South Asian Journal of Mathematics 2013, Vol. 3 (1): 67 ~ 70
www.sajm-online.com ISSN 2251-1512 RESEARCH ARTICLE

On the starlikeness for the class of multivalent non-Bazilevič functions

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Received: 11-5-2012; Accepted: 1-12-2013 *Corresponding author

Abstract In this paper we consider starlikeness of the class of multivalent non-Bazilevič functions.

Key Words Analytic functions, Multivalent non-Bazilevič functions, Multivalent starlike functions

MSC 2010 30C45

1 Introduction

Let $H(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, (p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1.1}$$

which are analytic and multivalent in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}(p) \subset H(p)$ be the class of normalized ananlytic function f in $\mathcal{U}$ such that $f(0) = f'(0) - 1 = 0$.

A function $f \in \mathcal{A}(p)$ is said to be in the class $\mathcal{S}^*(p, \beta)$ of multivalent starlike functions of order β in $\mathcal{U}$ if it satisfies the following inequality:

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, 0 \leqslant \beta < p, p \in \mathbb{N}, z \in \mathcal{U}. \tag{1.2}$$

In this paper we consider starlikeness of the class of functions $f \in \mathcal{A}(p)$ defined by the condition

$$\left|\frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\mu} - 1\right| < \lambda,\ 0 < \mu < \frac{1}{p}, 0 < \lambda < p, z \in \mathcal{U}. \tag{1.3}$$

We denote the class of all such functions by $\mathcal{N}(p, \mu, \lambda)$. In particular, $\mathcal{N}(1, \mu, \lambda)$ is the class of non-Bazilevič functions, $\mathcal{N}(1, \mu, \lambda)$ (see [1]). In recent years, Obradović and Owa (see [2]), Tuneski and Darus (see [3]), Wang et al. (see [4]) and Shanmugam et al. (see [5]) obtained many interesting results associated with non-Bazilevič functions.

Citation: L. Guo, Y. Ling, G. Bao, On the starlikeness for the class of multivalent non-Bazilevič functions, South Asian J Math, 2013, 3(1), 67-70.

To prove our main result, we need the following lemmas.

Lemma 1.1 (see [6]). *Let the function $w(z)$ be (non-constant) analytic in $\mathcal{U}$ with $w(0) = 0$. If $|w(z)|$ attsts its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, then*

$$z_0 w'(z_0) = kw(z_0), \tag{1.4}$$

where $k \geqslant 1$ is a real number.

Lemma 1.2 (see [7]). *Let $0 < \lambda_1 < \lambda < 1$ and let F be analytic in $\mathcal{U}$ satisfying*

$$F(z) \prec 1 + \lambda_1 z, \; F(0) = 1. \tag{1.5}$$

(1) If f is analytic in $\mathcal{U}$, $f(0) = 1$ and satisfies

$$F(z)[\alpha + (1-\alpha)f(z)] \prec 1 + \lambda z, \tag{1.6}$$

where

$$\alpha = \begin{cases} \frac{1-\lambda}{1+\lambda_1}, & if \; 0 < \lambda + \lambda_1 \leqslant 1, \\ \frac{1-(\lambda^2+\lambda_1^2)}{2(1-\lambda_1^2)}, & if \; \lambda^2 + \lambda_1^2 < 1 \leqslant \lambda + \lambda_1, \end{cases} \tag{1.7}$$

then $\mathbf{Re}\{f(z)\} > 0, z \in \mathcal{U}$.

(2) If f is analytic in $\mathcal{U}$, $f(0) = 1$ and satisfies

$$F(z)[1 + f(z)] \prec 1 + \lambda z, \tag{1.8}$$

then

$$|f(z)| \leqslant \frac{\lambda + \lambda_1}{1 - \lambda_1} \leqslant 1, \; \lambda + 2\lambda_1 \leqslant 1. \tag{1.9}$$

The bound (1.9) are the best possible.

2 Main Results and Their Consequences

Lemma 2.1. *Let $f(z)$ is analytic in $\mathcal{U}$ with $f(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$, $n \geqslant 1$, satisfy the condition*

$$f(z) - \frac{1}{p\mu} z f'(z) \prec 1 + \lambda z, \; 0 < \mu < \frac{1}{p}, \; 0 \leqslant \lambda < 1. \tag{2.1}$$

Then

$$f(z) \prec 1 + \lambda_1 z, \tag{2.2}$$

where

$$\lambda_1 = \frac{p\mu}{1 - p\mu} \lambda. \tag{2.3}$$

Proof. Let

$$f(z) = 1 + \lambda_1 w(z). \tag{2.4}$$

where λ_1 is given by (2.3). We want to show that $|w(z)| < 1, z \in \mathcal{U}$. If not, by Lemma 1.1 there exists a $z_0, |z_0| < 1$ such that $|w(z_0)| = 1, z_0 w'(z_0) = kw(z_0), k \geqslant 1$. If we put $w(z_0) = e^{i\theta}$, then we get

$$\begin{aligned} \left|f(z_0) - \frac{1}{p\mu} z_0 f'(z_0) - 1\right| &= \left|\lambda_1 w(z_0) - \frac{\lambda_1}{p\mu} z_0 w'(z_0)\right| \\ &= \left|\lambda_1 e^{i\theta} - \frac{\lambda_1}{p\mu} k e^{i\theta}\right| = \lambda_1 \left|1 - \frac{k}{p\mu}\right| \\ &\geqslant \lambda_1\left(\frac{1}{p\mu} - 1\right) = \lambda \end{aligned} \tag{2.5}$$

which is a contradiction to (2.1). Now, it means that $|w(z)| < 1, z \in \mathcal{U}$, and by (2.4) we have (2.2). □

Theorem 2.2. *If $f \in \mathcal{A}(p)$ satisfies the condition (1.3) with $0 < \mu < \frac{1}{p}$ and $0 < \lambda \leqslant \frac{1-p\mu}{\sqrt{(1-p\mu)^2 + (p\mu)^2}}$, then $f \in \mathcal{S}^*(p, 0)$.*

Proof. We use a technique in [5]. Since $f \in \mathcal{A}(p)$ satisfies (1.3), we can write

$$\frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\mu} \prec 1 + \lambda z. \tag{2.6}$$

We define the function F by

$$F(z) = \left(\frac{z^p}{f(z)}\right)^{\mu}, \tag{2.7}$$

then by some transformations and (1.3) we get

$$F(z) - \frac{1}{p\mu} zF'(z) = \frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\mu} \prec 1 + \lambda z. \tag{2.8}$$

From there by Lemma 2.1 we obtain

$$F(z) \prec 1 + \lambda_1 z, \lambda_1 = \frac{p\mu}{1 - p\mu}\lambda. \tag{2.9}$$

From the condition (1.3) and (2.9) we have

$$\left|\arg \frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\mu}\right| < \operatorname{argtg}\frac{\lambda}{\sqrt{1-\lambda^2}} \tag{2.10}$$

and

$$\left|\arg\left(\frac{f(z)}{z^p}\right)^{\mu}\right| = \left|\arg\left(\frac{z^p}{f(z)}\right)^{\mu}\right| < \operatorname{argtg}\frac{\lambda_1}{\sqrt{1-\lambda_1^2}}, \tag{2.11}$$

which give

$$\begin{aligned} \left|\arg \frac{zf'(z)}{pf(z)}\right| &\leqslant \left|\arg \frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\mu}\right| + \left|\arg\left(\frac{f(z)}{z^p}\right)^{\mu}\right| \\ &\leqslant \operatorname{argtg}\frac{\lambda}{\sqrt{1-\lambda^2}} + \operatorname{argtg}\frac{\lambda_1}{\sqrt{1-\lambda_1^2}} \\ &= \operatorname{argtg}\frac{\frac{\lambda}{\sqrt{1-\lambda^2}} + \frac{\lambda_1}{\sqrt{1-\lambda_1^2}}}{1 - \frac{\lambda\lambda_1}{\sqrt{1-\lambda^2}\sqrt{1-\lambda_1^2}}} \leqslant \frac{\pi}{2} \end{aligned} \tag{2.12}$$

since $1 - \frac{\lambda\lambda_1}{\sqrt{1-\lambda^2}\sqrt{1-\lambda_1^2}} \geqslant 0$ is true by hypothesis. It means that $f \in \mathcal{S}^*(p, 0)$. □

Especially for $\mu = \frac{1}{2p}$ we have

Corollary 2.3. *If $f \in \mathcal{A}(p)$ satisfies the condition*

$$\left|\frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\frac{1}{2p}} - 1\right| < \frac{\sqrt{2}}{2}, z \in \mathcal{U}. \tag{2.13}$$

then $f \in \mathcal{S}^*(p, 0)$.

By using Lemma 1.2 for $0 < \mu < \frac{1}{2p}$ we can get a better result as the following theorem shows.

Theorem 2.4. *If $f \in \mathcal{A}(p)$ satisfies the condition (1.3) with $0 < \mu < \frac{1}{2p}$. If λ_1 is given by (2.3), then*

(1) $f \in \mathcal{S}^*(p, \alpha)$, where

$$\alpha = \begin{cases} \frac{1-\lambda}{1+\lambda_1}, & if\ 0 < \lambda \leqslant 1 - p\mu, \\ \frac{1-(\lambda^2+\lambda_1^2)}{2(1-\lambda_1^2)}, & if\ 1 - p\mu < \lambda \leqslant \frac{1-p\mu}{\sqrt{(1-p\mu)^2+(p\mu)^2}}. \end{cases} \tag{2.14}$$

(2) $|\frac{zf'(z)}{f(z)} - 1| < \frac{\lambda}{1-p\mu-\lambda p\mu} \leqslant 1, z \in \mathcal{U}$, where

$$0 < \lambda \leqslant \frac{1 - p\mu}{1 + p\mu}. \tag{2.15}$$

Proof. Let $F(z) = \left(\frac{z^p}{f(z)}\right)^{\mu}, g(z) = \frac{zf'(z)}{pf(z)}, h(z) = \frac{zf'(z)}{pf(z)} - 1$. Then by (2.9) we have $F(z) \prec 1 + \lambda_1 z, \lambda_1 = \frac{p\mu}{1-p\mu}\lambda < \lambda < 1$, since $0 < \mu < \frac{1}{2p}$. Also, since the condition (1.3) is equivalent to

$$F(z)[\alpha + (1-\alpha)\frac{f(z) - \alpha}{1 - \alpha}] \prec 1 + \lambda z, \tag{2.16}$$

where α is given by (1.7) and as

$$F(z)[1 + h(z)] \prec 1 + \lambda z, \tag{2.17}$$

then the statements of the theorem directly follows from Lemma 1.2. □

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