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Bitopological spaces associated with digraphs

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Abstract In this paper we deal with bitopological spaces associated with digraphs.

Key Words digraph, quasi-pseudometric, bitopological spaces

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1 Introduction

The notion of bitopological space (X, τ_1, τ_2), that is, a set X equipped with two arbitrary topologies τ_1, τ_2 was first formulated by J.C Kelley. Kelley investigated non symmetric distance function, so called quasi pseudometric on $X \times X$ that generate two topologies on X that in general are independent of each other. In this paper we discuss bitopological spaces associated with digraphs. For concepts in bitopology, the references are [3] and [5].

2 Preliminaries

Let X be a nonempty set. Let p be a quasi pseudometric on X. Associated with p there is another quasi pseudometric q such that $q(x, y) = p(y, x)$. We say that p and q are conjugates. The collection of all open p - balls forms a basis for a topology on X. Let this topology be denoted by τ_1. Similarly the collection of all open q - balls forms a basis for another topology on X, denoted by τ_2. Thus we get the bitopological space (X, τ_1, τ_2). A set X together with two (arbitrary) topologies τ_1 and τ_2 is called a *bitopological space* and is denoted by (X, τ_1, τ_2). Throughout this paper X will denote a bitopological space with topologies τ_1 and τ_2.

A subset A of a bitopological space (X, τ_1, τ_2) is (i, j)-*dense subset* in X if $\tau_i cl(\tau_j cl A) = X$, where $i, j = 1, 2$. and (i, j)-*nowhere dense subset*(also called (i,j)- rare) if $\tau_j cl A$ contains no non empty i-open set. ie, if $\tau_i int(\tau_j cl A) = \phi$ where $(i, j = 1, 2)$.

Let (X, τ_1, τ_2) be a bitopological space. Then (X, τ_1, τ_2) is

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1. *zero dimensional bitopological space* if (X, τ_1) has a basis whose elements are τ_1 open and τ_2 closed and (X, τ_2) has a basis whose elements are τ_2 open and τ_1 closed.
2. $w-p-T_0$ (*weakly pairwise* T_0*)-space* if for every pair of distinct points there exists a τ_1 neighborhood or τ_2 neighborhood of one point not containing the other.
3. $w-p-T_1$ (*weakly pairwise* T_1*)-space* if for every pair of distinct points, at least one point has a τ_1 neighborhood not containing the other while the second point has a τ_2 neighborhood not containing the first.
4. $w-p-T_2$ (*weakly pairwise* T_2*-space*) if for any pair of distinct points x, y there exists τ_1 open set U and τ_2 open set V with $U \cap V = \phi$ such that $x \in U$ and $y \in V$ or $x \in V$ and $y \in U$.
5. $p-T_2$ (*pairwise Hausdorff*) if for each pair of distinct points $x, y \in X$ there exists a τ_1 open set U and a τ_2 open set V such that $x \in U, y \in V$ and $U \cap V = \phi$.
6. p- normal (*pairwise normal space*) if for every pair of disjoint sets A and B in X where A is τ_1 closed and B is τ_2 closed, there exist a τ_2 open set U and a τ_1 open set V such that $A \subseteq U$ and $B \subseteq V$ and $U \cap V = \phi$.
7. *pairwise perfectly normal space* if it is pairwise normal and every τ_1 closed set is a $\tau_1 - G_\delta$ and every τ_2 closed set is a $\tau_2 - G_\delta$.
8. p-connected(*pairwise connected space*) if X cannot be expressed as a union of two disjoint sets A and B such that $A \in \tau_1 \setminus \{\phi\}$ and $B \in \tau_2 \setminus \{\phi\}$.
9. *weakly totally disconnected space* if for each pair of distinct points there exists a disconnection $X = A \mid B$ ($ie, X = A \cup B$, where $A \in \tau_1 \setminus \{\phi\}, \quad B \in \tau_2 \setminus \{\phi\}$ and $A \cap B = \phi$) such that one point belongs to A, the other to B, and the role of the points need not be interchangeable.
10. p-regular (*pairwise regular space*) if τ_1 is regular with respect to τ_2 and τ_2 is regular with respect to τ_1. The topology τ_1 of X is said to be regular with respect to τ_2 if for each $x \in X$ and each τ_1 closed set F such that $x \notin F$, there exist τ_1 open set U and a τ_2 open set V such that $x \in U, F \subseteq V$ and $U \cap V = \phi$.
11. *pairwise completely regular space* if τ_1 is completely regular with respect to τ_2 and τ_2 is completely regular with respect to τ_1. τ_1 is *completely regular* with respect to τ_2 if for each τ_1 closed set C and each point $x \notin C$, there is a real valued function $f : X \to [0, 1]$ such that $f(x) = 0, f(C) = \{1\}$ and f is τ_1 upper semicontinuous and τ_2 lower semicontinuous.

If (X, τ_1, τ_2), pairwise normal, then τ_1 is completely regular with respect to τ_2 if and only if τ_2 is completely regular with respect to τ_1.

A *directed graph* or *digraph* D consists of a finite set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pairs of distinct vertices called *arcs*. We call $V(D)$, the *vertex set* and $A(D)$, the *arc set* of D. We will often denote $D = (V, A)$ which means that V and A are the vertex set and the arc set of D respectively. A digraph is said to be *symmetric digraph* if $(u, v) \in A(D)$ implies $(v, u) \in A(D)$. A *directed walk* or *walk* $v_0x_1v_1x_2v_2 \dots x_nv_n$ in a digraph is an alternating sequence of vertices and arcs in which each arc x_i is $v_{i-1}v_i$ and is called a (v_0, v_n)- walk. A *closed walk* has the same first and last vertices. . A *path* is a walk in which all the vertices are distinct. A *trail* is a walk in which all arcs are distinct. A *cycle* is a nontrivial closed walk with all vertices except the first and last are distinct.. A closed trail is called a *circuit*. A *semiwalk* is an alternating sequence of vertices and arcs $v_0x_1v_1x_2v_2 \dots x_nv_n$ in which each arc

x_i may be $v_{i-1}v_i$ or v_iv_{i-1}. A semiwalk is termed as a *semipath* if all the vertices are distinct. A vertex y is reachable from x if and only if the digraph has an (x, y)-path and is denoted by $A(x, y)$. A digraph is said to be *strongly connected* or *strong* if every two points are mutually reachable . It is called *unilaterally connected* or *unilateral* if for any two points at least one is reachable from the other. A digraph is *weakly connected* or *weak* if every two points are joined by a semipath. A digraph is *disconnected* if it is not even weakly connected. A *point basis* of a digraph D is a minimal collection of points from which all points are reachable. A *quasi pseudometric* on a nonempty set X is a nonnegative real valued function p on $X \times X$ such that

$$p(x, x) = 0, \ \forall \ x \in X$$

and

$$p(x, z) \leqslant p(x, y) + p(y, z), \ \forall \ x, y, z \in X$$

3 Main Results

3.1 Bitopological Spaces associated with digraphs

Let V be the vertex set of a digraph. The function $p : V \times V \to R$ defined by

$$p(x, y) = \begin{cases} 0 & \text{if } x \text{ is reachable from } y \\ 1 & \text{otherwise} \end{cases}$$

is a quasi pseudometric on V and therefore p induces a unique topology on V with $\{B_p(x, \epsilon); x \in V, \epsilon > 0\}$ as a basis. Let us denote this topology by τ_1.

Let $q : V \times V \to R$ be the function defined by

$$q(x, y) = \begin{cases} 0 & \text{if } y \text{ is reachable from } x \\ 1 & \text{otherwise} \end{cases}$$

Then q is also a quasi pseudometric on V and $p(x, y) = q(y, x)$. So p and q are conjugate to each other. Also q induces a topology on V with $\{B_q(x, \epsilon); x \in V, \epsilon > 0\}$ as a basis where

$$B_q(x, \epsilon) = \{y \in V : q(x, y) < \epsilon\} = \{y \in V : A(x, y)\}$$

. Let us denote this topology by τ_2. The above two topologies τ_1 and τ_2 give the bitopological space (V, τ_1, τ_2).

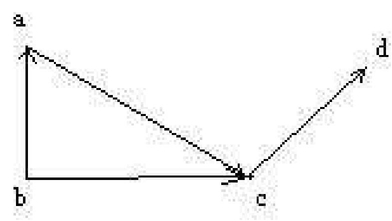

Figure 1

In figure 6.1, $\tau_1 = \{\{a,b\},\{a,b,c\},\{b\},\{a,b,c,d\}\}$ and
$\tau_2 = \{\{a,c,d\},\{a,b,c,d\},\{c,d\},\{d\}\}$.

Notation: Let $D = (V, A)$ be a digraph. For $x \in V$, $O_1(x)$ and $O_2(x)$ denote the minimum neighborhoods of x in τ_1 and τ_2 respectively and (V, τ_1, τ_2) is the bitopological space associated with D discussed above. Then $O_1(x) = B_p(x, \epsilon)$ and $O_2(x) = B_q(x, \epsilon)$, where $0 < \epsilon \leqslant 1$.

A subset $A \subseteq V$ belongs to τ_1 if and only if for every pair x, y with $y \in A$ and $x \notin A,\ (x, y) \notin A(D)$. Similarly $A \subseteq V$ belongs to τ_2 if and only if for every pair x, y with $x \in A, y \notin A, (x, y) \notin A(D)$. From this it is clear that $A \subseteq V$ belongs to τ_1 if and only if $V \setminus A$ belongs to τ_2. Hence $\tau_2 = co\tau_1$.

Remark 1. *If in-degree$(x) = 0$, then $O_1(x) = \{x\}$ and if out-degree$(x) = 0$, then $O_2(x) = \{x\}$.*

Remark 2. *If the digraph D is symmetric then $\tau_1 = \tau_2$. If it is connected and symmetric then both topologies are indiscrete topology. But if a symmetric digraph is not connected then both topologies are neither discrete nor indiscrete, whereas if D is an empty graph then both coincide with the discrete topology. For strongly connected digraphs both topologies coincide with the indiscrete topology. In particular for cycles both the topologies are indiscrete.*

Remark 3. *If the digraph D is connected but not symmetric then no nonempty proper subset belong to both τ_1 and τ_2.*

In this paper we are interested in connected digraphs which are not symmetric, for which the study of bitopology is important.

Theorem 1. *Let $D = (V, A)$ be a digraph. Then the topologies τ_1 and τ_2 are identical if and only if for every $a, b \in V$ such that $A(a, b)$ holds, $A(b, a)$ also hold.*

Proof. Let $\tau_1 = \tau_2$. Let $a, b \in V$ such that $A(a, b)$ holds, so that $a \in O_1(b)$ and $b \in O_2(a)$. But since $\tau_1 = \tau_2$, $O_1(a) = O_2(a)$. Hence $b \in O_1(a)$ which implies that $A(b, a)$ holds.

Conversely suppose the condition of the theorem holds. That is for every $a, b \in V$, with $A(a, b)$ holds, $A(b, a)$ holds. To prove that $\tau_1 = \tau_2$ it is enough to prove that $O_1(a) = O_2(a),\ \forall\ a \in V$. Suppose $O_1(a) \neq O_2(a)$, for some $a \in V$. Then there exists $b \in O_1(a)$ such that $b \notin O_2(a)$ or vice versa. Suppose that $b \in O_1(a)$ and $b \notin O_2(a)$. Since $b \in O_1(a),\ A(b, a)$ holds. Therefore by the assumption $A(a, b)$ holds, which implies that $b \in O_2(a)$, a contradiction. A similar contradiction arises in the other case also. $\square$

Theorem 2. *The bitopological space (V, τ_1, τ_2) associated with the digraph $D = (V, A)$ is zero dimensional.*

Proof. We know $\{O_1(x) : x \in V\}$ forms a basis for (V, τ_1). Since $O_1(x)$ belongs to τ_1, $V \setminus O_1(x)$ belongs to τ_2. Hence $O_1(x)$ is τ_2 closed. Hence (V, τ_1) has a basis whose elements are τ_1 open and τ_2 closed. Similarly $\{O_2(x) : x \in V\}$ is a basis for (V, τ_2) whose elements are τ_2 open and τ_1 closed. □

Note: Since every quasi pseudometrizable bitopological space is pairwise regular, pairwise normal, and pairwise perfectly normal[3] and since the bitopological space (V, τ_1, τ_2) associated with the digraph $D = (V, A)$ is quasi pseudometrizable, it is pairwise regular, pairwise normal, and pairwise perfectly normal. We can also prove that it is pairwise completely regular.

Theorem 3. *The bitopological space (V, τ_1, τ_2) associated with the digraph $D = (V, A)$ is pairwise completely regular.*

Proof. Suppose $a \in V$ and F is a closed set in V, with respect to τ_1 such that $a \notin F$. Then $V \setminus F = U$ is τ_1 open and $a \in U$. Then the characteristic function of F, $f = \chi_F$ is τ_1 upper semi continuous because, for any real number b,

$$\{x : f(x) < b\} = \begin{cases} \phi & ; \ b \in (-\infty, 0] \\ U & ; \ b \in (0, 1] \\ V & ; \ b \in (1, \infty) \end{cases}$$

Also f is τ_2 lower semi continuous, because,

$$\{x : f(x) > b\} = \begin{cases} V & ; \ b \in (-\infty, 0) \\ F & ; \ b \in [0, 1) \\ \phi & ; \ b \in [1, \infty) \end{cases}$$

Hence τ_1 is completely regular with respect to τ_2. Since (V, τ_1, τ_2) is pairwise normal, τ_2 is completely regular with respect to τ_1. Hence (V, τ_1, τ_2) is pairwise completely regular. □

3.2 Closure and Interior

Let D be a digraph and τ_1, the topology induced by the quasi pseudometric p on $V(D)$. If $A \subseteq V(D)$ we know that the closure, $\overline{A}$ of A with respect to τ_1 is

$$\overline{A} = \bigcap_{A \subseteq K} K,$$

where $V(D) \setminus K \in \tau_1$.

Since τ_1 is induced by the quasi pseudo metric p, and for $x \in V(D)$, $dist(x, A) = inf\{p(x, a) : a \in A\}$ we have,

$$\begin{aligned} \overline{A} &= \{x \in V(D) : dist(x, A) = 0\} \\ &= \{x \in V(D) : p(x, a) = 0 \text{ for at least one } \ a \in A\} \\ &= \text{the set of all points in } V(D) \text{ which are reachable from } A. \end{aligned}$$

Since points in A are reachable from A, $A \subseteq \overline{A}$ and we denote the closure of A with respect to τ_1 by $\tau_1(clA)$

We have the interior A° of A with respect to τ_1 is $A^\circ = \bigcup_{O \subseteq A} O$, where $O \in \tau_1$. Since τ_1 is induced by the quasi pseudometric p,

$$\begin{aligned} A^\circ &= \{x \in V(D) : dist(x, V(D) \setminus A) > 0\} \\ &= \{x \in V(D) : p(x,a) > 0 \text{ for all } \ a \in V(D) \setminus A\} \\ &= \text{the set of all points in } V(D) \text{ which are not reachable from } V(D) \setminus A. \end{aligned}$$

Since points in $V(D) \setminus A$ are reachable from $V(D) \setminus A$, we have $A^\circ \subseteq A$. We denote interior of A with respect to τ_1 by $\tau_1(intA)$.
Similarly, $\tau_2(clA)$ is the set of all points in $V(D)$ which are reachable to A and $\tau_2(intA)$ is the set of all points in $V(D)$ which are not reachable to $V(D) \setminus A$. Clearly $\tau_2(intA) \subseteq A \subseteq \tau_2(clA)$.

Proposition 1. *Let D be a digraph and τ_1, the topology induced by the quasi pseudometric p. Let $A \subseteq V(D)$. Then A is dense in $(V(D), \tau_1)$ if and only if given $a \in V(D) \setminus A$, there exists a path from some point of A to a.*

Proof. Suppose that A is dense in $(V(D), \tau_1)$. Then $\tau_1(clA) = V(D)$, so that every $a \in V(D) \setminus A$ belongs to $\tau_1(clA)$, which implies that there exists a path from some point of A to a.

Conversely suppose for every $a \in V(D) \setminus A$, $\exists$ a path from some point of A to a. So $V(D) \setminus A \subseteq \tau_1(clA)$. Also $A \subseteq \tau_1(clA)$. Hence $V(D) \subseteq \tau_1(clA)$, which implies that $V(D) = \tau_1(clA)$. □

Since $q(x,y) = p(y,x)$, analogues to the proposition 1 we have,

Corollary 1. *Let D be a digraph and τ_2, the topology induced by the quasi pseudometric q. Then $A \subseteq V(D)$ is dense in $(V(D), \tau_2)$ if and only if given $a \in V(D) \setminus A$, there exists a path from a to some point of A.*

For the graph in Figure 2(a), $\{a\}$ is a dense subset of the topological space $(V(D), \tau_1)$ but has no proper subset dense in $(V(D), \tau_2)$. The graph in Figure 2(b) has no proper subset of $V(D)$ which is dense in $(V(D), \tau_1)$ but the subset $\{a\}$ is dense in $(V(D), \tau_2)$.

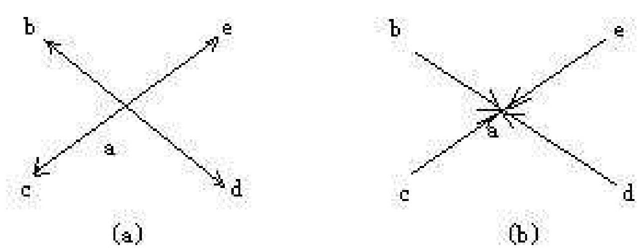

Figure 2

Remark 4. *If S is a point basis of the digraph D, $\tau_1 clS = V(D)$ and hence every point basis is dense in $(V(D), \tau_1)$.*

Definition 1. *A subset A of a topological space X is nowhere dense if $intcl(A) = \phi$.*

Suppose A is nowhere dense in $(V(D), \tau_1)$. Then $\tau_1 int(\tau_1 clA) = \phi$, that is, $\tau_1 clA$ contains no nonempty τ_1 open set. Therefore, $\forall a \in V(D)$,
$O_1(a) \not\subseteq \tau_1 clA$, which implies that $O_1(a) \cap (V(D) \setminus \tau_1 clA) \neq \phi$.

Conversely suppose $\forall a \in V(D)$, $O_1(a) \cap (V(D) \setminus \tau_1 clA) \neq \phi$. Therefore, $O_1(a) \not\subseteq \tau_1 clA$, for any $a \in V(D)$. So $\tau_1 clA$ contains no nonempty τ_1 open set, which implies that $\tau_1 int(\tau_1 clA) = \phi$. Thus we have, a set $A \subseteq V(D)$ is nowhere dense in $(V(D), \tau_1)$ if and only if for every $a \in V(D)$,
$O_1(a) \cap (V(D) \setminus \tau_1 clA) \neq \phi$.
Analoguesly we have, a set $A \subseteq V(D)$ is nowhere dense in $(V(D), \tau_2)$ if and only if $\forall a \in V(D)$, $O_2(a) \cap (V(D) \setminus \tau_2 clA) \neq \phi$.

Proposition 2. *Let $D = (V, A)$ be a connected digraph and $U \subseteq V$. Then U is nowhere dense in (V, τ_1) if every vertex in U has out-degree zero.*

Proof. Let us suppose that every vertex in U has out-degree zero. Then $\tau_1 clU = U$. Since D is connected, $\forall a \in U$, $\exists\, b \neq a$ such that $b \in O(a)$. Since out- degree of b is nonzero, $b \notin U$. In particular $O(a) \not\subseteq U$ for any $a \in U$. Hence $\tau_1 int(\tau_1 clU) = \tau_1 intU = \phi$. □

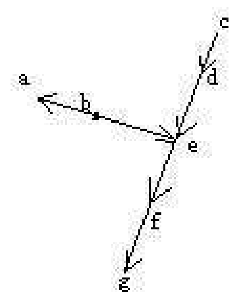

Figure 3

In Figure 3, $\{a, g\}$ is nowhere dense in $(V(D), \tau_1)$, by proposition 2.

But for every nowhere dense set in $V(D)$, all the vertices need not have out-degree zero. For example, the set $\{e, d\}$ is also a nowhere dense subset with respect to τ_1, even though out-degree of d is nonzero. Analogues to proposition 2 we have:

Proposition 3. *Let $D = (V, A)$ be a connected digraph and $U \subseteq V$. Then U is nowhere dense in $(V(D), \tau_2)$ if every vertex of U has in-degree zero.*

For example, in Figure 3, $\{b, c\}$ is nowhere dense in $(V(D), \tau_2)$, by proposition 3. Here also not all nowhere dense subsets of $V(D)$ with respect to τ_2 are of the type stated in proposition 3. For example the set $\{d, e\}$ is nowhere dense in $(V(D), \tau_2)$, but the in-degree of d is nonzero.

Proposition 4. *Let $D = (V, A)$ be a unilaterally connected digraph. Then any nonempty subset B of the bitopological space (V, τ_1, τ_2) is $(1, 2)$- dense and $(2, 1)$- dense in V.*

Proof. Let $y \in V \setminus \tau_2 cl(B)$. Since $\tau_2 cl(B)$ is τ_2 closed, it is τ_1 open. Therefore for every $x \in \tau_2 cl(B)$, $(y, x) \notin A(D)$, so that no point of $\tau_2 cl(B)$ is reachable from y. Hence $A(y, x)$ does not hold for any $x \in \tau_2 cl(B)$. But the digraph is unilateral. Hence $A(x, y)$ must hold for every $x \in \tau_2 cl(B)$ and $y \in \tau_1 cl(\tau_2 cl(B))$, which implies that $V \setminus \tau_2 cl(B) \subseteq \tau_1 cl(\tau_2 cl(B))$. Hence $\tau_1 cl(\tau_2 cl(B)) =$ V. Using similar arguments we can prove that B is $(2, 1)$- dense in V. □

Proposition 5. *Let $D = (V, A)$ be any digraph. A subset A of $V(D)$ is $(1, 2)$- nowhere dense in $V(D)$ if and only if $A = \phi$.*

Proof. Since $\tau_2 clA$ is τ_2 closed, it is τ_1 open. Therefore,
$\tau_1 int(\tau_2 clA) = \tau_2 clA = \phi$,
which is possible if and only if $A = \phi$. □

Analogues to proposition 5 we have the following proposition.

Proposition 6. *Let $D = (V, A)$ be any digraph. A subset A of $V(D)$ is $(2, 1)$- nowhere dense in $V(D)$ if and only if $A = \phi$.*

3.3 Separation Axioms in Bitopological Spaces

In this section we give the conditions under which the bitopological spaces associated with digraphs satisfy different bitopological separation axioms. In [5] it is proved that the bitopological space (V, τ_1, τ_2) associated with a digraph is $w - p - T_2$ if and only if the digraph contains no circuit. We give an alternate proof for this result.

Theorem 4. *The bitopological space (V, τ_1, τ_2) associated with the digraph $D = (V, A)$ is $w - p - T_2$ if and only if the digraph contains no circuit.*

Proof. Let (V, τ_1, τ_2) is $w - p - T_2$. So for $x \neq y$ there exists τ_1 open set U and τ_2 open set W such that $U \cap W = \phi$ and either $x \in U$ and $y \in W$ or $x \in W$ and $y \in U$. If possible let the graph contain a circuit and let x and y be two distinct vertices on the circuit. Then every τ_1 neighborhood of x and τ_2 neighborhood of x contain y, which is a contradiction.

Conversely let the graph contain no circuit. Let x and y be two distinct points in V. It is enough to prove that $O_1(x) \cap O_2(y) = \phi$ or $O_1(y) \cap O_2(x) = \phi$.
Suppose $O_1(x) \cap O_2(y) \neq \phi$ and $O_1(y) \cap O_2(x) \neq \phi$.
Let $z \in O_1(x) \cap O_2(y)$ and $k \in O_1(y) \cap O_2(x)$.

The point $z \in O_1(x) \cap O_2(y)$

$$\Leftrightarrow z \in O_1(x) \text{ and } z \in O_2(y)$$
$$\Leftrightarrow \exists \text{ a path from } z \text{ to } x \text{ and a path from } y \text{ to } z$$
$$\Leftrightarrow \exists \text{ a path from } y \text{ to } x$$

The point $k \in O_1(y) \cap O_2(x)$

$$\Leftrightarrow k \in O_1(y) \text{ and } k \in O_2(x)$$

$\Leftrightarrow \exists$ a path from k to y and a path from x to k

$\Leftrightarrow \exists$ a path from x to y

Hence the graph contains a circuit, a contradiction. □

Theorem 5. *Let (V, τ_1, τ_2) be the bitopological space associated with the digraph $D = (V, A)$. Then the following statements are equivalent.*

1. (V, τ_1, τ_2) *is* $w - p - T_2$.
2. (V, τ_1, τ_2) *is* $w - p - T_1$.
3. (V, τ_1, τ_2) *is* $w - p - T_0$.

Proof. The implications of (1) to (2) and (2) to (3) are obvious. Now suppose that (V, τ_1, τ_2) is $w-p-T_0$. Then the digraph contains no circuit. For if the digraph contains a circuit C, then every τ_1 neighborhood and τ_2 neighborhood of each point of C contains all other points of C, a contradiction. Hence (V, τ_1, τ_2) is $w - p - T_2$, by theorem 4. □

Theorem 6. *The bitopological space (V, τ_1, τ_2) associated with the digraph $D = (V, A)$ is $p - T_2$ if and only if the graph is an empty graph with vertex set V.*

Proof. Let (V, τ_1, τ_2) be $p - T_2$. Suppose $(x, y) \in A$ where $x \neq y$. Then every τ_1 neighborhood of y will contain x, a contradiction.
Converse part of the theorem holds trivially. □

3.4 Relation between some bitopological, properties and graph theoretical, properties

In this section we analyze the relation between some bitopological properties and graph theoretical properties.

Theorem 7. *The bitopological space (V, τ_1, τ_2) associated with the digraph $D = (V, A)$ is pairwise connected if and only if the digraph is strongly connected.*

Proof. Suppose that (V, τ_1, τ_2) is pairwise connected. Therefore $V \neq A \cup B$ whenever $A \cap B = \phi$ and either $A \in \tau_1 \setminus \{\phi\}$ and $B \in \tau_2 \setminus \{\phi\}$ or $A \in \tau_2 \setminus \{\phi\}$ and $B \in \tau_1 \setminus \{\phi\}$. Suppose the digraph is not strongly connected. Then there exists $x, y \in V$ such that at least one of $A(x, y)$ and $A(y, x)$ does not hold.

If $A(x, y)$ does not hold, there exists no path from x to y. Therefore $x \notin O_1(y)$ and $y \notin O_2(x)$. Since $x \notin O_1(y), O_1(y) \neq V$ and $V \setminus O_1(y) \neq \phi$. Also $V \setminus O_1(y) \in \tau_2$. Taking $A = O_1(y) \in \tau_1 \setminus \{\phi\}$ and $B = V \setminus O_1(y) \in \tau_2 \setminus \{\phi\}$, we get $V = A \cup B$ and $A \cap B = \phi$, a contradiction. A similar contradiction arises when $A(y, x)$ does not hold.

Conversely suppose the digraph D is strongly connected. In this case the topologies τ_1 and τ_2 coincide with the indiscrete topology of V and hence (V, τ_1, τ_2) is trivially pairwise connected. □

Corollary 2. *If (V, τ_1, τ_2) is pairwise connected then D is unilaterally connected and hence weakly connected.*

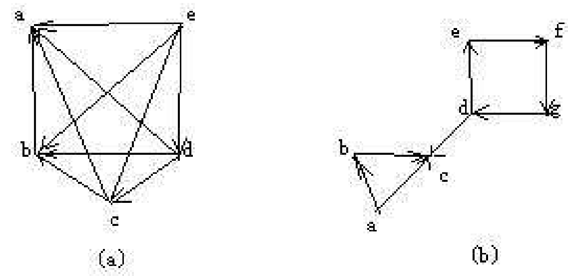

Figure 4

Remark 5. *The converse of corollary 2 need not be true. For example Figure 4(a) is unilaterally connected. But its associated bitopological space* (V, τ_1, τ_2) *where* $V = \{a, b, c, d, e\}$ *is not pairwise connected, since, for* $A = \{b, c, d, e\}$ *and* $B = \{a\}$, $V = A \cup B$, $A \in \tau_1 \setminus \{\phi\}$ *and* $B \in \tau_2 \setminus \{\phi\}$.
Now Figure 4 (b) is weakly connected. But the associated bitopological space (V, τ_1, τ_2) *where* $V = \{a, b, c, d, e, f, g\}$ *is not pairwise connected, since* $A = \{a\} \in \tau_1 \setminus \{\phi\}, B = \{b, c, d, e, f, g\} \in \tau_2 \setminus \{\phi\}$ *are such that* $V = A \cup B$ *and* $A \cap B = \phi$

Definition 2. *A digraph D is* strictly unilaterally connected *connected if* $\forall x, y \in V(D)$, *exactly one of* $A(x, y)$ *and* $A(y, x)$ *holds.*

Theorem 8. *If the digraph* $D = (V, A)$ *is strictly unilaterally connected then the bitopological space* (V, τ_1, τ_2) *associated with the digraph is weakly totally disconnected.*

Proof. Let the digraph be strictly unilaterally connected. Then for every $x, y \in V$, exactly one of $A(x, y)$ and $A(y, x)$ holds.
If $A(x, y)$ holds, then $x \in O_1(y)$ but $x \notin O_2(y)$.
Take $A = O_2(y) \in \tau_2$ and $B = V \setminus O_2(y) \in \tau_1$. Since $y \in O_2(y)$ and $x \notin O_2(y)$, $O_2(y) \neq \phi$ and $X \setminus O_2(y) \neq \phi$.
Therefore $V = A \cup B$, $A \in \tau_2 \setminus \{\phi\}$, $B \in \tau_1 \setminus \{\phi\}$, $A \cap B = \phi$, $x \in B$ and $y \in A$. Hence(V, τ_1, τ_2) is weakly totally disconnected.
If $A(y, x)$ holds, a similar proof can be given. □

Definition 3. *A bitopological space* (X, τ_1, τ_2) *is strictly totally disconnected if given* $x, y \in X, x \neq y$, *exactly one of the following holds.*

1. *There exist a pair* A, B *of disjoint subsets of* X *such that* $X = A \cup B$, $A \in \tau_1 \setminus \{\phi\}$, $B \in \tau_2 \setminus \{\phi\}$, $x \in A$ *and* $y \in B$.

2. *There exist a pair* C, D *of disjoint subsets of* X *such that* $X = C \cup D$, $C \in \tau_2 \setminus \{\phi\}$, $D \in \tau_1 \setminus \{\phi\}$, $x \in C$ *and* $y \in D$.

Theorem 9. *The bitopological space* (V, τ_1, τ_2) *associated with the digraph* $D = (V, A)$ *is strictly totally disconnected if and only if the digraph is strictly unilaterally connected.*

Proof. Let (V, τ_1, τ_2) be strictly totally disconnected. Suppose the digraph is not strictly unilaterally connected. Then there exists $x, y \in V$ such that, one of the following happens.

1. Both $A(x, y)$ and $A(y, x)$ hold.

2. Neither $A(x, y)$ nor $A(y, x)$ hold.

Case1 In this case we suppose that $A(x, y)$ and $A(y, x)$ hold for some x and y in X. So every τ_1 neighborhood of x contains y and every τ_2 neighborhood of y contains x. Also every τ_1 neighborhood of y contains x and every τ_2 neighborhood of x contains y. Therefore for this pair x, y there does not exist a separation, which is a contradiction.

Case2 Neither $A(x, y)$ nor $A(y, x)$ holds. In this case there exists no path from x to y and from y to x. Thus $x \notin O_1(y)$ and $y \notin O_2(x)$.

Similarly $y \notin O_1(x)$ and $x \notin O_2(y)$.

Let $A = O_1(y) \in \tau_1 \setminus \{\phi\}$ and $B = X \setminus O_1(y) \in \tau_2 \setminus \{\phi\}$.

Then $X = A \cup B,\ A \cap B = \phi,\ y \in A,\ x \in B$.

Take $C = O_2(y)$ and $D = X \setminus O_2(y)$. Then $X = C \cup D,\ C \cap D = \phi,\ y \in C \in \tau_2 \setminus \{\phi\}$ and $x \in D \in \tau_1 \setminus \{\phi\}$, a contradiction.

Conversely let the digraph be strictly unilaterally connected. Then for every x, y in V, exactly one of $A(x, y)$ and $A(y, x)$ holds. Without loss of generality assume that $A(x, y)$ holds. Then every τ_1 neighborhood of y contains x, but $x \notin O_2(y)$. Take $A = O_2(y)$ and $B = X \setminus O_2(y)$.

Then $V = A \cup B$ where $A \in \tau_2 \setminus \{\phi\},\ B \in \tau_1 \setminus \{\phi\},\ A \cap B = \phi,\ y \in A$ and $x \in B$. As $A(x, y)$ holds, x belongs to every τ_1 neighborhood of y. Hence we cannot find a τ_1 neighborhood of y not containing x and a τ_2 neighborhood of x not containing y. Hence (V, τ_1, τ_2) is strictly totally disconnected. ☐

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