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Conformal Kropina change of a Finsler space with $(\alpha,\beta)-$metric of Douglas type

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Abstract A change of Finsler metric $L(\alpha,\beta) \to \overline{L}(\overline{\alpha},\overline{\beta}) = e^{\sigma(x)} \left\{ \frac{L^2(\alpha,\beta)}{\beta} \right\}$ is called conformal Kropina change where σ is a function of position x^i only, α is Riemannian metric and β is a differentiable one-form. M. Matsumoto has found several conditions under which a Finsler space with $(\alpha,\beta)-$metric is of Douglas type ([2], [8]). The purpose of the present paper is to find the condition that conformal Kropina change of Finsler space with $(\alpha,\beta)-$metric of Douglas type yields a space of Douglas type.

Key Words $(\alpha,\beta)-$metric, Douglas space, conformal change

MSC 2010 53B40, 53C60

1 Introduction

The theory of Finsler space with $(\alpha,\beta)-$metric has been developed into faithful branch of Finsler Geometry. For the first time M. Matsumoto introduced $(\alpha,\beta)-$metric in 1972 while studying C-reducible Finsler space [5] and in 1991 he studied about its Berwald connection [6]. The notion of Douglas space and the condition that the Finsler space with $(\alpha,\beta)-$metric be of Douglas type has been given by Matsumoto, M. and Bacso, S. ([2], [8]). Ichijyo, Y. and Hashiguchi, M. [3] have studied the conformal change of $(\alpha,\beta)-$metric.

The concept of Douglas space ([1], [2] and [8]) has been introduced by M. Matsumoto and S. Bacso as a generalization of Berwald space from the view-point of geodesic equations. A Finsler space is said to be Douglas space if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomial of degree three in y^i, It is remarkable that a Finsler space is Douglas space or is of Douglas type if and only if the Douglas tensor vanishes identically.

2 Preliminaries

Let $\alpha(x,y) = \sqrt{a_{ij}(x)\, y^i\, y^j}$ be Riemannian metric and $\beta(x,y) = b_i(x)\, y^i$ be a differentiable one-form in an n-dimensional differentiable manifold M^n. If the Finsler metric function $L(\alpha,\beta)$ is positively homogeneous of degree one in α and β in M^n, then $F^n = (M^n, L(\alpha,\beta))$ is called a Finsler space with $(\alpha,\beta)-$metric [6].

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The space $R^n = (M^n, \alpha)$ is called a Riemannian space associated with F^n [3] and Christoffel symbol of R^n are indicated by γ^i_{jk} and covariant differentiation with respect to $\gamma^i_{jk}(x)$ by ∇.

We shall use the symbols as follows:

$$r_{ij} = \frac{1}{2}(\nabla_j b_i + \nabla_i b_j), \quad s_{ij} = \frac{1}{2}(\nabla_j b_i - \nabla_i b_j), \quad s^i_j = a^{ir} s_{rj}, \quad s_j = b_r s^r_j. \tag{2.1}$$

It is to be noted that $s_{ij} = \frac{1}{2}(\partial_j b_i - \partial_i b_j)$. Throughout the paper the symbols ∂_i and $\dot{\partial}_i$ stand for $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^i}$ respectively. We are concerned with the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$, which given by

$$2G^i(x,y) = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F), \quad \text{where} \quad F = \frac{L^2}{2}, \quad G^i_j = \dot{\partial}_j G^i \quad \text{and} \quad G^i_{jk} = \dot{\partial}_k G^i_j.$$

The Finsler space F^n is said to be of Douglas type (or Douglas space) [8] if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomial of degree three in y^i. We shall denote the "homogeneous polynomial of degree r in y^i" by $hp(r)$.

For a Finsler space F^n with $(\alpha, \beta)-$metric ([4], [6]), we have

$$2G^i = \gamma^i_{00} + 2B^i \tag{2.2}$$

where

$$B^i = \frac{E}{\alpha} y^i + \frac{\alpha L_\beta}{L_\alpha} s^i_0 - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} C^* \left(\frac{y^i}{\alpha} - \frac{\alpha}{\beta} b^i \right), \quad E = \frac{\beta L_\beta}{L} C^*,$$

$$C^* = \frac{\alpha\beta(r_{00} L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}, \quad b^i = a^{ij} b_j, \quad \gamma^2 = b^2\alpha^2 - \beta^2, \tag{2.3}$$

$$b^2 = a^{ij} b_i b_j$$

and the subscripts α and β in L denote the partial differentiation with respect to α and β respectively. Since $\gamma^i_{00} = \gamma^i_{jk}(x) y^j y^k$ is homogeneous polynomial degree two in y^i, we have [8]:

Proposition 2.1. *A Finsler space with $(\alpha, \beta)-$metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are $hp(3)$. Equation (2.3) gives*

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i). \tag{2.4}$$

3 Conformal Kropina change of Finsler spaces with $(\alpha, \beta)-$metric of Douglas type

Let $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L})$ be two Finsler spaces on the same underlying manifold M^n. If we have a function $\sigma(x)$ in each co-ordinate neighbourhoods of M^n such that $\overline{L}(\overline{\alpha}, \overline{\beta}) = e^\sigma \left[\frac{L^2(\alpha,\beta)}{\beta} \right]$, then F^n is called conformal Kropina to $\overline{F}^n$ and the change $L \to \overline{L}$ of metric is called conformal Kropina change of $(\alpha, \beta)-$metric. A conformal change of $(\alpha, \beta)-$metric is expressed as $(\alpha, \beta) \to (\overline{\alpha}, \overline{\beta})$, where $\overline{\alpha} = e^{\sigma}\alpha$, $\overline{\beta} = e^{\sigma}\beta$. We have

$$\overline{y}^i = y^i, \quad \overline{y}_i = e^{2\sigma}\, y_i, \quad \overline{a}_{ij} = e^{2\sigma} a_{ij}, \quad \overline{b}_i = e^{\sigma} b_i, \quad \overline{a}^{ij} = e^{-2\sigma} a^{ij},$$
$$\overline{b}^i = \overline{e}^{\sigma} b^i \qquad \text{and} \quad \overline{b}^2 = b^2. \tag{3.1}$$

Therefore we have

Proposition 3.1. *In a Finsler space with $(\alpha, \beta)-$metric the length b of b_i with respect to the Riemannian α is invariant under any conformal change of metric.*

From (3.1) it follows that the conformal change of Christoffel symbols is given by

$$\overline{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma^i a_{jk}, \tag{3.2}$$

where $\sigma_j = \partial_j \sigma$ and $\sigma^i = a^{ij}\sigma_j$.

From (2.1), (3.1) and (3.2) we have the following conformal changes

(a) $\overline{\nabla}_j \overline{b}_i = e^{\sigma}(\nabla_j b_i + \rho a_{ij} - \sigma_i b_j)$, (b) $\overline{r}_{ij} = e^{\sigma}[r_{ij} - \frac{1}{2}(b_i\sigma_j + b_j\sigma_i) + \rho a_{ij}]$,

(c) $\overline{s}_{ij} = e^{\sigma}[s_{ij} + \frac{1}{2}(b_i\sigma_j - b_j\sigma_i)]$, (d) $\overline{s}^i_j = \overline{e}^{\sigma}[s^i_j + \frac{1}{2}(b^i\sigma_j - b_j\sigma^i)]$, (3.3)

(e) $\overline{s}_j = s_j + \frac{1}{2}(b^2\sigma_j - \rho b_j)$, where $\rho = \sigma_r b^r$.

From (3.2) and (3.3) we can easily obtain the following:

(a) $\overline{\gamma}^i_{00} = \gamma^i_{00} + 2\sigma_0 y^i - \alpha^2\sigma^i$, (b) $\overline{r}_{00} = e^{\sigma}(r_{00} + \rho\alpha^2 - \sigma_0\beta)$,

(c) $\overline{s}^i_0 = e^{-\sigma}[s^i_0 + \frac{1}{2}(b^i\sigma_0 - \beta\sigma^i)]$, (d) $\overline{s}_0 = s_0 + \frac{1}{2}(b^2\sigma_0 - \rho\beta)$. (3.4)

To find the conformal Kropina change of B^{ij} given in (2.4), we first find the conformal Kropina change of C^* given in (2.3).

Since $\overline{L}(\overline{\alpha}, \overline{\beta}) = e^{\sigma}\left[\frac{L^2(\alpha,\beta)}{\beta}\right]$, we have

$$\overline{L}_{\overline{\alpha}} = \frac{2L}{\beta} L_{\alpha}, \quad \overline{L}_{\overline{\alpha}\overline{\alpha}} = e^{-\alpha}\frac{2}{\beta}[LL_{\alpha\alpha} + (L_{\alpha})^2], \quad \overline{L}_{\overline{\beta}} = \frac{2\beta LL_{\beta} - L^2}{\beta^2}, \quad \overline{\gamma}^2 = e^{2\sigma}\gamma^2. \tag{3.5}$$

From (2.3), (3.4) and (3.5), we have

$$\overline{C}^* = e^{\sigma}(C^* + D^*), \tag{3.6}$$

where

$$D^* = \frac{\alpha L(\beta^2 L_{\alpha} + \alpha\gamma^2 L_{\alpha\alpha})[\beta(\rho\alpha^2 - \sigma_0\beta)L_{\alpha} - \alpha\beta(b^2\sigma_0 - \rho\beta)L_{\beta} + }{2(\beta^2 L_{\alpha} + \alpha\gamma^2 L_{\alpha\alpha})\{(\beta^2 L_{\alpha} + \alpha\gamma^2 L_{\alpha\alpha})L + \alpha\gamma^2 (L_{\alpha})^2\}}$$
$$\frac{\alpha L\{s_0 + \frac{1}{2}(b^2\sigma_0 - \rho\beta)\}] - \alpha^2\beta\gamma^2(L_{\alpha})^2(r_{00}L_{\alpha} - 2\alpha s_0 L_{\beta})}{} \tag{3.7}$$

Hence the conformal Kropina change of B^{ij} is written in the form

$$\overline{B}^{ij} = B^{ij} + C^{ij}, \tag{3.8}$$

where

$$C^{ij} = \frac{\alpha L(2\beta L_\beta - L)\{\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i)\} - 2\alpha L^2(s_0^i y^j - s_0^j y^i) +}{4\beta L L_\alpha}$$
$$\frac{4\alpha^2\{(L_\alpha)^2 C^* + [L L_{\alpha\alpha} + (L_\alpha)^2]\, D^*\}\,(b^i y^j - b^j y^i)}{} \tag{3.9}$$

Theorem 3.1. *A Douglas space with $(\alpha,\beta)-$metric is transformed to a Douglas space with $(\alpha,\beta)-$metric under conformal Kropina change if and only if C^{ij} defined in equation (3.9) is $hp(3)$.*

In the following three sections we deal with conformal Kropina change of Finsler spaces with three particular $(\alpha,\beta)-$metrics.

4 Riemannian Metric

For a Riemannian metric we have $L = \alpha,$ so that

$$L_\alpha = 1, \qquad L_\beta = 0 \qquad \text{and} \qquad L_{\alpha\alpha} = 0.$$

Hence the values of C^*, D^* and C^{ij} given by equations (2.3), (3.7) and (3.9) respectively reduce to

$$C^* = \frac{\alpha r_{00}}{2\beta}, \quad D^* = \frac{\beta^3(\rho\alpha^2 - \sigma_0\beta) + \alpha^2\beta^2 s_0 + \frac{1}{2}\alpha^2\beta^2(b^2\sigma_0 - \rho\beta) - b^2\alpha^2\beta r_{00} + \beta^3 r_{00}}{2b^2\alpha\beta^2}$$

$$C^{ij} = \frac{\alpha^2}{4}(\sigma^i y^j - \sigma^j y^i) - \frac{\alpha^2}{2\beta}(s_0^i y^j - s_0^j y^i) + \left(\frac{\rho\alpha^2}{4b^2} - \frac{\beta}{2b^2}\sigma_0 + \frac{\alpha^2 s_0}{2b^2\beta} + \frac{r_{00}}{4b^2}\right)(b^i y^j - b^j y^i). \tag{4.1}$$

Since $\frac{\alpha^2}{4}(\sigma^i y^j - \sigma^j y^i$ and $\left(\frac{\rho\alpha^2}{4b^2} - \frac{\beta\sigma_0}{2b^2} + \frac{r_{00}}{4b^2}\right)(b^i y^j - b^j y^i)$ are $hp(3)$, these terms of (4.1) may be neglected in our discussion and we treat only of

$$V^{ij}_{(3)} = \frac{\alpha^2 s_0}{2b^2\beta}(b^i y^j - b^j y^i) - \frac{\alpha^2}{2\beta}(s_0^i y^j - s_0^j y^i), \quad \text{where} \quad V^{ij}_{(3)} \text{ is } hp(3). \tag{4.2}$$

The equation (4.2) can be written as

$$2b^2\beta V^{ij}_{(3)} - \alpha^2 s_0(b^i y^j - b^j y^i + b^2\alpha^2(s_0^i y^j - s_0^j y^i) = 0. \tag{4.3}$$

Take $n > 2$, $\alpha^2 \not\equiv 0 \pmod{\beta}$ [8]. The terms of (4.3), which seemingly do not contain β are $b^2\alpha^2(s_0^i y^j - s_0^j y^i) - \alpha^2 s_0(b^i y^j - b^j y^i)$. Hence we must have $hp(1)$ $V^{ij}_{(1)}$ such that the above expression is equal to $\alpha^2\beta V^{ij}_{(1)}$. Thus

$$b^2(s_0^i y^j - s_0^j y^i) - s_0(b^i y^j - b^j y^i) = \beta V^{ij}_{(1)}. \tag{4.4}$$

By putting $V^{ij}_{(1)} = V^{ij}_k(x)\, y^k$, the equation (4.4) is written as

$$b^2[s_h^i\delta_k^j + s_k^i\delta_h^j - s_h^j\delta_k^i - s_k^j\delta_h^i] - [(s_h\delta_k^j + s_k\delta_h^j)b^i - (s_h\delta_k^i + s_k\delta_h^i)b^j] = b_h V^{ij}_k + b_k V^{ij}_h \tag{4.5}$$

Contracting (4.5) by $j = k$, we get

$$nb^2 s_h^i - nb^i s_h = b_h V^{ir}_r + b_r V^{ir}_h. \tag{4.6}$$

Next, transvecting (4.5) by $b_j b^h$, we have

$$b^2(b^2 s^i_k - s^i b_k - s_k b^i) = b^2 b_r V^{ir}_k + b_k b_r V^{ir}_s b^s. \tag{4.7}$$

Transvecting (4.7) by b^k, we get

$$-2b^4 s^i = 2b^2 b_r V^{ir}_s b^s \quad \text{which gives}$$

$$b_r V^{ir}_s b^s = -b^2 s^i, \quad \text{provided} \quad b^2 \neq 0. \tag{4.8}$$

Putting the value of $b_r V^{ir}_s b^s$ from (4.8) in (4.7), we get

$$b_r V^{ir}_k = b^2 s^i_k - s_k b^i. \tag{4.9}$$

Substituting the value of $b_r V^{ir}_h$ from (4.9) in (4.6), we get

$$b^2 s^i_h = \frac{1}{(n-1)} V^{ir}_r b_h + b^i s_h. \tag{4.10}$$

If we put $v^i = \frac{1}{n-1} V^{ir}_r$, then equation (4.10) gives $b^2 s^i_h = v^i b_h + b^i s_h$ which implies $b^2 s_{ij} = v_i b_j + b_i s_j$, where $v_i = a_{ij} v^j$. Since s_{ij} is skew-symmetric tensor, we have $v_i = -s_i$ easily. Thus

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i). \tag{4.11}$$

Hence, we have

Theorem 4.1. *A Finsler space $\overline{F}^n$ $(n > 2)$ which is obtained by conformal Kropina change of a Riemannian space F^n with $b^2 \neq 0$ is of Douglas type if and only if (4.11) is satisfied.*

5 Randers Metric

For a Randers metric we have $L = \alpha + \beta,$ so that

$$L_\alpha = 1, \qquad L_\beta = 1 \qquad \text{and} \qquad L_{\alpha\alpha} = 0.$$

We know that [8] Finsler space with Randers metric is Douglas space if and only if $s_{ij} = 0$. Under this condition the values of C^*, D^* and C^{ij} given by equation (2.3), (3.7) and (3.9) respectively reduce to

$$C^* = \frac{\alpha r_{00}}{2\beta}, \quad D^* = \frac{\alpha\beta(\alpha+\beta)\{(b^2\alpha^2 - b^2\alpha\beta - 2\beta^2)\,\sigma_0 + \rho\beta(\alpha^2+\alpha\beta)\} - 2\alpha^2(b^2\alpha^2 - \beta^2) r_{00}}{4\beta(b^2\alpha^3 + \beta^3)}$$

and

$$C^{ij} = \frac{\alpha(\beta-\alpha)\{\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i)\}(b^2\alpha^3+\beta^3) +}{4\beta(b^2\alpha^3+\beta^3)}$$

$$\frac{\alpha^3[2\beta r_{00} + (b^2\alpha^2 - b^2\alpha\beta - 2\beta^2)\sigma_0 + \rho\beta(\alpha^2+\alpha\beta)](b^i y^j - b^j y^i)}{}. \tag{5.1}$$

The equation (5.1) can be written as

$$4(b^2\alpha^3+\beta^3)C^{ij}+(2\alpha^3\beta-\alpha\beta^3+\alpha^2\beta^2)\sigma_0(b^iy^j-b^jy^i)-\{b^2\alpha^5+\alpha^2\beta^3-b^2\alpha^4\beta-\alpha\beta^4\}\times$$
$$(\sigma^iy^j-\sigma^jy^i)-(2\alpha^3r_{00}+\rho\alpha^5+\rho\beta\alpha^4)(b^iy^j-b^jy^i)=0. \tag{5.2}$$

Since α is an irrational function in y^i, the equation (5.2) gives rise to two equations as follows:

$$4\beta^2C^{ij}+\alpha^2\beta\sigma_0(b^iy^j-b^jy^i)+\alpha^2(b^2\alpha^2-\beta^2)(\sigma^iy^j-\sigma^jy^i)$$
$$-\alpha^4\rho(b^iy^j-b^jy^i)=0 \tag{5.3}$$

and

$$4b^2\alpha^2C^{ij}+\beta(2\alpha^2-\beta^2)\sigma_0(b^iy^j-b^jy^i)-(b^2\alpha^4-\beta^4)(\sigma^iy^j-\sigma^jy^i)$$
$$-\alpha^2(2r_{00}+\rho\alpha^2)(b^iy^j-b^jy^i)=0. \tag{5.4}$$

Take $n>2$, $\alpha^2\not\equiv 0\ (\mathrm{mod}\,\beta)$. The terms β of (5.4), which seemingly do not contain α^2 are $-\beta^3\sigma_0(b^iy^j-b^jy^i)+\beta^4(\sigma^iy^j-\sigma^jy^i)$. Hence we must have $hp(0)$, $M^{ij}(x)$ such that the above expression is equal to $\alpha^2\beta^3M^{ij}(x)$. Therefore we have

$$-\sigma_0(b^iy^j-b^jy^i)+\beta(\sigma^iy^j-\sigma^jy^i)=\alpha^2M^{ij}(x). \tag{5.5}$$

The equation (5.5) can be written as

$$-[(\sigma_h\delta^j_k+\sigma_k\delta^j_h)b^i-(\sigma_k\delta^i_h+\sigma_h\delta^i_k)b^j]+[(b_h\delta^j_k+b_k\delta^j_h)\sigma^i-(b_h\delta^i_k+b_k\delta^i_h)\sigma^j]=a_{hk}M^{ij}. \tag{5.6}$$

Contracting (5.6) by $j=h$, we get

$$n(b_k\sigma^i-b^i\sigma_k)=M^i_k \quad \text{which implies}$$
$$M_{ij}(x)=n(b_j\sigma_i-b_i\sigma_j). \tag{5.7}$$

Thus, we have

Theorem 5.1. *A Finsler space $\overline{F}^n$ $(n>2)$ which is obtained by conformal Kropina change of a Randers space of Douglas type remains to be of Douglas type if and if (5.7) is satisfied.*

6 Kropina Metric

For a Kropina metric we have $L=\frac{\alpha^2}{\beta}$, so that

$$L_\alpha=\frac{2\alpha}{\beta},\qquad L_\beta=-\frac{\alpha^2}{\beta^2}\qquad\text{and}\qquad L_{\alpha\alpha}=\frac{2}{\beta}.$$

Hence the values of C^*, D^* and C^{ij} given by equation (2.3), (3.7) and (3.9) respectively reduce to

$$C^*=\frac{\beta r_{00}+\alpha^2s_0}{2\beta^2\alpha},$$

$$D^* = \frac{b^2\alpha^2\{\rho\beta\alpha^2 + (3b^2\alpha^2 - 4\beta^2)\sigma_0 + 2\alpha^2 s_0\} - 8b^2\alpha^2\beta r_{00} + 8\beta^3 r_{00} - 8b^2\alpha^4 s_0 + 8\alpha^2\beta^2 s_0}{8b^2\alpha(3b^2\alpha^2 - 2\beta^2)}$$

and

$$8b^2\beta(3b^2\alpha^2 - \beta^2)\, C^{ij} = \{8\beta^3 r_{00} + 2\alpha^2 s_0(3b^2\alpha^2 + 4\beta^2) + 3b^2\alpha^4\rho\beta - 6b^2\alpha^2\beta^2\sigma_0\}\times$$
$$(b^i y^j - b^j y^i) + 3b^2\alpha^2\beta(3b^2\alpha^2 - 2\beta^2)(\sigma^i y^j - \sigma^j y^i) - 2b^2\alpha^2(3b^2\alpha^2 - 2\beta^2)(s_0^i y^j - s_0^j y^i). \tag{6.1}$$

Take $n > 2$, $\alpha^2 \not\equiv 0 \pmod{\beta}$. The terms in (6.1), which seemingly do not contain β are

$$6b^2\alpha^4 s_0(b^i y^j - b^j y^i) - 6b^4\alpha^4(s_0^i y^j - s_0^j y^i).$$

Hence we must have $hp(1)\, V_{(1)}^{ij}$ such that the above expression is equal to $6b^2\alpha^4\beta V_{(1)}^{ij}$. Therefore we have

$$s_0(b^i y^j - b^j y^i) - b^2(s_0^i y^j - s_0^j y^i) = \beta V_{(1)}^{ij}. \tag{6.2}$$

By putting $V_{(1)}^{ij} = V_k^{ij}(x) y^k$, the equation (6.2) can be written as

$$(s_h\delta_k^j + s_k\delta_h^j)b^i - (s_h\delta_k^i + s_k\delta_h^i)b^j - b^2[s_h^i\delta_k^j + s_k^i\delta_h^j - s_h^j\delta_k^i - s_k^j\delta_h^i] = b_h V_k^{ij} + b_k V_h^{ij} \tag{6.3}$$

Contracting (6.3) by $j = k$, we get

$$nb^i s_h - nb^2 s_h^i = b_h V_r^{ir} + b_r V_h^{ir}. \tag{6.4}$$

Next transvecting (6.3) by $b_j b^h$, we have

$$-b^2(b^2 s_k^i - s^i b_k - s_k b^i) = b^2 b_r V_k^{ir} + b_k b_r V_s^{ir} b^s. \tag{6.5}$$

Transvecting (6.5) by b^k, we get

$$2b^4 s^i = 2b^2 b_r V_s^{ir} b^s, \quad \text{which gives}$$
$$b_r V_s^{ir} b^s = b^2 s^i, \quad \text{provided} \ \ b^2 \neq 0. \tag{6.6}$$

Substituting the value of $b_r V_s^{ir} b^s$ from (6.6) in (6.5), we get

$$b_r V_h^{ir} = b^i s_h - b^2 s_h^i. \tag{6.7}$$

Substituting the value of $b_r V_h^{ir}$ from (6.7) in (6.4), we get

$$b^2 s_h^i = b^i s_h - \frac{1}{(n-1)}\, V_r^{ir} b_h. \tag{6.8}$$

If we put $v^i = \frac{1}{n-1} V_r^{ir}$, then equation (6.8) gives $b^2 s_h^i = b^i s_h - v^i b_h$ which implies $b^2 s_{ij} = b_i s_j - v_i b_j$, where $v_i = a_{ij} v^j$.

Since s_{ij} is skew-symmetric tensor, we have $V_i = s_i$ easily. Hence

$$s_{ij} = \frac{1}{b^2}\,(b_i s_j - b_j s_i). \tag{6.9}$$

Thus, we have

Theorem 6.1. *A Finsler space $\overline{F}^n$ $(n > 2)$ which is obtained by conformal Kropina change of a Kropina space F^n with $b^2 \neq 0$ is of Douglas type if and only if (6.9) is satisfied.*

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