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RESEARCH ARTICLE

Generalization of Preece's identity and some contigous results

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Abstract The aim of this paper is to establish generalizations of the well known Preece's identity and other identities involving product of generalized hypergeometric series by using new and very short method in the line of author Rathie Arjun K. and Choi Junesang [8].

Key Words class number, equivalence classes, ring of integers, quadratic field

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1 Introduction

The generalized hypergeometric function with p numerator and q denominator parameters is defined by Rainville E.D.[9], as

$$ {}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;x) = \sum_{m,n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_p)_n n!} x^m \tag{1.0} $$

where $(a)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined by $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)(n = 1,2,3,\cdots)$, for any complex number a. The generalized hypergeometric function is also defined by Saran N., Sharma S.D. and Trivedi T.N.[11] and Sharma J.N. and Gupta R.K.[12] in their literature as same as (1.0).

From the theory of differential equations, Professor Preece C.T.[4] established the following very interesting identity involving product of generalized hypergeometric series:

$$ {}_1F_1(a;2a;x)\times {}_1F_1(1-a;2-2a;x) = e^x\, {}_1F_2(1/2;a+1/2,3/2-a;\frac{x^2}{4}) \tag{1.1} $$

The author [5] has given a very short proof of (1.1). In another paper author [6] has also obtained the following two results contiguous to (1.1):

$$ {}_1F_1(a;2a+1;x)\times {}_1F_1(1-a;2-2a;x) $$

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$$= e^x\{{}_1F_2(1/2; a+1/2, 3/2-a; \frac{x^2}{4}) - x/(2(2a+1))\,{}_2F_3(3/2, 1; 3/2-a, a+3/2; \frac{x^2}{4})\} \tag{1.2}$$

and

$${}_1F_1(a; 2a-1; x) \times {}_1F_1(1-a; 2-2a; x) = e^x\{{}_1F_2(1/2; a-1/2, 3/2-a; \frac{x^2}{4})$$

$$-x/(2(2a-1))\,{}_1F_2(1/2; 3/2-a, a+1/2; \frac{x^2}{4})\}. \tag{1.3}$$

The well-known Kummer's first and second theorems [3] are

$${}_1F_1(a; c; x) = e^x\,{}_1F_1(c-a; c; -x), \tag{1.4}$$

$$e^{-x}\,{}_1F_1(a; 2a; x) = {}_0F_1(-; a+1/2; \frac{x^2}{16}). \tag{1.5}$$

The Rathie Arjun K. and Nagar V. [7] have obtained the following two interesting results contiguous to (1.5):

$$e^{-x/2}\,{}_1F_1(a; 2a+1; x) = {}_0F_1(-; a+1/2; \frac{x^2}{16}) - x/(2(2a+1))\,{}_0F_1(-; a+3/2; \frac{x^2}{16}), \tag{1.6}$$

and

$$e^{-x/2}\,{}_1F_1(a; 2a-1; x) = {}_0F_1(-; a-1/2; \frac{x^2}{16}) + x/(2(2a-1))\,{}_0F_1(-; a+1/2; \frac{x^2}{16}). \tag{1.7}$$

The following is an interesting result due to Bailey [1]:

$${}_0F_1(-; a; x) \times {}_0F_1(-; b; x) = {}_2F_3(1/2(a+b), 1/2(a+b-1); a, b, a+b-1; 4x). \tag{1.8}$$

We also recall the following interesting formula due to author [10] (p. 322):

$${}_1F_1(a; 2a; x) \times {}_1F_1(b; 2b; -x) = {}_2F_3(1/2(a+b), 1/2(a+b+1); a+1/2, b+1/2, a+b; \frac{x^2}{4}). \tag{1.9}$$

We recall the following interesting results due to Rathie Arjun K. and Choi Junesang [8] (p. 341)

$${}_1F_1(a; 2a; x) \times {}_1F_1(b; 2b; x) = e^x\{{}_2F_3(1/2(a+b), 1/2(a+b+1); a+1/2, b+1/2, a+b; \frac{x^2}{4})\}, \tag{1.10}$$

$${}_1F_1(a; 2a+1; x) \times {}_1F_1(b; 2b; x) = e^x\{{}_2F_3(1/2(a+b+1), 1/2(a+b); a+1/2, b+1/2, a+b; \frac{x^2}{4})$$

$$-x/(2(2a+1)) \times {}_2F_3(1/2(a+b+2), 1/2(a+b+1); a+3/2, b+1/2, a+b+1; \frac{x^2}{4})\} \tag{1.11}$$

$${}_1F_1(a; 2a-1; x) \times {}_1F_1(b; 2b; x) = e^x\{{}_2F_3(1/2(a+b-1), 1/2(a+b); b+1/2, a-1/2, a+b-1; \frac{x^2}{4})$$

$$-x/(2(2a-1)) \times {}_2F_3(1/2(a+b+1), 1/2(a+b); b+1/2, a+1/2, a+b; \frac{x^2}{4})\}. \tag{1.12}$$

The following is an interesting result due to Rathie Arjun K. and Choi Junesang [8] (p. 343):

$${}_1F_1(a; 2a; x) \times {}_1F_1(2-a; 4-2a; x) = e^x\{{}_2F_3(1, 3/2; a+1/2, 5/2-a, 2; \frac{x^2}{4})\} \tag{1.13}$$

$${}_1F_1(a; 2a; x) \times {}_1F_1(3-a; 6-2a; x) = e^x\{{}_2F_3(3/2, 2; a+1/2, 7/2-a, 3; \frac{x^2}{4})\}. \tag{1.14}$$

The aim of this paper is to establish a generalization of the well-known Preece's identity (1.1) by a similar method given by author [5], also in the line of Rathie Arjun K. and Choi Junesang [8].

2 Main Results

In this section, we shall establish the following generalization of the well-known Preece's identity (1.1) and other results:

Theorem $${}_1F_1(a;2a;x)\times {}_1F_1(2-a;4-2a;-x) = {}_2F_3(1,3/2;a+1/2,5/2-a,2;\tfrac{x^2}{4}) \tag{2.1}$$

$$\begin{aligned}{}_1F_1(a;2a+1;x)\times {}_1F_1(2-a;4-2a;-x) &={}_2F_3(3/2,1;a+1/2,5/2-a,2;\frac{x^2}{4})\\ &-x/(2(2a+1))_2F_3(2,3/2;a+3/2,5/2-a,3;\frac{x^2}{4})\end{aligned} \tag{2.2}$$

$$\begin{aligned}{}_1F_1(a;2a-1;x)\times {}_1F_1(2-a;4-2a;-x) &={}_2F_3(1/2,1;5/2-a,a-1/2,1;\frac{x^2}{4})\\ &+x/(2(2a-1))_2F_3(3/2,1;5/2-a,a+1/2,2;\frac{x^2}{4})\end{aligned} \tag{2.3}$$

$${}_1F_1(a;2a;x)\times {}_1F_1(3-a;6-2a;-x) ={}_2F_3(3/2,2;a+1/2,7/2-a,3;\frac{x^2}{4}) \tag{2.4}$$

$$\begin{aligned}{}_1F_1(a;2a+1;x)\times {}_1F_1(3-a;6-2a;-x) &={}_2F_3(2,3/2;a+1/2,7/2-a,3;\frac{x^2}{4})\\ &-x/(2(2a+1))_2F_3(5/2,2;a+3/2,7/2-a,4;\frac{x^2}{4})\end{aligned} \tag{2.5}$$

$$\begin{aligned}{}_1F_1(a;2a-1;x)\times {}_1F_1(3-a;6-2a;-x) &={}_2F_3(3/2,1;a-1/2,7/2-a,2;\frac{x^2}{4})\\ &+x/(2(2a-1))_2F_3(2,3/2;a+1/2,7/2-a,3;\frac{x^2}{4})\end{aligned} \tag{2.6}$$

3 Proofs

Proof of (2.1): Using equation (1.4), we get

$${}_1F_1(a;2a;x)\times {}_1F_1(2-a;4-2a;-x) = e^{-x}\,{}_1F_1(a;2a;x)\,{}_1F_1(2-a;4-2a;x)$$

Using (1.13), we have

$$= e^{-x}.e^{x}_2F_3(1,3/2;a+1/2,5/2-a,2;\frac{x^2}{4}) ={}_2F_3(1,3/2;a+1/2,5/2-a,2;\frac{x^2}{4}) \tag{3.1}$$

In this way (2.1) is proved.

Proof of (2.2): In order to prove (2.2), it is sufficient to show that

$$\begin{aligned}{}_1F_1(a;2a+1;x)\times {}_1F_1(2-a;4-2a;-x) &={}_2F_3(3/2,1;a+1/2,5/2-a,2;\frac{x^2}{4})\\ &-x/(2(2a+1))_2F_3(2,3/2;a+3/2,5/2-a,3;\frac{x^2}{4})\end{aligned} \tag{3.2}$$

Now, start with the left-hand side of equation (3.2):

$$L.H.S. = {}_1F_1(a;2a+1;x)\times {}_1F_1(2-a;4-2a;-x)$$

Using equation (1.4), we have

$$= e^{-x}\,{}_1F_1(a; 2a+1; x) \times {}_1F_1(2-a; 4-2a; x) = e^{-x/2}\,{}_1F_1(a; 2a+1; x) \times e^{-x/2}\,{}_1F_1(2-a; 4-2a; x)$$

using equation (1.6) in the first expression and equation (1.5) in the second expression, we get

$$= {}_0F_1(-; a+1/2; \frac{x^2}{16}) - x/(2(2a+1))_0F_1(-; a+3/2; \frac{x^2}{16})_0F_1(-; 5/2-a; \frac{x^2}{16})$$

$$=_0 F_1(-; a+1/2; \frac{x^2}{16}) \times_0 F_1(-; 5/2-a; \frac{x^2}{16})x/(2(2a+1))_0F_1(-; a+3/2; \frac{x^2}{16}) \times_0 F_1(-; 5/2-a; \frac{x^2}{16})$$

using equation (1.8), we get the required result

$$=_2 F_3(3/2, 1; a+1/2, 5/2-a, 2; \frac{x^2}{4}) - x/(2(2a+1))_2F_3(2, 3/2; a+3/2, 5/2-a, 3; \frac{x^2}{4})$$

In this way (2.2) is proved.

Proof of (2.3): In order to prove (2.3), it is suffcient to show that

$$ {}_1F_1(a; 2a-1; x) \times {}_1F_1(2-a; 4-2a; -x) =_2 F_3(1/2, 1; 5/2-a, a-1/2, 1; \frac{x^2}{4})$$

$$+x/(2(2a-1))_2F_3(3/2, 1; 5/2-a, a+1/2, 2; \frac{x^2}{4}) \tag{3.3}$$

Now, start with the left-hand side of equation (3.3): Using equation (1.4), we have

$$L.H.S. = {}_1F_1(a; 2a-1; x) \times {}_1F_1(2-a; 4-2a; -x)$$

Using equation (1.4), we have

$$= e^{-x}\,{}_1F_1(a; 2a-1; x) \times {}_1F_1(2-a; 4-2a; x) = e^{-x/2}\,{}_1F_1(a; 2a-1; x) \times e^{-x/2}\,{}_1F_1(2-a; 4-2a; x)$$

using equation (1.7) in the first expression and equation (1.5) in the second expression, we get

$$= {}_0F_1(-; a-1/2; \frac{x^2}{16}) + x/(2(2a-1))_0F_1(-; a+1/2; \frac{x^2}{16})_0F_1(-; 5/2-a; \frac{x^2}{16})$$

$$=_0 F_1(-; a-1/2; \frac{x^2}{16}) \times_0 F_1(-; 5/2-a; \frac{x^2}{16}) + x/(2(2a+1))_0F_1(-; a+1/2; \frac{x^2}{16}) \times_0 F_1(-; 5/2-a; \frac{x^2}{16})$$

using equation (1.8), we get the required result

$$=_2 F_3(1/2, 1; 5/2-a, a-1/2, 1; \frac{x^2}{4}) + x/(2(2a-1))_2F_3(3/2, 1; 5/2-a, a+1/2, 2; \frac{x^2}{4})$$

In this way (2.3) is proved.

Proof of (2.4): Using equation (1.4), we get

$$ {}_1F_1(a; 2a; x) \times {}_1F_1(3-a; 6-2a; -x) = e^{-x}\,{}_1F_1(a; 2a; x) \times {}_1F_1(3-a; 6-2a; x)$$

Using equation (1.14), we have

$$= e^{-x}.e^x_2F_3(3/2, 2; a+1/2, 7/2-a, 3; \frac{x^2}{4}) =_2 F_3(3/2, 2; a+1/2, 7/2-a, 3; \frac{x^2}{4}) \tag{3.4}$$

In this way (2.4) is proved.

Proof of (2.5): In order to prove (2.5), it is suffcient to show that

$$ {}_1F_1(a;2a+1;x)\times {}_1F_1(3-a;6-2a;-x) ={}_2F_3(2,3/2;a+1/2,7/2-a,3;\frac{x^2}{4}) $$
$$ -x/(2(2a+1))_2F_3(5/2,2;a+3/2,7/2-a,4;\frac{x^2}{4}) \tag{3.5} $$

Now, start with the left-hand side of equation (3.5):

$$ L.H.S. = {}_1F_1(a;2a+1;x)\times {}_1F_1(3-a;6-2a;-x) $$

Using equation (1.4), we have

$$ = e^{-x}\,{}_1F_1(a;2a+1;x)\times {}_1F_1(3-a;6-2a;x) = e^{-x/2}\,{}_1F_1(a;2a+1;x)\times e^{-x/2}\,{}_1F_1(3-a;6-2a;x) $$

using equation (1.6) in the first expression and equation (1.5) in the second expression, we get

$$ = {}_0F_1(-;a+1/2;\frac{x^2}{16}) - x/(2(2a+1))_0F_1(-;a+3/2;\frac{x^2}{16})\,{}_0F_1(-;7/2-a;\frac{x^2}{16}) $$

$$ ={}_0F_1(-;a+1/2;\frac{x^2}{16})\times{}_0F_1(-;7/2-a;\frac{x^2}{16})x/(2(2a+1))_0F_1(-;a+3/2;\frac{x^2}{16})\times{}_0F_1(-;7/2-a;\frac{x^2}{16}) $$

using equation (1.8), we get the required result

$$ ={}_2F_3(2,3/2;a+1/2,7/2-a,3;\frac{x^2}{4}) - x/(2(2a+1))_2F_3(5/2,2;a+3/2,7/2-a,4;\frac{x^2}{4}) $$

In this way (2.5) is proved.

Proof of (2.6): In order to prove (2.6), it is suffcient to show that

$$ {}_1F_1(a;2a-1;x)\times {}_1F_1(3-a;6-2a;-x) ={}_2F_3(3/2,1;a-1/2,7/2-a,2;\frac{x^2}{4}) $$
$$ +x/(2(2a-1))_2F_3(2,3/2;a+1/2,7/2-a,3;\frac{x^2}{4}) \tag{3.6} $$

Now, start with the left-hand side of equation (3.6):

$$ L.H.S. = {}_1F_1(a;2a-1;x)\times {}_1F_1(3-a;6-2a;-x) $$

Using equation (1.4), we have

$$ = e^{-x}\,{}_1F_1(a;2a-1;x)\times {}_1F_1(3-a;6-2a;x) = e^{-x/2}\,{}_1F_1(a;2a-1;x)\times e^{-x/2}\,{}_1F_1(3-a;6-2a;x) $$

using equation (1.7) in the first expression and equation (1.5) in the second expression, we get

$$ = {}_0F_1(-;a-1/2;\frac{x^2}{16}) + x/(2(2a-1))_0F_1(-;a+1/2;\frac{x^2}{16})\,{}_0F_1(-;7/2-a;\frac{x^2}{16}) $$

$$ ={}_0F_1(-;a-1/2;\frac{x^2}{16})\times{}_0F_1(-;7/2-a;\frac{x^2}{16}) + x/(2(2a+1))_0F_1(-;a+1/2;\frac{x^2}{16})\times{}_0F_1(-;7/2-a;\frac{x^2}{16}) $$

using equation (1.8), we get the required result

$$ ={}_2F_3(3/2,1;a-1/2,7/2-a,2;\frac{x^2}{4}) + x/(2(2a-1))_2F_3(2,3/2;a+1/2,7/2-a,3;\frac{x^2}{4}) $$

In this way (2.6) is proved. □

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