}),\\ & (e_2, \{d_1/[0,0], d_2/[0.3,0.5], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}),\\ & (e_3, \{d_1/[0,0], d_2/[0,0], d_3/[0.8,1], d_4/[0,0], d_5/[0,0]\}),\\ & (e_4, \{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0.7,0.9], d_5/[0,0]\}),\\ & (e_5, \{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0.6,0.8]\})\}\end{aligned}$$

and then the relation form of $(\bar{F}, E)$ is written by,

$$\begin{aligned}R_E \quad = \quad & \{(\{(\{d_1/[0.8,1], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}, e_1),\\ & (\{d_1/[0,0], d_2/[0.3,0.5], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}, e_2),\\ & (\{d_1/[0,0], d_2/[0,0], d_3/[0.8,1], d_4/[0,0], d_5/[0,0]\}, e_3),\\ & (\{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0.7,0.9], d_5/[0,0]\}, e_4),\\ & (\{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0.6,0.8]\}, e_5)\}\end{aligned}$$

Now the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.8,1] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0.3,0.5] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0.8,1] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0.7,0.9] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0.6,0.8] \end{pmatrix}$$

Hence by definition $(\bar{a}_{ij})$ is a diagonal $IVFS$-matrix.
Similarly,

$$(\bar{s}_{ij}) = \begin{pmatrix} [a,b] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [a,b] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [a,b] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [a,b] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [a,b] \end{pmatrix}$$

is a scalar $IVFS$-matrix (where $a, b \in [0,1]$) which may be denoted as $([a,b])_5$ and

$$(\bar{d}_{ij}) = \begin{pmatrix} [1,1] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [1,1] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [1,1] \end{pmatrix}$$

is a unit or identity $IVFS$-matrix which may be simply denoted as $\bar{I}_5$.

Definition 3.9 (Triangular $IVFS$-Matrix) A square $IVFS$-matrix $\bar{a}_{ij}$ of order $n \times n$ is said to be an **upper triangular $IVFS$-matrix** if all the elements below the leading diagonal are $[0,0]$, ie., $a_{ij} = [0,0]$ if $i > j$. A square $IVFS$-matrix $\bar{a}_{ij}$ of order $n \times n$ is said to be an **lower triangular $IVFS$-matrix** if all the elements above the leading diagonal are $[0,0]$, ie., $a_{ij} = [0,0]$ if $i < j$.

Example 3.9. Suppose the initial universe set $U = \{d_1, d_2, d_3, d_4, d_5\}$ and the parameter set $E = \{e_1, e_2, e_3, e_4, e_5\}$. Let $\bar{F} : E \longrightarrow P(U)$ s.t,

$$\bar{F}(e_1) = \{d_1/[0.8,1], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0,0]\},$$
$$\bar{F}(e_2) = \{d_1/[0.2,0.5], d_2/[0.3,0.5], d_3/[0,0], d_4/[0,0], d_5/[0,0]\},$$
$$\bar{F}(e_3) = \{d_1/[0.4,0.6], d_2/[0.9,1], d_3/[0.8,1], d_4/[0,0], d_5/[0,0]\},$$
$$\bar{F}(e_4) = \{d_1/[0.1,0.3], d_2/[0.7,0.9], d_3/[0.2,0.4], d_4/[0.7,0.9], d_5/[0,0]\},$$
$$\bar{F}(e_5) = \{d_1/[0.5,0.7], d_2/[0.1,0.4], d_3/[0.6,0.8], d_4/[0.3,0.5], d_5/[0.6,0.8]\}.$$

Then the interval-valued fuzzy soft set

$$\begin{aligned} (\bar{F}, E) \quad = \quad & \{(e_1, \{d_1/[0.8,1], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}), \\ & (e_2, \{d_1/[0.2,0.5], d_2/[0.3,0.5], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}), \\ & (e_3, \{d_1/[0.4,0.6], d_2/[0.9,1], d_3/[0.8,1], d_4/[0,0], d_5/[0,0]\}), \\ & (e_4, \{d_1/[0.1,0.3], d_2/[0.7,0.9], d_3/[0.2,0.4], d_4/[0.7,0.9], d_5/[0,0]\}), \\ & (e_5, \{d_1/[0.5,0.7], d_2/[0.1,0.4], d_3/[0.6,0.8], d_4/[0.3,0.5], d_5/[0.6,0.8]\})\} \end{aligned}$$

and so the relation form of $(\bar{F}, E)$ is,

$$\begin{aligned} R_E \quad = \quad & \{(\{(\{d_1/[0.8,1], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}, e_1), \\ & (\{d_1/[0.2,0.5], d_2/[0.3,0.5], d_3/[0,0], d_4/[0,0], d_5/[0,0]\}, e_2), \\ & (\{d_1/[0.4,0.6], d_2/[0.9,1], d_3/[0.8,1], d_4/[0,0], d_5/[0,0]\}, e_3), \\ & (\{d_1/[0.1,0.3], d_2/[0.7,0.9], d_3/[0.2,0.4], d_4/[0.7,0.9], d_5/[0,0]\}, e_4), \\ & (\{d_1/[0.5,0.7], d_2/[0.1,0.4], d_3/[0.6,0.8], d_4/[0.3,0.5], d_5/[0.6,0.8]\}, e_5)\} \end{aligned}$$

Therefore the $IVFS$-matrix $(\bar{a}_{ij})$ is written by,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.8,1] & [0.2,0.5] & [0.4,0.6] & [0.1,0.3] & [0.5,0.7] \\ [0,0] & [0.3,0.5] & [0.9,1] & [0.7,0.9] & [0.1,0.4] \\ [0,0] & [0,0] & [0.8,1] & [0.2,0.4] & [0.6,0.8] \\ [0,0] & [0,0] & [0,0] & [0.7,0.9] & [0.3,0.5] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0.6,0.8] \end{pmatrix}$$

Here $a_{ij} = [0,0]$ if $i > j$, hence $(\bar{a}_{ij})$ is an upper triangular $IVFS$-matrix. Now let $\bar{G} : E \longrightarrow P(U)$ s.t,

$$\begin{aligned}(\bar{G}, E) \quad = \quad & \{(e_1, \{d_1/[0.8,1], d_2/[0.2,0.5], d_3/[0.4,0.6], d_4/[0.1,0.3], d_5/[0.5,0.7]\}),\\ & (e_2, \{d_1/[0,0], d_2/[0.3,0.5], d_3/[0.8,1], d_4/[0.6,0.8], d_5/[0.4,0.6]\}),\\ & (e_3, \{d_1/[0,0], d_2/[0,0], d_3/[0.8,0.9], d_4/[0.2,0.5], d_5/[0.7,0.9]\}),\\ & (e_4, \{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0.6,0.9], d_5/[0.2,0.5]\}),\\ & (e_5, \{d_1/[0,0], d_2/[0,0], d_3/[0,0], d_4/[0,0], d_5/[0.6,0.8]\})\}\end{aligned}$$

Then similarly the associated $IVFS$-matrix will be,

$$(\bar{b}_{ij}) = \begin{pmatrix} [0.8,1] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0.2,0.5] & [0.3,0.5] & [0,0] & [0,0] & [0,0] \\ [0.4,0.6] & [0.8,1] & [0.8,0.9] & [0,0] & [0,0] \\ [0.1,0.3] & [0.6,0.8] & [0.2,0.5] & [0.6,0.9] & [0,0] \\ [0.5,0.7] & [0.4,0.6] & [0.7,0.9] & [0.2,0.5] & [0.6,0.8] \end{pmatrix}$$

Here $b_{ij} = [0,0]$ if $i < j$, therefore $(\bar{b}_{ij})$ is a lower triangular $IVFS$-matrix.

Definition 3.10 (Equality of $IVFS$-Matrices) Let A and B be two $IVFS$-matrices under the same universe U and set of parameters E. Now A and B are said to be **conformable for equality**, if they be of the same order. Now the $IVFS$-matrices A and B with same order are said to be **equal**, if and only if the corresponding elements of A and B be equal.

Definition 3.11 (Transpose of a square $IVFS$-Matrix) The **transpose** of a square $IVFS$-matrix A of order $n \times n$ is another square $IVFS$-matrix of the same order obtained from A by interchanging its rows and columns. It is denoted by A^T. Now if $A = (\bar{a}_{ij})_{n\times n}$, then its transpose A^T is defined by $A^T = (\bar{b}_{ij})_{n\times n}$, where $b_{ij} = a_{ji}$. Therefore the interval-valued fuzzy soft set associated with A^T becomes a new interval-valued fuzzy soft set over the same universe and over the same set of parameters.

Note: Transpose of a non-square $IVFS$-Matrix cannot be defined, as it does not carry any physical meaning.

Example 3.10. Consider the example 3.1. Here $(\bar{F}_A, E)$ be an interval-valued fuzzy soft set over the universe U and over the set of parameters E, given by,

$$\begin{aligned}(\bar{F}_A, E) \quad = \quad & \{\text{highly polluted city } = \{C_1/[.2,.4], C_2/[.8,.9], C_3/[.4,.5], C_4/[.6,.7], C_5/[.6,.8]\},\\ & \text{immensely polluted city} = \{C_1/[0,.1], C_2/[.9,1], C_3/[.3,.4], C_4/[.4,.6], C_5/[.6,.7]\},\\ & \text{moderately polluted city} = \{C_1/[.3,.5], C_2/[.4,.5], C_3/[.8,1], C_4/[.1,.2], C_5/[.3,.5]\},\\ & \text{less polluted city} = \{C_1/[.9,1], C_2/[.1,.2], C_3/[.5,.7], C_4/[.3,.5], C_5/[.1,.2]\}\}\end{aligned}$$

whose associated $IVFS$-matrix is,

$$(\bar{a}_{ij}) = \begin{pmatrix} [\,.2,\,.4\,] & [\,0,\,.1\,] & [\,.3,\,.5\,] & [0,0] & [\,.9,\,1\,] \\ [\,.8,\,.9\,] & [\,.9,\,1\,] & [\,.4,\,.5\,] & [0,0] & [\,.1,\,.2] \\ [\,.4,\,.5] & [\,.3,\,.4] & [\,.8,\,1\,] & [0,0] & [\,.5,\,.7\,] \\ [\,.6,\,.7\,] & [\,.4,\,.6\,] & [\,.1,\,.2\,] & [0,0] & [\,.3,\,.5\,] \\ [.6,\,.8\,] & [.6,\,.7] & [.3,\,.5] & [0,0] & [.1,\,.2] \end{pmatrix}$$

Now its transpose $IVFS$-matrix is,

$$(\bar{a}_{ij})^T = \begin{pmatrix} [\ .2,\ .4\ ] & [\ .8,\ .9\ ] & [\ .4,\ .5\ ] & [.6,.7] & [\ .6,\ .8\ ] \\ [\ 0,\ .1\ ] & [\ .9,\ 1\ ] & [\ .3,\ .4\ ] & [.4,.6] & [\ .6,\ .7] \\ [\ .3,\ .5] & [\ .4,\ .5] & [\ .8,\ 1\ ] & [.1,.2] & [\ .3,\ .5\ ] \\ [\ 0,\ 0\ ] & [\ 0,\ 0\ ] & [\ 0,\ 0\ ] & [0,0] & [\ 0,\ 0\ ] \\ [.9,\ 1\ ] & [.1,\ .2] & [.5,\ .7] & [.3,.5] & [.1,\ .2] \end{pmatrix}$$

Therefore the interval-valued fuzzy soft set associated with $(\bar{a}_{ij})^T$ is,

$$\begin{aligned} (\bar{G}_B, E) = \{ & \text{highly polluted city } = \{C_1/[.2,.4], C_2/[0,.1], C_3/[.3,.5], C_4/[0,0], C_5/[.9,1]\}, \\ & \text{immensely polluted city} = \{C_1/[.8,.9], C_2/[.9,1], C_3/[.4,.5], C_4/[0,0], C_5/[.1,.2]\}, \\ & \text{moderately polluted city} = \{C_1/[.4,.5], C_2/[.3,.4], C_3/[.8,1], C_4/[0,0], C_5/[.3,.5]\}, \\ & \text{average polluted city} = \{C_1/[.6,.7], C_2/[.4,.6], C_3/[.1,.2], C_4/[0,0], C_5/[.3,.5]\}, \\ & \text{less polluted city} = \{C_1/[.6,.8], C_2/[.6,.7], C_3/[.3,.5], C_4/[0,0], C_5/[.1,.2]\}\} \end{aligned}$$

Definition 3.12 (Choice Matrix) It is a square matrix whose rows and columns both indicate parameters. If ξ is a choice matrix, then its element $\xi(i,j)$ is defined as follows:

$$\xi(i,j) = \begin{cases} [1,1] \text{ when } i^{th} \text{ and } j^{th} \text{ parameters are both } \textbf{choice} \text{ parameters of the decision makers} \\ [0,0] \text{ when atleast one of the } i^{th} \text{ or } j^{th} \text{ parameters be } \textbf{not under choice} \text{ of the decision maker} \end{cases}$$

Any Greek letter may be used to denote a choice matrix. There are different types of choice matrices according to the number of decision makers. We may realize this by the following example.

Example 3.11. Suppose that U be a set of four factories, say, $U = \{f_1, f_2, f_3, f_4\}$ Let E be a set of parameters, given by, $E = \{$ costly, excellent work culture, assured production, good location, cheap$\} = \{e_1, e_2, e_3, e_4, e_5\}$(say). Now let the interval-valued fuzzy soft set $(\bar{F}, A)$ describing "the quality of the factories", is given by,

$$\begin{aligned} (\bar{F}, E) = \{ & \text{ costly factories } = \{f_1/[0.9,1], f_2/[0.2,0.5], f_3/[0.4,0.5], f_4/[0.8,1]\}, \\ & \text{factories with excellent work culture } = \{f_1/[0.8,1], f_2/[0.3,0.5], f_3/[0.5,0.7], f_4/[0.4,0.5]\}, \\ & \text{factories with assured production } = \{f_1/[0.9,1], f_2/[0.2,0.5], f_3/[0.4,0.6], f_4/[0.8,1]\}, \\ & \text{factories with good location} = \{f_1/[0.7,0.9], f_2/[0.9,1], f_3/[0.4,0.7], f_4/[0.8,0.9]\}, \\ & \text{cheap factories} = \{f_1/[0.1,0.4], f_2/[0.7,0.9], f_3/[0.5,0.7], f_4/[0.2,0.5]\}\} \end{aligned}$$

Suppose Mr.X wants to buy a factory on the basis of his choice parameters excellent work culture, assured production and cheap which form a subset P of the parameter set E. Therefore $P = \{e_2, e_3, e_5\}$. Now **the choice matrix of Mr.X** is,

$$(\xi_{ij})_P = e_P \begin{pmatrix} & & e_P & & \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \end{pmatrix}$$

Now suppose Mr.X and Mr.Y together wants to buy a factory according to their choice parameters. Let the choice parameter set of Mr.Y be, $Q = \{e_1, e_2, e_3, e_4\}$. Then **the combined choice matrix of Mr.X and Mr.Y** is

$$(\xi_{ij})_{(P,Q)} = \begin{array}{cc} & e_Q \\ e_P & \begin{pmatrix} [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [1,1] & [1,1] & [1,1] & [1,1] & [0,0] \\ [1,1] & [1,1] & [1,1] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [1,1] & [1,1] & [1,1] & [1,1] & [0,0] \end{pmatrix} \end{array}$$

Here the entries $e_{ij} = [1, 1]$ indicates that e_i is a choice parameter of Mr.X and e_j is a choice parameter of Mr.Y. Now $e_{ij} = [0, 0]$ indicates either e_i fails to be a choice parameter of Mr.X or e_j fails to be a choice parameter of Mr.Y.] Again the above combined choice matrix of Mr.X and Mr.Y may be also presented in its transpose form as,

$$(\xi_{ij})_{(Q,P)} = \begin{array}{cc} & e_P \\ e_Q & \begin{pmatrix} [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix} \end{array}$$

Now let us see the form of the combined choice matrix associated with three decision makers. Suppose that Mr.Z is willing to buy a factory together with Mr.X and Mr.Y on the basis of his choice parameters excellent work culture, assured production and good location which form a subset R of the parameter set E. Therefore $R = \{e_2, e_3, e_4\}$. Then **the combined choice matrix of Mr.X, Mr.Y and Mr.Z** will be of three different types which are as follows,

$$\text{i) } (\xi_{ij})_{(R,P\wedge Q)} = \begin{array}{cc} & e_{(P\wedge Q)} \\ e_R & \begin{pmatrix} [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix} \end{array}$$

Since the set of common choice parameters of Mr.X and Mr.Y is, $P \wedge Q = \{e_2, e_3\}$. Here the entries $e_{ij} = [1, 1]$ indicates that e_i is a choice parameter of Mr.Z and e_j is a common choice parameter of Mr.X and Mr.Y. Now $e_{ij} = [0, 0]$ indicates either e_i fails to be a choice parameter of Mr.Z or e_j fails to be a common choice parameter of Mr.X and Mr.Y.

$$\text{ii) } (\xi_{ij})_{(P,Q\wedge R)} = \begin{array}{cc} & e_{(Q\wedge R)} \\ e_P & \begin{pmatrix} [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [1,1] & [0,0] \\ [0,0] & [1,1] & [1,1] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [1,1] & [0,0] \end{pmatrix} \end{array} \text{[Since } Q \wedge R = \{e_2, e_3, e_4\}]$$

$$\text{iii) } (\xi_{ij})_{(Q,R\wedge P)} = e_Q \begin{pmatrix} & & e_{(R\wedge P)} & & \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix} [\text{ Since } R\wedge P = \{e_2, e_3\}]$$

Definition 3.13 (Symmetric $IVFS$-Matrix A square $IVFS$-matrix A of order $n \times n$ is said to be a **symmetric $IVFS$-matrix**, if its transpose be equal to it, i.e., if $A^T = A$. Hence the $IVFS$-matrix $(\bar{a}_{ij})$ is symmetric, if $a_{ij} = a_{ji}, \forall i, j$. Therefore if $(\bar{a}_{ij})$ be a symmetric $IVFS$-matrix then the interval-valued fuzzy soft sets associated with $(\bar{a}_{ij})$ and $(\bar{a}_{ij})^T$ both be the same.

Example 3.12. Let the set of universe $U = \{u_1, u_2, u_3, u_4\}$ and the set of parameters $E = \{e_1, e_2, e_3, e_4\}$. Now suppose that, $A \subseteq E$ and $\bar{F}_A : E \longrightarrow P(U)$ s.t, $(\bar{F}_A, E)$ forms an interval-valued fuzzy soft set given by,

$$\begin{aligned} (\bar{F}_A, E) \quad = \quad & \{(e_1, \{u_1/[0.2, 0.4], u_2/[0.3, 0.5], u_3/[0.8, 1], u_4/[0.5, 0.7]\}), \\ & (e_2, \{u_1/[0.3, 0.5], u_2/[0.6, 0.8], u_3/[0.1, 0.3], u_4/[0.7, 1]\}), \\ & (e_3, \{u_1/[0.8, 1], u_2/[0.1, 0.3], u_3/[0.7, 0.9], u_4/[0.2, 0.5]\}), \\ & (e_4, \{u_1/[0.5, 0.7], u_2/[0.7, 1], u_3/[0.2, 0.5], u_4/[0.4, 0.6]\})\} \end{aligned}$$

The $IVFS$-matrix associated with this interval-valued fuzzy soft set $(\bar{F}_A, E)$ is,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.2,0.4] & [0.3,0.5] & [0.8,1] & [0.5,0.7] \\ [0.3,0.5] & [0.6,0.8] & [0.1,0.3] & [0.7,1] \\ [0.8,1] & [0.1,0.3] & [0.7,0.9] & [0.2,0.5] \\ [0.5,0.7] & [0.7,1] & [0.2,0.5] & [0.4,0.6] \end{pmatrix}.$$

Definition 3.14 (Addition of $IVFS$-Matrices) Two $IVFS$-matrices A and B are said to be **conformable for addition**, if they be of the same order and after addition the obtained sum also be an $IVFS$-matrix of the same order. Now if $A = (\bar{a}_{ij})$ and $B = (\bar{b}_{ij})$ of the same order $m \times n$, then the **addition** of A and B is denoted by, $A \oplus B$ and is defined by,

$$(\bar{a}_{ij}) \oplus (\bar{b}_{ij}) \quad = \quad (\bar{c}_{ij}), \text{ where } c_{ij} = [\sup\{\mu^-_{a_{ij}}, \mu^-_{b_{ij}}\}, \sup\{\mu^+_{a_{ij}}, \mu^+_{b_{ij}}\}] \forall i, j.$$

Example 3.13. Consider the $IVFS$-matrix of example 3.1,

$$(\bar{a}_{ij}) = \begin{pmatrix} [\ .2,\ .4\ ] & [\ 0,\ .1\ ] & [\ .3,\ .5\ ] & [0,0] & [\ .9,\ 1\ ] \\ [\ .8,\ .9\ ] & [\ .9,\ 1\ ] & [\ .4,\ .5\ ] & [0,0] & [\ .1,\ .2] \\ [\ .4,\ .5] & [\ .3,\ .4] & [\ .8,\ 1\ ] & [0,0] & [\ .5,\ .7\ ] \\ [\ .6,\ .7\ ] & [\ .4,\ .6\ ] & [\ .1,\ .2\ ] & [0,0] & [\ .3,\ .5\ ] \\ [.6,\ .8\ ] & [.6,\ .7] & [.3,\ .5] & [0,0] & [.1,\ .2] \end{pmatrix}$$

Now consider another $IVFS$-matrix $(\bar{b}_{ij})$ associated with the interval-valued fuzzy soft set $(\bar{G}_B, E)$. (also describing "the pollution of the cities") over the same universe U. Let $B = \{e_1, e_4, e_5\} \subset E$ and

$$(\bar{G}, B) \quad = \quad \{\text{highly polluted city } = \{C_1/[.3, .5], C_2/[.9, 1], C_3/[.4, .5], C_4/[.7, .9], C_5/[.6, .9]\},$$

average polluted city $= \{C_1/[.2, .5], C_2/[.3, .5], C_3/[.7, .9], C_4/[.2, .4], C_5/[.3, .6]\}$,
less polluted city $= \{C_1/[.8, 1], C_2/[.2, .3], C_3/[.6, .8], C_4/[.3, .5], C_5/[.2, .4]\}\}$

Hence the $IVFS$-matrix $(\bar{b}_{ij})$ is written by,

$$(\bar{b}_{ij}) = \begin{pmatrix} [\,.3,\ .5\,] & [0,0] & [0,0] & [.2,.5] & [\,.8,\ 1\,] \\ [\,.9,\ 1\,] & [0,0] & [0,0] & [.3,.5] & [\,.2,\ .3] \\ [\,.4,\ .5] & [0,0] & [0,0] & [.7,.9] & [\,.6,\ .8\,] \\ [\,.7,\ .9\,] & [0,0] & [0,0] & [.2,.4] & [\,.3,\ .5\,] \\ [.6,\ .9\,] & [0,0] & [0,0] & [.3,.6] & [.2,\ .4] \end{pmatrix}.$$

Therefore the sum of the $IVFS$-matrices $(\bar{a}_{ij})$ and $(\bar{b}_{ij})$ is,

$$(\bar{a}_{ij}) \oplus (\bar{b}_{ij}) = \begin{pmatrix} [\,.3,\ .5\,] & [\,0,\ .1\,] & [\,.3,\ .5\,] & [.2,.5] & [\,.9,\ 1\,] \\ [\,.9,\ 1\,] & [\,.9,\ 1\,] & [\,.4,\ .5\,] & [.3,.5] & [\,.2,\ .3] \\ [\,.4,\ .5] & [\,.3,\ .4] & [\,.8,\ 1\,] & [.7,.9] & [\,.6,\ .8\,] \\ [\,.7,\ .9\,] & [\,.4,\ .6\,] & [\,.1,\ .2\,] & [.2,.4] & [\,.3,\ .5\,] \\ [.6,\ .9\,] & [.6,\ .7] & [.3,\ .5] & [.3,.6] & [.2,\ .4] \end{pmatrix}.$$

Definition 3.15 (Subtraction of $IVFS$-Matrices) Two $IVFS$-matrices A and B are said to be **conformable for subtraction**, if they be of the same order and after subtraction the obtained result also be an $IVFS$-matrix of the same order. Now if $A = (\bar{a}_{ij})$ and $B = (\bar{b}_{ij})$ of order $m \times n$, then **subtraction** of B from A is denoted by, $A \ominus B$ and is defined by,

$$(\bar{a}_{ij}) \ominus (\bar{b}_{ij}) \quad = \quad (\bar{c}_{ij}),\ \text{where } c_{ij} = [\inf\{\mu^-_{a_{ij}}, \mu^-_{b^o_{ij}}\}, \inf\{\mu^+_{a_{ij}}, \mu^+_{b^o_{ij}}\}] \forall i, j$$

where $(\bar{b}^o_{ij})$ is the complement of $(\bar{b}_{ij})$

Example 3.14. Consider the $IVFS$-matrices $(\bar{a}_{ij})$ and $(\bar{b}_{ij})$ of example 3.13. Now

$$(\bar{a}_{ij}) = \begin{pmatrix} [\,.2,\ .4\,] & [\,0,\ .1\,] & [\,.3,\ .5\,] & [0,0] & [\,.9,\ 1\,] \\ [\,.8,\ .9\,] & [\,.9,\ 1\,] & [\,.4,\ .5\,] & [0,0] & [\,.1,\ .2] \\ [\,.4,\ .5] & [\,.3,\ .4] & [\,.8,\ 1\,] & [0,0] & [\,.5,\ .7\,] \\ [\,.6,\ .7\,] & [\,.4,\ .6\,] & [\,.1,\ .2\,] & [0,0] & [\,.3,\ .5\,] \\ [.6,\ .8\,] & [.6,\ .7] & [.3,\ .5] & [0,0] & [.1,\ .2] \end{pmatrix}$$

and

$$(\bar{b}_{ij})^o = \begin{pmatrix} [\,.5,\ .7\,] & [1,1] & [1,1] & [.5,.8] & [\,0,\ .2\,] \\ [\,0,\ .1\,] & [1,1] & [1,1] & [.5,.7] & [\,.7,\ .8] \\ [\,.5,\ .6] & [1,1] & [1,1] & [.1,.3] & [\,.2,\ .4\,] \\ [\,.1,\ .3\,] & [1,1] & [1,1] & [.6,.8] & [\,.5,\ .7\,] \\ [.1,\ .4\,] & [1,1] & [1,1] & [.4,.7] & [.6,\ .8] \end{pmatrix}.$$

Therefore the subtraction of the $IVFS$-matrix $(\bar{b}_{ij})$ from the $IVFS$-matrix $(\bar{a}_{ij})$ is,

$$(\bar{a}_{ij}) \ominus (\bar{b}_{ij}) = \begin{pmatrix} [\ .2,\ .4\ ] & [\ 0,\ .1\ ] & [\ .3,\ .5\ ] & [0,0] & [\ 0,\ .2\ ] \\ [\ 0,\ .1\ ] & [\ .9,\ 1\ ] & [\ .4,\ .5\ ] & [0,0] & [\ .1,\ .2] \\ [\ .4,\ .5] & [\ .3,\ .4] & [\ .8,\ 1\ ] & [0,0] & [\ .2,\ .4\ ] \\ [\ .1,\ .3\ ] & [\ .4,\ .6\ ] & [\ .1,\ .2\ ] & [0,0] & [\ .3,\ .5\ ] \\ [.1,\ .4\ ] & [.6,\ .7] & [.3,\ .5] & [0,0] & [.1,\ .2] \end{pmatrix}.$$

Properties. *Let A and B be two $IVFS$-matrices of order $m \times n$. Then*

i) $A \oplus B = B \oplus A$

ii) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

iii) $A \ominus B \neq B \ominus A$

iv) $(A \ominus B) \ominus C \neq A \ominus (B \ominus C)$

v) $A \oplus A^o \neq \bar{C}_A$

vi) $A \ominus A \neq \bar{\Phi}$.

Proof: The proofs of (i)-(vi) are directly obtained from the definitions of addition, subtraction and complement. □

Theorem 1. *If A be an $IVFS$-square matrix of order $n \times n$, then $(A^T)^T = A$.*

Proof: Let $A = (\bar{a}_{ij})_{n\times n}$. Then by definition, $A^T = (\bar{b}_{ij})_{n\times n}$ where $b_{ij} = a_{ji} \forall i, j$. ie., $A^T = (\bar{a}_{ji})_{n\times n}$. Therefore $(A^T)^T = (\bar{c}_{ij})_{n\times n}$ where $c_{ij} = a_{ij}$ ie., $(A^T)^T = (\bar{a}_{ij})_{n\times n} = A$.

Theorem 2. *If A and B be two $IVFS$-square matrices of order $n \times n$, then $(A \oplus B)^T = A^T \oplus B^T$.*

Proof: Let $A = (\bar{a}_{ij})_{n\times n}$ and $B = (\bar{b}_{ij})_{n\times n}$. Then

$$\begin{aligned} L.H.S &= (A \oplus B)^T = C^T \text{ where } C = (\bar{c}_{ij})_{n\times n} \\ &= (\bar{c}_{ji})_{n\times n} \text{ where } c_{ji} = [\sup\{\mu^-_{a_{ji}}, \mu^-_{b_{ji}}\}, \sup\{\mu^+_{a_{ji}}, \mu^+_{b_{ji}}\}] \forall i, j \end{aligned}$$

and

$$\begin{aligned} R.H.S &= A^T \oplus B^T = (\bar{a}_{ji})_{n\times n} \oplus (\bar{b}_{ji})_{n\times n} \\ &= (\bar{d}_{ji})_{n\times n} \text{ where } d_{ji} = [\sup\{\mu^-_{a_{ji}}, \mu^-_{b_{ji}}\}, \sup\{\mu^+_{a_{ji}}, \mu^+_{b_{ji}}\}] \forall i, j \\ &= C^T = L.H.S \end{aligned}$$

Hence $(A \oplus B)^T = A^T \oplus B^T$. □

Theorem 3. *If A be an $IVFS$-square matrix of order $n \times n$, then $(A \oplus A^T)$ is symmetric.*

Proof: Let $A = (\bar{a}_{ij})_{n\times n}$. Then by definition, $A^T = (\bar{a}_{ji})_{n\times n}$. Now

$$\begin{aligned} A \oplus A^T &= (\bar{a}_{ij})_{n\times n} \oplus (\bar{a}_{ji})_{n\times n} \\ &= (\bar{c}_{ij})_{n\times n} \text{ where } c_{ij} = [\sup\{\mu^-_{a_{ij}}, \mu^-_{a_{ji}}\}, \sup\{\mu^+_{a_{ij}}, \mu^+_{a_{ji}}\}] \forall i, j. \end{aligned}$$

Now $c_{ji} = [\sup\{\mu^-_{a_{ji}}, \mu^-_{a_{ij}}\}, \sup\{\mu^+_{a_{ji}}, \mu^+_{a_{ij}}\}] = c_{ij} \forall i, j$. Therefore $(\bar{c}_{ij})_{n\times n}$ ie., $(A \oplus A^T)$ is symmetric. □

Theorem 4. *If A and B be two $IVFS$-square matrices of order $n \times n$ and if A and B be symmetric, then $A \oplus B$ is symmetric.*

Proof: Since A and B be symmetric, $A^T = A$ and $B^T = B$. Therefore $A^T \oplus B^T = A \oplus B$. Thus from theorem-2 we have, $(A \oplus B)^T = A^T \oplus B^T = A \oplus B$. Hence $A \oplus B$ is symmetric. □

Definition 3.16 (Product of an $IVFS$-Matrix with a Choice Matrix) Let U be the set of universe and E be the set of parameters. Suppose that A be an $IVFS$-matrix and β be the choice matrix over (U, E). The product of an $IVFS$-matrix A with a choice matrix β is denoted by $A \otimes \beta$. Now A and β are said to be conformable for product $A \otimes \beta$ when the number of columns of A be equal to the number of rows of β and the product $A \otimes \beta$ becomes also another $IVFS$-matrix. If $A = (\bar{a}_{ij})_{m\times n}$ and $\beta = (\bar{\beta}_{jk})_{n\times p}$, then the product $A \otimes \beta$ is defined as

$$A \otimes \beta \quad = \quad (\bar{c}_{ik}) \text{ where } c_{ik} = [\sup_{j=1}^{n} \inf\{\mu^-_{a_{ij}}, \mu^-_{\beta_{jk}}\}, \sup_{j=1}^{n} \inf\{\mu^+_{a_{ij}}, \mu^+_{\beta_{jk}}\}]$$

Properties. *i) $\beta \otimes A$ cannot be defined here. ii) If $A \otimes \beta = \bar{\Phi}$, then we cannot say, as in scalar algebra, that either A is a null $IVFS$-matrix or, β is a null matrix.*

Example 3.15. Let U be the set of four dresses, given by, U = $\{d_1, d_2, d_3, d_4\}$. Let E be the set of parameters, given by, E = { cheap, beautiful, comfortable, gorgeous } = $\{e_1, e_2, e_3, e_4\}$(say). Suppose that the set of choice parameters of Mr.X be, $A = \{e_1, e_3\}$. Now let according to the choice parameters of Mr.X, we have the interval-valued fuzzy soft set $(\bar{F}, A)$ which describes "the attractiveness of the dresses"and the $IVFS$-matrix of the interval-valued fuzzy soft set $(\bar{F}, A)$ be,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.8,1] & [0.2,0.3] & [0.7,0.9] & [0.3,0.5] \\ [0.3,0.4] & [0.7,0.9] & [0.4,0.6] & [0.8,1] \\ [0.7,1] & [0.4,0.5] & [0.5,0.8] & [0.6,0.7] \\ [0.5,0.7] & [0.1,0.3] & [0.9,1] & [0.2,0.5] \end{pmatrix}$$

Again the choice matrix of Mr.X is,

$$‘\ (\xi_{ij})_A = e_A \begin{array}{c} e_A \\ \begin{pmatrix} [1,1] & [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [1,1] & [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix} \end{array}$$

Since the number of columns of the $IVFS$-matrix $(\bar{a}_{ij})$ is equal to the number of rows of the choice matrix $(\xi_{ij})_A$, they are conformable for the product. Therefore

$$U \begin{array}{c} e_A \\ \begin{pmatrix} [0.8,1] & [0.2,0.3] & [0.7,0.9] & [0.3,0.5] \\ [0.3,0.4] & [0.7,0.9] & [0.4,0.6] & [0.8,1] \\ [0.7,1] & [0.4,0.5] & [0.5,0.8] & [0.6,0.7] \\ [0.5,0.7] & [0.1,0.3] & [0.9,1] & [0.2,0.5] \end{pmatrix} \end{array} \otimes e_A \begin{array}{c} e_A \\ \begin{pmatrix} [1,1] & [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [1,1] & [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix} \end{array}$$

$$= U \begin{array}{c} E \\ \begin{pmatrix} [0.8,1] & [0,0] & [0.8,1] & [0,0] \\ [0.4,0.6] & [0,0] & [0.4,0.6] & [0,0] \\ [0.7,1] & [0,0] & [0.7,1] & [0,0] \\ [0.9,1] & [0,0] & [0.9,1] & [0,0] \end{pmatrix} \end{array}.$$

4 Group Decisions

Group Decision is the academic and professional field that aims to improve collective decision process by taking into account the needs and opinions of every group members. Group decision includes the development and study of methods for assisting groups or individuals within groups, as they interact and collaborate to reach a collective decision. But till now there does not exist any method to solve interval-valued fuzzy soft set(IVFSs) based group decision making problem. In the following subsection at first a general IVFSs based decision making problem is defined, then a new approach is developed to solve such types of problems.

A Generalized Interval-Valued Fuzzy Soft Set Based Group Decision Making Problem: Let N number of decision makers want to select an object jointly from the m number of objects which have n number of features ie., parameters(E). Suppose that each decision maker has freedom to take his decision of inclusion and evaluation of parameters associated with the selected object, ie., each decision maker has his own choice parameters belonging to the parameter set E and has his own view of evaluation. Here it is assumed that parameter evaluation of the objects must be interval-valued fuzzy. Now the problem is to find out the object out of these m objects which satisfies all the choice parameters of all decision makers jointly as much as possible.

A New Approach to Solve $IVFS$-Matrix Based Group Decision Making Problems: This new approach is specially based on choice matrices and its operations. The stepwise procedure of this new approach is presented in the following $IVFSM$-algorithm.

$IVFSM$-**Algorithm**:

Step-I: First construct the combined choice matrices with respect to the choice parameters of the decision makers.

Step-II: Compute the product $IVFS$-matrices by multiplying each given $IVFS$-matrix with the combined choice matrix as per the rule of multiplication of $IVFS$-matrices.

Step-III: Compute the sum of these product $IVFS$-matrices to have the resultant $IVFS$-matrix($\bar{R}_{IVFS}$).

Step-IV: Then compute the weight of each object(O_i) by adding the membership values of the entries of its concerned row(i-th row) of $\bar{R}_{IVFS}$ and denote it as $W(O_i)$.

Step-V: $\forall O_i \in U$, compute the score r_i of O_i such that,

$$r_i = \sum_{O_j \in U} ((\mu_i^- - \mu_j^-) + (\mu_i^+ - \mu_j^+)$$

Step-VI: The object having the highest score becomes the jointly selected object according to all decision makers. If more than one object have the highest score then any one of these highest scorers may be chosen as the jointly selected object.

To illustrate the basic idea of the $IVFSM$-algorithm, now we apply it to the following $IVFS$-matrix based decision making problem.

Example 4.1. Let the set of universe U consist of four cities C_1, C_2, C_3, C_4 and the set of parameters $E = \{$ industrialization, greeneries, modern facilities, government pollution control policy $\} = \{e_1, e_2, e_3, e_4\}$. Now two Mr.X and Mrs.X wants to live in a city among these four. Mr.X likes an industrialized city but having greeneries and Government pollution control policy. Whereas Mrs.X likes greeneries but a city with modern facilities. So the sets of choice parameters of Mr.X and Mrs.X are respectively, $A = \{e_1, e_2, e_4\} \subset E$ and $B = \{e_2, e_3\} \subset E$. Now let according to the choice parameters of Mr.X and Mrs.X, we have the interval-valued fuzzy soft sets $(\bar{F}_A, E)$ and $(\bar{G}_B, E)$, both describing "the attractiveness of the cities "according to Mr.X and Mrs.X respectively. Let the $IVFS$-matrices of the interval-valued fuzzy soft sets $(\bar{F}_A, E)$ and $(\bar{G}_B, E)$ are respectively,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.8,1] & [0.1,0.3] & [0,0] & [0.3,0.5] \\ [0,0.2] & [0.7,0.9] & [0,0] & [0.6,0.7] \\ [0.6,0.9] & [0.3,0.4] & [0,0] & [0.4,0.6] \\ [0.7,0.8] & [0.1,0.2] & [0,0] & [0.2,0.3] \end{pmatrix}, \ (\bar{b}_{ik}) = \begin{pmatrix} [0,0] & [0.2,0.3] & [0.5,0.7] & [0,0] \\ [0,0] & [0.7,0.9] & [0.3,0.6] & [0,0] \\ [0,0] & [0.3,0.5] & [0.5,0.6] & [0,0] \\ [0,0] & [0.2,0.3] & [0.6,0.8] & [0,0] \end{pmatrix}$$

Now the problem is to select the city among the four cities which satisfies the choice parameters of Mr.X and Mrs.X as much as possible.

1) The combined choice matrix of Mr.X and Mrs.X is,

$$\begin{array}{cc} & e_A \\ e_B & \begin{pmatrix} [0,0] & [0,0] & [0,0] & [0,0] \\ [1,1] & [1,1] & [0,0] & [1,1] \\ [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix} \end{array}$$

or it may be presented in its transposed form as,

$$\begin{array}{cc} & e_B \\ e_A & \begin{pmatrix} [0,0] & [1,1] & [1,1] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] \end{pmatrix} \end{array}$$

2) Corresponding product $IVFS$-matrices are,

$$\begin{array}{cc} & e_A \\ U & \begin{pmatrix} [0.8,1] & [0.1,0.3] & [0,0] & [0.3,0.5] \\ [0,0.2] & [0.7,0.9] & [0,0] & [0.6,0.7] \\ [0.6,0.9] & [0.3,0.4] & [0,0] & [0.4,0.6] \\ [0.7,0.8] & [0.1,0.2] & [0,0] & [0.2,0.3] \end{pmatrix} \end{array} \otimes \begin{array}{cc} & e_B \\ e_A & \begin{pmatrix} [0,0] & [1,1] & [1,1] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [1,1] & [1,1] & [0,0] \end{pmatrix} \end{array}$$

$$= \begin{array}{cc} & E \\ U & \begin{pmatrix} [0,0] & [0.8,1] & [0.8,1] & [0,0] \\ [0,0] & [0.7,0.9] & [0.7,0.9] & [0,0] \\ [0,0] & [0.6,0.9] & [0.6,0.9] & [0,0] \\ [0,0] & [0.7,0.8] & [0.7,0.8] & [0,0] \end{pmatrix} \end{array}$$

$$\begin{array}{cc} & e_B \\ U & \begin{pmatrix} [0,0] & [0.2,0.3] & [0.5,0.7] & [0,0] \\ [0,0] & [0.7,0.9] & [0.3,0.6] & [0,0] \\ [0,0] & [0.3,0.5] & [0.5,0.6] & [0,0] \\ [0,0] & [0.2,0.3] & [0.6,0.8] & [0,0] \end{pmatrix} \end{array} \otimes \begin{array}{cc} & e_A \\ e_B & \begin{pmatrix} [0,0] & [0,0] & [0,0] & [0,0] \\ [1,1] & [1,1] & [0,0] & [1,1] \\ [1,1] & [1,1] & [0,0] & [1,1] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix} \end{array}$$

$$= U \begin{array}{c} E \\ \begin{pmatrix} [0,0] & [0.4,0.7] & [0.4,0.7] & [0,0] \\ [0,0] & [0.7,0.9] & [0.7,0.9] & [0,0] \\ [0,0] & [0.5,0.6] & [0.5,0.6] & [0,0] \\ [0,0] & [0.6,0.8] & [0.6,0.8] & [0,0] \end{pmatrix} \end{array}$$

3) The sum of these product $IVFS$-matrices is,

$$\begin{pmatrix} [0,0] & [0.8,1] & [0.8,1] & [0,0] \\ [0,0] & [0.7,0.9] & [0.7,0.9] & [0,0] \\ [0,0] & [0.6,0.9] & [0.6,0.9] & [0,0] \\ [0,0] & [0.7,0.8] & [0.7,0.8] & [0,0] \end{pmatrix} \oplus \begin{pmatrix} [0,0] & [0.4,0.7] & [0.4,0.7] & [0,0] \\ [0,0] & [0.7,0.9] & [0.7,0.9] & [0,0] \\ [0,0] & [0.5,0.6] & [0.5,0.6] & [0,0] \\ [0,0] & [0.6,0.8] & [0.6,0.8] & [0,0] \end{pmatrix}$$

$$= \begin{pmatrix} [0,0] & [0.8,1] & [0.8,1] & [0,0] \\ [0,0] & [0.7,0.9] & [0.7,0.9] & [0,0] \\ [0,0] & [0.6,0.9] & [0.6,0.9] & [0,0] \\ [0,0] & [0.7,0.8] & [0.7,0.8] & [0,0] \end{pmatrix} = R_{IVFS}$$

4) Now the weights of the cities are,

(i) $W(C_1) = [(0 + 0.8 + 0.8 + 0), (0 + 1 + 1 + 0)] = [1.6, 2]$
(ii) $W(C_2) = [(0 + 0.7 + 0.7 + 0), (0 + 0.9 + 0.9 + 0)] = [1.4, 1.8]$
(iii) $W(C_3) = [(0 + 0.6 + 0.6 + 0), (0 + 0.9 + 0.9 + 0)] = [1.2, 1.8]$
(iv) $W(C_4) = [(0 + 0.7 + 0.7 + 0), (0 + 0.8 + 0.8 + 0)] = [1.4, 1.6]$

5) Now the scores for the cities are,

(i) $r_1 = (0.2 + 0.4 + 0.2) + (0.2 + 0.2 + 0.4) = 1.6$
(ii) $r_2 = (-0.2 + 0.2 + 0) + (-0.2 + 0 + 0.2) = 0$
(iii) $r_3 = (-0.4 - 0.2 - 0.2) + (-0.2 + 0 + 0.2) = -0.8$
(iv) $r_4 = (-0.2 + 0 + 0.2) + (-0.4 - 0.2 - 0.2) = -0.8$

6) Since the score r_1 is maximum (1.6), the highest scorer city C_1 will be their selected city which fulfill the choice parameters of Mr.X and Mrs.X as much as possible.

5 Purpose of introducing $IVFSM$-Algorithm

Till now researchers [17, 19] have worked on finding solution of the interval-valued fuzzy soft set (IVFSs) based decision making problems involving only one decision maker. There does not exist any algorithm for solving an IVFSs-based group decision making problem. $IVFSM$-algorithm can solve the IVFSs-based decision making problems involving any number of decision maker.

Moreover the existing methods for solving IVFSs-based decision making problems have some drawbacks. Feng et al.[19] have pointed out the drawback of Yang's method[17] and shown that their proposed method is more efficient than it. Now by the following example we will find out the drawbacks of Feng's method and show that our proposed $IVFSM$-algorithm is more deterministic than Feng's method.

Example 5.1. Let the set of universe U consist of three dresses d_1, d_2, d_3 and the set of parameters E = { cheap, comfortable, beautiful, costly } = $\{e_1, e_2, e_3, e_4\}$. Now two Mr.X wants to buy a dress among these three and his set of choice parameters is, $A = \{e_1, e_2, e_3\}$. Suppose that, according to the choice parameters of Mr.X, the interval-valued fuzzy soft set $(\bar{F}_A, E)$ describing "the attractiveness of the dresses "according to Mr.X is given by,

$$\begin{aligned}(\bar{F}_A, E) \quad = \quad &\{ \text{ cheap dresses} = \{d_1/[.7, .9], d_2/[.5, .7], d_3/[.6, .8]\},\\ &\text{comfortable dresses} = \{d_1/[.6, .8], d_2/[.8, 1], d_3/[.5, .7]\},\\ &\text{beautiful dresses} = \{d_1/[.1, .3], d_2/[.5, .7], d_3/[.7, .9]\}\}\end{aligned}$$

Now at first we solve this problem by Feng's method and then by $IVFSM$-Algorithm.

By Feng's Method:

For a neutral estimation of uncertain membership values we have the Neutral Reduct Fuzzy Soft set(NRFSs) of $(\bar{F}_A, E)$ in the tabular form as,

Table-4(a) : Tabular representation of the NRFSs of $(\bar{F}_A, E)$

	e_1	e_2	e_3
d_1	0.8	0.7	0.2
d_2	0.6	0.9	0.6
d_3	0.7	0.6	0.8

Using Top Level Soft Set:

Table-4(b)

	e_1	e_2	e_3	choice value
d_1	1	0	0	1
d_2	0	1	0	1
d_3	0	0	1	1

As the choice values of all the dresses are same, according to Feng's method, Mr.X may select any one of the three dresses d_1, d_2, d_3. Using Mid Level Soft Set:

Table-4(c)

	e_1	e_2	e_3	choice value
d_1	1	1	0	2
d_2	0	1	1	2
d_3	1	0	1	2

Here also the choice values of all the dresses be the same. Hence **according to Feng's method Mr.X may select either d_1 or, d_2 or, d_3.**

By $IVFSM$-Algorithm:

The interval-valued fuzzy soft matrix associated with $(\bar{F}_A, E)$ is,

$$(\bar{a}_{ij}) = \begin{pmatrix} [0.7,0.9] & [0.6,0.8] & [0.1,0.3] & [0,\ 0] \\ [0.5,0.7] & [0.8,1] & [0.5,0.7] & [0,\ 0] \\ [0.6,0.8] & [0.5,0.7] & [0.7,0.9] & [0,\ 0] \end{pmatrix}$$

The choice matrix of Mr.X is,

$$\begin{array}{cc} & e_A \\ e_A & \begin{pmatrix} [1,1] & [1,1] & [1,1] & [0,0] \\ [1,1] & [1,1] & [1,1] & [0,0] \\ [1,1] & [1,1] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix} \end{array}$$

The product $IVFS$-matrix is,

$$\begin{array}{cc} & e_A \\ U & \begin{pmatrix} [0.7,0.9] & [0.6,0.8] & [0.1,0.3] & [0,\ 0] \\ [0.5,0.7] & [0.8,1] & [0.5,0.7] & [0,\ 0] \\ [0.6,0.8] & [0.5,0.7] & [0.7,0.9] & [0,\ 0] \end{pmatrix} \end{array} \otimes \begin{array}{cc} & e_A \\ e_A & \begin{pmatrix} [1,1] & [1,1] & [1,1] & [0,0] \\ [1,1] & [1,1] & [1,1] & [0,0] \\ [1,1] & [1,1] & [1,1] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix} \end{array}$$

$$= \begin{array}{cc} & E \\ U & \begin{pmatrix} [0.7,0.9] & [0.7,0.9] & [0.7,0.9] & [0,\ 0] \\ [0.8,1] & [0.8,1] & [0.8,1] & [0,\ 0] \\ [0.7,0.9] & [0.7,0.9] & [0.7,0.9] & [0,\ 0] \end{pmatrix} \end{array}$$

Now the weights of the dresses are,

(i) $W(d_1) = [(0.7 + 0.7 + 0.7 + 0), (0.9 + 0.9 + 0.9 + 0)] = [2.1, 2.7]$
(ii) $W(d_2) = [(0.8 + 0.8 + 0.8 + 0), (1 + 1 + 1 + 0)] = [2.4, 3]$
(iii) $W(d_3) = [(0.7 + 0.7 + 0.7 + 0), (0.9 + 0.9 + 0.9 + 0)] = [2.1, 2.7]$

Now the scores for the dresses are,

(i) $r_1 = (-0.3 + 0) + (-0.3 + 0) = -0.6$
(ii) $r_2 = (0.3 + 0.3) + (0.3 + 0.3) = 1.2$
(iii)$r_3 = (-0.3 + 0) + (-0.3 + 0) = -0.6$

Since the score r_2 is maximum (1.2), **Mr.X select the dress** d_2 which fulfill his choice parameters as much as possible. From this example we can see that $IVFSM$**-Algorithm is more deterministic than Feng's method** as it can properly solve the problem by giving an unique result. Feng's method also may give this unique result but for obtaining this we have to find out the proper level to form the level soft set by try and error method which is laborious.

6 Conclusion

In this paper first we have proposed the concept of $IVFS$-matrix and defined different types of matrices in interval-valued fuzzy soft set theory. Then we have introduced some new operations on these matrices and discussed all these definitions and operations by appropriate examples. Moreover we have proven some theorems along with few properties on these matrices. At last the new efficient $IVFSM$-

algorithm has been developed to solve interval-valued fuzzy soft set based real life group decision making problems.

References

1 D.Chen, E.C.C. Tang, D.S.Yeung and X.Wang, The parameterization Reduction of Soft Set and It's Application, Comput Math Appl. 49(2005) 757-763.
2 D.Molodtsov, Soft Set Theory-First Results, Comput Math Appl. 37(1999) 19-31.
3 F.Feng, Y. B. Jun, X. Zhao, Soft Semirings, Comput Math Appl. 56(2008) 2621-2628.
4 F.Feng, Y. Li, V.L-Fotea, Application of level soft sets in decision making based on interval-valued fuzzy soft sets, Comput Math Appl. 60(2010) 1756-1767.
5 H. Aktas, N. Cagman, Soft Sets and Soft Groups, Information Sciences. 177(2007) 2726-2735.
6 M.B.Gorzalczany, A method of inference in approximate reasoning based on interval valued fuzzy sets, Fuzzy Sets and Systems. 21 (1987) 1-17.
7 N.Cagman, S.Enginoglu, Soft Matrix Theory and It's Decision Making, Comput Math Appl. 59(2010) 3308-3314.
8 N.Cagman, S.Enginoglu, Soft Set Theory and Uni-int Decision Making, Eur J Oper Res. 207(2010)848-855.
9 P.K. Maji, R.Biswas, and A.R.Roy, An application of soft sets in a decision making problem, Comput Math Appl. 44(2002) 1077-1083.
10 P.K. Maji, R.Biswas and A.R.Roy, Soft Set Theory, Comput Math Appl. 45(2003) 555-562.
11 P.K. Maji, R.Biswas, and A.R.Roy, Fuzzy Soft Sets, The Journal of Fuzzy Mathematics. Vol 9, No.3(2001) 589-602.
12 T.M. Basu, N.K. Mahapatra and S.K. Mondal, A Balanced Solution of a Fuzzy Soft Set Based Decision Making Problem in Medical Science, Appl. Soft Comput. 12(2012) 3260-3275.
13 X.Yang, D. Yu, J. Yang, C. Wu, Generalization of Soft Set Theory: From Crisp to Fuzzy Case, Fuzzy Information and Engineering (ICFIE). ASC 40 (2007) 345-354.
14 X.Yang, T. Y. Lin, J. Yang, Y.Li, D. Yu, Combination of interval-valued fuzzy set and soft set, Comput Math Appl. 58(2009) 521-527.
15 Y. Zou, Z. Xiao, Data analysis approaches of soft sets under incomplete information, Knowl-Based Syst. 21 (2008) 941-945.
16 Y.Zou, Y.Chen, Research on Soft Set Theory and Parameters Reduction Based on Relational Algebra. Second International Symposium on Information Technology Application, (2008)DOI 10.1109/IITA 264.
17 Y.B.Jun, Soft BCK/BCI-algebras, Comput Math Appl. 56(2008) 1408-1413.
18 Z. Kong, L. Gao, L. Wang, and S. Li, The normal parameter reduction of soft sets and its algorithm, Comput Math Appl. 56(2008) 3029-3037.