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Fixed point theorem for A-Contraction mappings of integral type

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Abstract In the present paper, we shall establish a fixed point theorem for A-Contraction type mappings in integral type by using contractive condition. Our result is motivated by Mantu Saha & Debashis Dey [18].

Key Words Fixed point, general Contractive condition, integral type

MSC 2010 47H10, 54H25

1 Introduction

An elementary account of the Banach contraction principle and some applications, including its role in solving nonlinear ordinary differential equations, is in [6]. The contraction mapping theorem is used to prove the inverse function theorem in [15]. A beautiful application of contraction mappings to the construction of fractals is in [16]. After the classical result by Banach, Kannan [9] gave a substantially new Contractive mapping to prove the fixed point theorem. Since then there have been many theorems emerged as generalizations under various contractive conditions. Such conditions involve linear and nonlinear expressions. The intrested reader who wants to know more about this matter is recommended to go deep into the survey articles by Rhoades ([12], [13], [14]) and Bianchini [4], and into the references there in.

Theorem 1.1. *Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ be a mapping such that for each $x, y \in X$,*

$$d(fx, fy) \leqslant cd(x, y), \tag{1.1}$$

then f has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

In 2002, A. Branciari[5] analysed the existence of fixed point for mapping T defined on a complete metric space (X, d) satisfying a general contractive condition of integral type in the following theorem:

Theorem 1.2. *Let (X, d) be a complete metric space, $c \in (0, 1)$ and $T : X \to X$ be a mapping such that for each $x, y \in X$,*

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$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leqslant c \int_0^{d(x,y)} \varphi(t)dt, \tag{1.2}$$

where $\varphi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable on each compact subset of R^+ non negative and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t)dt > 0$, then T has a unique fixed point $a \in X$, such that for each $x \in X, \lim_{n\to\infty} T^n = a$.

After the paper of Branciari, a lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades [11] extending the result of Branciari by replacing the condition (1.2) by the following

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leqslant c \int_0^{max\{d(x,y),d(x,Tx),d(y,Ty),\frac{[d(x,Ty)+d(y,Tx)]}{2}\}} \varphi(t)dt \tag{1.3}$$

(1.3) for each $x, y \in X$, with some $c \in [0, 1)$.

Definition 1.3. *A-Contraction Akram et al.[2] introduced a new class of contraction maps, called A-contraction, which is proper super class of Kannan's [9] , Bianchini's [4] and Reich's [10] type contractions. Akram et al.[2] defined A-contractions as follows:*

Let a non- empty set A consisting of all $\alpha : R_+^3 \to R_+$ functions satisfying

(A_1): α is continuous on the set R_+^3 of all triplets of non-negative reals. (with respect to the Euclidean metric on R^3).

(A_2): $a \leqslant kb$ for some $k \in [0, 1)$ whenever $a \leqslant \alpha(a, b, b)$ or $a \leqslant \alpha(b, a, b)$ or $a \leqslant \alpha(b, b, a)$ for all a, b.

Definition 1.4. *A self -map T on a metric space X is said to be A-contraction, if it satisfies the condition:*

$$d(Tx, Ty) \leqslant \alpha(d(x, y), d(x, Tx), d(y, Ty))$$

for each $x, y \in X$, with some $\alpha \in A$.

Example 1.5. Let a self -map T on a metric space (X, d) satisfying: $d(Tx, Ty) \leqslant \beta\ max\{d(Tx, x) + d(Ty, y), d(Ty, y) + d(x, y), d(Tx, x) + d(x, y)\}$ for each $x, y \in X$, with some $\beta \in [0, 1/2)$, is an A-contraction.

In a very recent paper, Dey et al.[7] proved some fixed point theorems for mixed type of contraction mappings of integral type in complete metric space. Motivated and inspired by these consequent works, we introduce the analogues of some fixed point results for A-contraction mappings in integral setting which in turn generalize several known results. Also we have analysed the existence of fixed point of mapping over two related metrics due to Theorem of [1] in integral setting. Our results substantially extend, improve, and generalize comparable results in the literature.

2 Main Results

Theorem 2.1. *Let T be a self- mapping of a complete metric space (X, d) satisfying the following condition:*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leqslant \alpha(\int_0^{d(x,Tx)} \varphi(t)dt, \int_0^{d(y,Ty)} \varphi(t)dt)$$

$$+\beta(\int_0^{d(x,y)+d(y,Tx)}\varphi(t)dt, \int_0^{d(x,Tx)+d(y,Tx)}\varphi(t)dt, \int_0^{max\{d(x,Ty),d(y,Tx)\}}\varphi(t)dt) \tag{2.1}$$

for each $x, y \in X$, with some $\alpha \in A$, where Where $\varphi : R^+ \to R^+$ is a Lebesgue - integrable mapping which is summable on each compact subset of R^+ non negative and such that

$$for\ each\ \varepsilon > 0, \int_0^{\varepsilon}\varphi(t)dt > 0, \tag{2.2}$$

Then T has a unique fixed point $z \in X$, such that for each $x \in X, \lim_{n\to\infty} T^n x = z$.

Proof. Let $x_0 \in X$ be arbitrary and, for brevity, define $x_{n+1} = Tx_n$. For each integer $n \geqslant 1$, from (2.1), we have get

$$\int_0^{d(x_n,x_{n+1})}\varphi(t)dt = \int_0^{d(Tx_{n-1},Tx_n)}\varphi(t)dt \leqslant \alpha(\int_0^{d(x_{n-1},x_n)}\varphi(t)dt, \int_0^{d(x_{n-1},Tx_{n-1})}\varphi(t)dt, \int_0^{d(x_n,Tx_n)}\varphi(t)dt)$$

$$+\beta(\int_0^{d(x_{n-1},x_n)+d(x_n,Tx_{n-1})}\varphi(t)dt, \int_0^{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_{n-1})}\varphi(t)dt), \int_0^{max\{d(x_{n-1},Tx_n),d(x_n,Tx_{n-1})\}}\varphi(t)dt$$

$$\leqslant \alpha(\int_0^{d(x_{n-1},x_n)}\varphi(t)dt, \int_0^{d(x_{n-1},x_n)}\varphi(t)dt, \int_0^{d(x_n,x_{n+1})}\varphi(t)dt)$$

$$+\beta(\int_0^{d(x_{n-1},x_n)+d(x_n,x_n)}\varphi(t)dt, \int_0^{d(x_{n-1},x_n)+d(x_n,x_n)}\varphi(t)dt), \int_0^{max\{d(x_{n-1},x_{n+1}),d(x_n,x_n)\}}\varphi(t)dt)$$

$$= \alpha(\int_0^{d(x_{n-1},x_n)}\varphi(t)dt, \int_0^{d(x_{n-1},Tx_{n-1})}\varphi(t)dt, \int_0^{d(x_n,Tx_n)}\varphi(t)dt)$$

$$+\beta(\int_0^{d(x_{n-1},x_n)+d(x_n,Tx_{n-1})}\varphi(t)dt, \int_0^{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_{n-1})}\varphi(t)dt)).$$

Then by axiom (A_2) of function α,

$$\int_0^{d(x_n,x_{n+1})}\varphi(t)dt \leqslant k\int_0^{d(x_{n-1},x_n)}\varphi(t)dt + L\int_0^{d(x_{n-1},x_n)}\varphi(t)dt. \tag{2.3}$$

For some $k, L \in [0, 1)$ with $k + L \in [0, 1)$ as $\alpha \in A$. In this fashion, one can obtain

$$\int_0^{d(x_n,x_{n+1})}\varphi(t)dt \leqslant k\int_0^{d(x_{n-1},x_n)}\varphi(t)dt + L\int_0^{d(x_{n-1},x_n)}\varphi(t)dt$$

$$= (k+L)\int_0^{d(x_n,x_{n+1}}\varphi(t)dt) \leqslant (k+L)^2\int_0^{d(x_{n-2},x_{n-1}}\varphi(t)dt \leqslant \cdots \leqslant (k+L)^n\int_0^{d(x_0,x_1}\varphi(t)dt.$$

Taking limit as $n \to \infty$, we get $\lim_{n\to\infty}\int_0^{d(x_n,x_{n+1})}\varphi(t)dt = 0$ as $k + L \in [0, 1)$ Which from (2.2) implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{2.4}$$

We now establish that $\{x_n\}$ is a Cauchy sequence. Suppose it is not so. Then, there exists an $\varepsilon > 0$ and subsequence's $\{x_{m(p)}\}$ and $\{x_{n(p)}\}$ such that $m(p) < n(p) < m(p + 1)$ with

$$d(x_{m(p)}, x_{n(p)}) \geqslant \varepsilon,\ d(x_{m(p)}, x_{n(p)-1}) < \varepsilon. \tag{2.5}$$

Now

$$d(x_{m(p)}, x_{n(p)-1}) \leqslant d(x_{m(p)-1}, x_{m(p)-1}) + d(x_{m(p)}, x_{n(p)-1}) < d(x_{m(p)-1}, x_{m(p)}) + \varepsilon \tag{2.6}$$

So by (2.4) and (2.6) we get

$$\lim_{p\to\infty} \int_0^{d(x_{m(p)-1}, x_{n(p)-1}} \varphi(t)dt \leqslant \int_0^{\varepsilon} \varphi(t)dt \tag{2.7}$$

Using (2.3), (2.5) and (2.7) we get

$$\int_0^{\varepsilon} \varphi(t)dt \leqslant \int_0^{d(x_{m(p)}, x_{n(p)}} \varphi(t)dt \leqslant (k+L) \int_0^{d(x_{m(p)-1}, x_{n(p)-1}} \varphi(t)dt \leqslant (k+L) \int_0^{\varepsilon} \varphi(t)dt,$$

which is a contradiction, since $k + L \in [0, 1)$. Since (X, d) be a complete metric space, $\{x_n\}$ converges to some $z \in X$, that is $\lim_{n\to\infty} x_n = z$. From (2.1) we get

$$\int_0^{d(Tz,x_{n+1})} \varphi(t)dt = \int_0^{d(Tz,x_n)} \varphi(t)dt \leqslant \alpha(\int_0^{d(z,x_n)} \varphi(t)dt), \int_0^{d(z,Tz)} \varphi(t)dt, \int_0^{d(x_n,x_{n+1})} \varphi(t)dt$$
$$+\beta(\int_0^{d(z,x_n)+d(x_n,Tz)} \varphi(t)dt, \int_0^{d(z,Tz)+d(x_n,Tz)} \varphi(t)dt, \int_0^{max\{d(z,x_{n+1}),d(x_n,Tz)\}} \varphi(t)dt)$$

Taking limit as $n \to \infty$, we get

$$\int_0^{d(Tz,x_{n+1})} \varphi(t)dt = \int_0^{d(Tz,x_n)} \varphi(t)dt \leqslant \alpha(\int_0^{d(z,x_n)} \varphi(t)dt, \int_0^{d(z,Tz)} \varphi(t)dt, \int_0^{d(x_n,x_{n+1})} \varphi(t)dt)$$
$$+\beta(\int_0^{d(z,x_n)+d(x_n,Tz)} \varphi(t)dt), \int_0^{d(z,Tz)+d(x_n,Tz)} \varphi(t)dt, \int_0^{\{d(z,x_{n+1}),d(x_n,Tz)\}} \varphi(t)dt))$$

Taking limit as $n \to \infty$, we get

$$\int_0^{d(Tz,z)} \varphi(t)dt \leqslant \alpha(0, \int_0^{d(Tz,z)} \varphi(t)dt, 0) + \beta(\int_0^{d(z,Tz)} \varphi(t)dt, \int_0^{2d(z,Tz)} \varphi(t)dt), \int_0^{d(Tz,z)} \varphi(t)dt$$

So by axiom A_2 of function α,

$$\int_0^{d(Tz,z)} \varphi(t)dt \leqslant k.0 + \int_0^{d(Tz,z)} \varphi(t)dt$$

Which from (2.2), implies that $d(Tz, z) = 0$ or $Tz = z$. Next suppose that $w(\neq z)$ be another fixed point of T. Then from (2.1) we have,

$$\int_0^{d(z,w)} \varphi(t)dt = \int_0^{d(Tz,Tw)} \varphi(t)dt \leqslant \alpha(\int_0^{d(z,w)} \varphi(t)dt, \int_0^{d(z,Tz)} \varphi(t)dt, \int_0^{d(w,Tw)} \varphi(t)dt)$$
$$+\beta(\int_0^{d(z,w)+d(w,Tz)} \varphi(t)dt, \int_0^{d(z,Tz)+d(w,Tz)} \varphi(t)dt, \int_0^{max\{d(z,Tw),d(w,Tz)\}} \varphi(t)dt)$$
$$= \alpha(\int_0^{d(z,w)} \varphi(t)dt, \int_0^{d(z,z)} \varphi(t)dt, \int_0^{d(w,w)} \varphi(t)dt)$$

$$+\beta(\int_0^{d(z,w)+d(w,z)} \varphi(t)dt, \int_0^{d(z,z)+d(w,z)} \varphi(t)dt, \int_0^{max\{d(z,w),d(w,z)\}} \varphi(t)dt)$$
$$= \alpha(\int_0^{d(z,w)} \varphi(t)dt, 0, 0) + \beta(\int_0^{2d(z,w)} \varphi(t)dt, \int_0^{d(w,z)} \varphi(t)dt).$$

So by axiom A_2 of function α,

$$\int_0^{d(z,w)} \varphi(t)dt \leqslant k.0 + L \int_0^{d(z,w)} \varphi(t)dt$$

which from (2.2), implies that $d(z,w) = 0$ or $z = w$ and so the fixed point is unique. $\square$

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