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Some extension of Hardy-Hilbert's integral inequality

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Abstract New kinds of Hardy-Hilbert's integral inequalities are presented.

Key Words Hardy-Hilbert's integral inequality, integral inequality

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1 Introduction

The hypergeometric function $F(\alpha, \beta, \gamma, x)$ is defined by ([1])

$$F(\alpha, \beta, \gamma, x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \frac{x^r}{r!} \tag{1}$$

where $(\alpha)_r$ is defined by

$$(\alpha)_r = \alpha (\alpha + 1) ... (\alpha + r - 1).$$

The series in (1) converges for $|x| < 1$and diverges for $|x| > 1$. For $x = 1$, the series converges if $\alpha + \beta < \gamma$ and in this case we have

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\,\Gamma(\lambda - \beta)}. \tag{2}$$

The hypergeometric function satisfies the following integral representation

$$F(\alpha, \beta, \gamma, x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma - \beta)} \int_0^1 t^{\beta - 1} (1 - t)^{\gamma - \beta - 1} (1 - xt)^{\alpha - 1}\, dt, \quad \gamma > \beta > 0\,. \tag{3}$$

Let $f, g \geqslant 0$ satisfy

$$0 < \int_0^\infty f^2(t)\, dt < \infty \ \ and \ \ 0 < \int_0^\infty g^2(t)\, dt < \infty\ ,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)\, g(y)}{x + y}\, dx\, dy \ < \ \pi \left(\int_0^\infty f^2(t)\, dt \int_0^\infty g^2(t)\, dt \right)^{1/2}, \tag{4}$$

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where the constant factor π is the best possible (cf. Hardy et al.[3]).Inequality (4) is well known as Hilbert's integral inequality. This inequality had been extended by Hardy [3] as follows:

If p>1, $\frac{1}{p}+\frac{1}{q}=1,\ \ f,g\geqslant 0$ satisfy

$$0<\int_0^\infty f^p(t)\,dt\ <\infty\ \ and\ \ \int_0^\infty g^q(t)\,dt\ <\infty\ ,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{x+y}\,dx\,dy\ <\ \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty g^q(t)\,dt\right)^{1/q}, \tag{5}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy-Hilbert's integral inequality and is important in analysis and application (cf. Mitrinovic et al. [4]).

B. Yang gave the following extension of (2) as follows:

Theorem 1. *If* $\lambda>2-\min\{p,q\}\,,\ \ f,g\geqslant 0,$ *satisfy*

$$0<\int_0^\infty t^{1-\lambda}f^p(t)\,dt\ <\infty\ \ and\ \ \int_0^\infty t^{1-\lambda}g^q(t)\,dt\ <\infty,$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{(x+y)^\lambda}\,dx\,dy\ <k_\lambda(p)\left(\int_0^\infty t^{1-\lambda}f^p(t)\,dt\right)^{1/p}\left(\int_0^\infty t^{1-\lambda}g^q(t)\,dt\right)^{1/q}, \tag{6}$$

where the constant factor $k_\lambda(p)\ =\ B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ is the best possible, B is the beta function.

2 Results

Theorem 2. *Let* $f,\,g\geqslant 0,\quad h>0,\quad b>a+1/2,\quad \alpha,\beta,\gamma,\delta>0,\quad 0<\lambda<1,\quad p>1,\quad \frac{1}{p}+\frac{1}{q}=1.$

Then

$$\begin{aligned}
&\int_a^b\int_a^b \frac{f(x)\,g(y)}{(1+2a-(x+y))^{1-\lambda}}\,dx\,dy\\
\leqslant\ &K\left(\int_a^b \frac{K(\lambda,\alpha,\beta,x)f^p(x)\,dx}{(x-a)^{(\gamma-1)\,(p-1)}(b-x)^{(\delta-\gamma-1)\,(p-1)}}\right)^{1/p}\\
&\times\left(\int_a^b \frac{K(\lambda,\gamma,\delta,y)\,g^q(y)}{(y-a)^{(\alpha-1)\,(q-1)}(b-y)^{(\beta-\alpha-1)\,(q-1)}}\,dy\right)^{1/q}
\end{aligned} \tag{7}$$

where

$$K(\lambda,\alpha,\beta,x)\ =\ \int_a^b \frac{(y-a)^{\alpha-1}(b-y)^{\beta-\alpha-1}}{(1+2a-h(x,y))^{1-\lambda}}\,dy,$$

and

$$\int_a^b \frac{1}{(y-a)^{(1-\alpha)}(b-y)^{(1+\alpha-\beta)}}\left(\int_a^b \frac{K(\lambda,\gamma,\delta,y)^{-1/q} f(x)}{(1+2a-(x+y))^{\lambda}}dx\right)^p dy$$
$$\leqslant K^p \int_a^b \frac{K(\lambda,\alpha,\beta,x)\, f^p(x)\, dx}{(x-a)^{(\gamma-1)(p-1)}(b-x)^{(\delta-\gamma-1)(p-1)}} \tag{8}$$

The inequalities (7) *and* (8) *are equivalent* .

Proof.

$$\int_a^b\int_a^b \frac{f(x)\, g(y)}{(1+2a-h(x,y))^{1-\lambda}}\, dx\, dy$$
$$= \int_a^b\int_a^b \frac{f(x)\,(y-a)^{\frac{\alpha-1}{p}}\,(b-y)^{\frac{\beta-\alpha-1}{p}}}{(x-a)^{\frac{\gamma-1}{q}}\,(b-x)^{\frac{\delta-\gamma-1}{q}}\,(1+2a-h(x,y))^{\frac{1-\lambda}{p}}}$$
$$\times \frac{g(y)\,(x-a)^{\frac{\gamma-1}{q}}\,(b-x)^{\frac{\delta-\gamma-1}{q}}}{(y-a)^{\frac{\alpha-1}{p}}\,(b-y)^{\frac{\beta-\alpha-1}{p}}\,(1+2a-h(x,y))^{\frac{1-\lambda}{q}}}\, dx\, dy$$
$$\leqslant \left(\int_a^b\int_a^b \frac{f^p(x)\,(y-a)^{\alpha-1}\,(b-y)^{\beta-\alpha-1}}{(x-a)^{\frac{p}{q}(\gamma-1)}\,(b-x)^{\frac{\delta-\gamma-1}{q}p}\,(1+2a-h(x,y))^{1-\lambda}}\, dx\, dy\right)^{1/p}$$
$$\times \left(\int_a^b\int_a^b \frac{g^q(y)\,(x-a)^{\gamma-1}\,(b-x)^{\delta-\gamma-1}}{(y-a)^{\frac{q}{p}(\alpha-1)}\,(b-y)^{\frac{\beta-\alpha-1}{p}q}\,(1+2a-h(x,y))^{1-\lambda}}\, dx\, dy\right)^{1/q}$$
$$= M^{1/p}N^{1/q}.$$

$$M = \int_a^b \frac{f^p(x)\, dx}{(x-a)^{(\gamma-1)(p-1)}(b-x)^{(\delta-\gamma-1)(p-1)}}\int_a^b \frac{(y-a)^{\alpha-1}(b-y)^{\beta-\alpha-1}}{(1+2a-h(x,y))^{1-\lambda}}\, dy$$
$$= \int_a^b \frac{K(\lambda,\alpha,\beta,x) f^p(x)\, dx}{(x-a)^{(\gamma-1)(p-1)}(b-x)^{(\delta-\gamma-1)(p-1)}}$$

Similarly,

$$N = \int_a^b \frac{K(\lambda,\gamma,\delta,y)\, g^q(y)}{(y-a)^{(1-\alpha)(q-1)}(b-y)^{(\beta-\alpha-1)(q-1)}}\, dy.$$

Therefore, we have

$$\int_a^b\int_a^b \frac{f(x)\, g(y)}{(1+2a-h(x,y))^{1-\lambda}}\, dx\, dy \leqslant K\left(\int_a^b \frac{K(\lambda,\alpha,\beta,x) f^p(x)\, dx}{(x-a)^{(\gamma-1)(p-1)}(b-x)^{(\delta-\gamma-1)(p-1)}}\right)^{1/p}$$

$$\times \left(\int_a^b \frac{K(\lambda,\gamma,\delta,y)\, g^q(y)}{(y-a)^{(\alpha-1)\,(q-1)}(b-y)^{(\beta-\alpha-1)\,(q-1)}}\, dy \right)^{1/q}.$$

In order to prove the equivalence of (7) and (8), suppose (8) is satisfied, then

$$\begin{aligned}
&\int_a^b \int_a^b \frac{f(x)\, g(y)}{(1+2a-h(x,y))^{1-\lambda}}\, dx\, dy \\
= \;& \int_a^b \frac{K(\lambda,\gamma,\delta,y)^{1/q} g(y)}{(y-a)^{\frac{(\alpha-1)\,(q-1)}{q}}(b-y)^{\frac{(\beta-\alpha-1)\,(q-1)}{q}}} \\
&\times \int_a^b \frac{K(\lambda,\gamma,\delta,y)^{-1/q} f(x)}{(y-a)^{\frac{(1-\alpha)\,(q-1)}{q}}(b-y)^{\frac{(1+\alpha-\beta)\,(q-1)}{q}}\,(1+2a-h(x,y))^{1-\lambda}} dx\, dy \\
\leqslant \;& \left(\int_a^b \frac{K(\lambda,\gamma,\delta,y) g^q(y)}{(y-a)^{(\alpha-1)\,(q-1)}(b-y)^{(\beta-\alpha-1)\,(q-1)}}\, dy \right)^{1/q} \times \left(\int_a^b \frac{1}{(y-a)^{(1-\alpha)}\,(b-y)^{(1+\alpha-\beta)}} \right. \\
&\left. \times \left(\int_a^b \frac{K(\lambda,\gamma,\delta,y)^{-1/q} f(x)}{(1+a-y)^{\frac{\lambda-1}{q}}\,(1+2a-h(x,y))^{1-\lambda}} dx \right)^p dy \right)^{1/p} \\
\leqslant \;& K \left(\int_a^b \frac{K(\lambda,\gamma,\delta,y) g^q(y)}{(y-a)^{(\alpha-1)\,(q-1)}(b-y)^{(\beta-\alpha-1)\,(q-1)}}\, dy \right)^{1/q} \\
&\times \left(\int_a^b \frac{K(\lambda,\alpha,\beta,x) f^p(x)\, dx}{(x-a)^{(\gamma-1)\,(p-1)}(b-x)^{(\delta-\gamma-1)\,(p-1)}} \right)^{1/p}.
\end{aligned}$$

Now suppose that (7) holds, then

$$\begin{aligned}
&\int_a^b \frac{1}{(y-a)^{(1-\alpha)}\,(b-y)^{(1+\alpha-\beta)}} \left(\int_a^b \frac{K(\lambda,\gamma,\delta,y)^{-1/q} f(x)}{(1+2a-h(x,y))^{1-\lambda}} dx \right)^p dy \\
= \;& \int_a^b \int_a^b \frac{K(\lambda,\gamma,\delta,y)^{-1/q} f(x)}{(y-a)^{(1-\alpha)}\,(b-y)^{(1+\alpha-\beta)}\,(1+2a-h(x,y))^{1-\lambda}} \times \left(\int_a^b \frac{K(\lambda,\gamma,\delta,y)^{-1/q} f(x)}{(1+2a-h(x,y))^{1-\lambda}} dx \right)^{p-1} dxdy \\
= \;& \int_a^b \int_a^b \frac{f(x)}{(1+2a-h(x,y))^{1-\lambda}} \times \\
& \left(\int_a^b \frac{K(\lambda,\gamma,\delta,y)^{-1} f(x)}{(y-a)^{(\alpha-1)(q-1)}\,(b-y)^{(1+\alpha-\beta)(q-1)}\,(1+2a-h(x,y))^{1-\lambda}} dx \right)^{p/q} dxdy \\
\leqslant \;& K \left(\int_a^b \frac{K(\lambda,\alpha,\beta,x) f^p(x)\, dx}{(x-a)^{(\gamma-1)\,(p-1)}(b-x)^{(\delta-\gamma-1)\,(p-1)}} \right)^{1/p} \times \left(\int_a^b \frac{K(\lambda,\gamma,\delta,y)}{(y-a)^{(\alpha-1)(q-1)}\,(b-y)^{(1+\alpha-\beta)(q-1)}} \times \right.
\end{aligned}$$

$$\left(\int_a^b \frac{K(\lambda,\gamma,\delta,y)^{-1}f(x)}{(1+a-y)^{\lambda-1}(y-a)^{(1-\alpha)(q-1)}(b-y)^{1+\alpha-\beta}\left(1+2a-h(x,y)\right)^{1-\lambda}}dx\right)^p dy\Bigg)^{1/q}$$

$$= K\left(\int_a^b \frac{K(\lambda,\alpha,\beta,x)\,f^p(x)\,dx}{(x-a)^{(\gamma-1)(p-1)}(b-x)^{(\delta-\gamma-1)(p-1)}}\right)^{1/p}$$

$$\times\left(\int_a^b \frac{1}{(y-a)^{(1-\alpha)}(b-y)^{(1+\alpha-\beta)}}\left(\int_a^b \frac{K(\lambda,\gamma,\delta,y)^{-1/q}f(x)}{(1+a-y)^{\frac{\lambda-1}{q}}\left(1+2a-(x+y)\right)^{1-\lambda}}dx\right)^p dy\right)^{1/q}$$

which implies

$$\int_a^b \frac{1}{(y-a)^{(1-\alpha)}(b-y)^{(1+\alpha-\beta)}}\left(\int_a^b \frac{K(\lambda,\gamma,\delta,y)^{-1/q}f(x)}{(1+a-y)^{\frac{\lambda-1}{q}}\left(1+2a-h(x,y)\right)^{1-\lambda}}dx\right)^p dy$$

$$\leqslant K^p\int_a^b \frac{K(\lambda,\alpha,\beta,x)\,f^p(x)\,dx}{(x-a)^{(\gamma-1)(p-1)}(b-x)^{(\delta-\gamma-1)(p-1)}}.$$

□

For the special case, by taking $h(x,y)=x+y$, we obtain the following result:

Theorem 3. *Let* $f,\ g\geqslant 0,\ \ b>a+1/2,\ \beta>\alpha>0,\ \delta>\gamma>0,\ \ 0<\lambda<1,\ p>1,\ \frac{1}{p}+\frac{1}{q}=1.$ *Then*

$$\int_a^b\int_a^b \frac{f(x)\,g(y)}{\left(1+2a-(x+y)\right)^{1-\lambda}}\,dx\,dy$$

$$\leqslant K\left(\int_a^b \frac{F\left(\lambda,\alpha,\beta,\frac{b-a}{1+a-x}\right)f^p(x)\,dx}{(x-a)^{(\gamma-1)(p-1)}(b-x)^{(\delta-\gamma-1)(p-1)}(1+a-x)^{1-\lambda}}dx\right)^{1/p}$$

$$\times\left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)g^q(y)}{(y-a)^{(\alpha-1)(q-1)}(b-y)^{(\beta-\alpha-1)(q-1)}(1+a-y)^{1-\lambda}}\,dy\right)^{1/q} \tag{9}$$

where

$$K = (b-a)^{\frac{\beta-1}{p}+\frac{\delta-1}{q}}\left(\frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)}\right)^{1/p}\left(\frac{\Gamma(\lambda)\Gamma(\delta-\lambda)}{\Gamma(\delta)}\right)^{1/q},$$

and

$$\int_a^b \frac{1}{(y-a)^{(1-\alpha)}(b-y)^{(1+\alpha-\beta)}}\left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)^{-1/q}f(x)}{(1+a-y)^{\frac{\lambda-1}{q}}\left(1+2a-(x+y)\right)^{\lambda}}dx\right)^p dy$$

$$\leqslant K^p\int_a^b \frac{F\left(\lambda,\alpha,\beta,\frac{b-a}{1+a-x}\right)f^p(x)\,dx}{(x-a)^{(\gamma-1)(p-1)}(b-x)^{(\delta-\gamma-1)(p-1)}(1+a-x)^{1-\lambda}}. \tag{10}$$

The inequalities (9) *and* (10) *are equivalent .*

Proof.

$$
\begin{aligned}
& \int_a^b\int_a^b \frac{f(x)\,g(y)}{(1+2a-(x+y))^{1-\lambda}}\,dx\,dy \\
= \;& \int_a^b\int_a^b \frac{f(x)\,(y-a)^{\frac{\alpha-1}{p}}\,(b-y)^{\frac{\beta-\alpha-1}{p}}}{(x-a)^{\frac{\gamma-1}{q}}\,(b-x)^{\frac{\delta-\gamma-1}{q}}\,(1+2a-(x+y))^{\frac{1-\lambda}{p}}} \\
& \times \frac{g(y)\,(x-a)^{\frac{\gamma-1}{q}}\,(b-x)^{\frac{\delta-\gamma-1}{q}}}{(y-a)^{\frac{\alpha-1}{p}}\,(b-y)^{\frac{\beta-\alpha-1}{p}}\,(1+2a-(x+y))^{\frac{1-\lambda}{q}}}\,dx\,dy \\
\leqslant \;& \left(\int_a^b\int_a^b \frac{f^p(x)\,(y-a)^{\alpha-1}\,(b-y)^{\beta-\alpha-1}}{(x-a)^{\frac{p}{q}(\gamma-1)}\,(b-x)^{\frac{\delta-\gamma-1}{q}p}\,(1+2a-(x+y))^{1-\lambda}}\,dx\,dy\right)^{1/p} \\
& \times \left(\int_a^b\int_a^b \frac{g^q(y)\,(x-a)^{\gamma-1}\,(b-x)^{\delta-\gamma-1}}{(y-a)^{\frac{q}{p}(\alpha-1)}\,(b-y)^{\frac{\beta-\alpha-1}{p}q}\,(1+2a-(x+y))^{1-\lambda}}\,dx\,dy\right)^{1/q} \\
= \;& P^{1/p}Q^{1/q}\,.
\end{aligned}
$$

$$
\begin{aligned}
P \;=\;& \int_a^b \frac{f^p(x)\,dx}{(x-a)^{(\gamma-1)\,(p-1)}(b-x)^{(\delta-\gamma-1)\,(p-1)}}\int_a^b \frac{(y-a)^{\alpha-1}(b-y)^{\beta-\alpha-1}}{(1+2a-(x+y))^{1-\lambda}}\,dy \\
=\;& \int_a^b \frac{f^p(x)\,dx}{(x-a)^{(\gamma-1)\,(p-1)}(b-x)^{(\delta-\gamma-1)\,(p-1)}(1+a-x)^{\lambda}}\int_a^b \frac{(y-a)^{\alpha-1}(b-y)^{\beta-\alpha-1}}{\left(1-\frac{y-a}{1+a-x}\right)^{1-\lambda}}\,dy\,.
\end{aligned}
$$

By putting $u=\dfrac{y-a}{b-a}$, $0\leqslant u\leqslant 1$, we obtain

$$
\begin{aligned}
P \;=\;& \int_a^b \frac{f^p(x)\,dx}{(x-a)^{(\gamma-1)\,(p-1)}(b-x)^{(\delta-\gamma-1)\,(p-1)}(1+a-x)^{1-\lambda}} \\
& \times\int_0^1 u^{\alpha-1}(1-u)^{\beta-\alpha-1}\left(1-\left(\frac{b-a}{1+a-x}\right)u\right)^{\lambda-1}du \\
=\;& (b-a)^{\beta-1}\left(\frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\beta)}\right)^{1/p}\int_a^b \frac{F\left(\lambda,\alpha,\beta,\frac{b-a}{1+a-x}\right)f^p(x)\,dx}{(x-a)^{(\gamma-1)\,(p-1)}(b-x)^{(\delta-\gamma-1)\,(p-1)}(1+a-x)^{1-\lambda}}
\end{aligned}
$$

Similarly,

$$
Q=\;(b-a)^{\beta-1}\left(\frac{\Gamma(\lambda)\Gamma(\delta-\gamma)}{\Gamma(\delta)}\right)^{1/q}\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)g^q(y)}{(y-a)^{(1-\alpha)\,(q-1)}(b-y)^{(\beta-\alpha-1)\,(q-1)}(1+a-y)^{1-\lambda}}\,dy.
$$

Therefore, we have

$$
\int_a^b\int_a^b \frac{f(x)\,g(y)}{(1+2a-(x+y))^{1-\lambda}}\,dx\,dy \;\leqslant\; K\left(\int_a^b \frac{F\left(\lambda,\alpha,\beta,\frac{b-a}{1+a-x}\right)f^p(x)\,dx}{(x-a)^{(\gamma-1)\,(p-1)}(b-x)^{(\delta-\gamma-1)\,(p-1)}(1+a-x)^{1-\lambda}}\right)^{1/p}
$$

$$\times\left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right) g^q(y)}{(y-a)^{(\alpha-1)\,(q-1)}(b-y)^{(\beta-\alpha-1)\,(q-1)}(1+a-y)^{1-\lambda}}\,dy\right)^{1/q}$$

In order to prove the equivalence of (9) and (10), suppose (10) is satisfied, then

$$\begin{aligned}
&\int_a^b\int_a^b \frac{f(x)\,g(y)}{(1+2a-(x+y))^{1-\lambda}}\,dx\,dy\\
=\;& \int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)^{1/q} g(y)}{(y-a)^{\frac{(\alpha-1)\,(q-1)}{q}}(b-y)^{\frac{(\beta-\alpha-1)\,(q-1)}{q}}(1+a-y)^{\frac{1-\lambda}{q}}}\\
&\times\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)^{-1/q} f(x)}{(y-a)^{\frac{(1-\alpha)\,(q-1)}{q}}(b-y)^{\frac{(1+\alpha-\beta)\,(q-1)}{q}}(1+a-y)^{\frac{\lambda-1}{q}}(1+2a-(x+y))^{1-\lambda}}dx\,dy\\
\leqslant\;& \left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right) g^q(y)}{(y-a)^{(\alpha-1)\,(q-1)}(b-y)^{(\beta-\alpha-1)\,(q-1)}(1+a-y)^{1-\lambda}}\,dy\right)^{1/q}\\
&\times\left(\int_a^b \frac{1}{(y-a)^{(1-\alpha)}\,(b-y)^{(1+\alpha-\beta)}}\left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)^{-1/q} f(x)}{(1+a-y)^{\frac{\lambda-1}{q}}(1+2a-(x+y))^{1-\lambda}}dx\right)^p dy\right)^{1/p}\\
\leqslant\;& K\left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right) g^q(y)}{(y-a)^{(\alpha-1)\,(q-1)}(b-y)^{(\beta-\alpha-1)\,(q-1)}(1+a-y)^{1-\lambda}}\,dy\right)^{1/q}\\
&\times\left(\int_a^b \frac{F\left(\lambda,\alpha,\beta,\frac{b-a}{1+a-x}\right) f^p(x)\,dx}{(x-a)^{(\gamma-1)\,(p-1)}(b-x)^{(\delta-\gamma-1)\,(p-1)}(1+a-x)^{1-\lambda}}\right)^{1/p}.
\end{aligned}$$

Now suppose that (9) holds, then

$$\begin{aligned}
&\int_a^b \frac{1}{(y-a)^{(1-\alpha)}\,(b-y)^{(1+\alpha-\beta)}}\left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)^{-1/q} f(x)}{(1+a-y)^{\frac{\lambda-1}{q}}(1+2a-(x+y))^{1-\lambda}}dx\right)^p dy\\
=\;& \int_a^b\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)^{-1/q} f(x)}{(y-a)^{(1-\alpha)}\,(b-y)^{(1+\alpha-\beta)}\,(1+a-y)^{\frac{\lambda-1}{q}}(1+2a-(x+y))^{1-\lambda}}\\
&\times\left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)^{-1/q} f(x)}{(1+a-y)^{\frac{\lambda-1}{q}}(1+2a-(x+y))^{1-\lambda}}dx\right)^{p-1} dxdy\\
=\;& \int_a^b\int_a^b \frac{f(x)}{(1+2a-(x+y))^{1-\lambda}}\times\\
&\left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)^{-1} f(x)}{(y-a)^{(\alpha-1)(q-1)}\,(b-y)^{(1+\alpha-\beta)(q-1)}\,(1+a-y)^{\lambda-1}(1+2a-(x+y))^{1-\lambda}}dx\right)^{p/q} dxdy
\end{aligned}$$

$$\leqslant K\left(\int_a^b \frac{F\left(\lambda,\alpha,\beta,\frac{b-a}{1+a-x}\right) f^p(x)\,dx}{(x-a)^{(\gamma-1)\,(p-1)}(b-x)^{(\delta-\gamma-1)\,(p-1)}(1+a-x)^{1-\lambda}}\right)^{1/p} \times$$

$$\left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)}{(y-a)^{(\alpha-1)(q-1)}\,(b-y)^{(1+\alpha-\beta)(q-1)}\,(1+a-y)^{1-\lambda}}\right.$$

$$\left.\times\left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)^{-1} f(x)}{(1+a-y)^{\lambda-1}(y-a)^{(1-\alpha)(q-1)}(b-y)^{1+\alpha-\beta}\,(1+2a-(x+y))^{1-\lambda}}dx\right)^p dy\right)^{1/q}$$

$$= K\left(\int_a^b \frac{F\left(\lambda,\alpha,\beta,\frac{b-a}{1+a-x}\right) f^p(x)\,dx}{(x-a)^{(\gamma-1)\,(p-1)}(b-x)^{(\delta-\gamma-1)\,(p-1)}(1+a-x)^{1-\lambda}}\right)^{1/p}$$

$$\times\left(\int_a^b \frac{1}{(y-a)^{(1-\alpha)}\,(b-y)^{(1+\alpha-\beta)}}\right.$$

$$\left.\times\left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)^{-1/q} f(x)}{(1+a-y)^{\frac{\lambda-1}{q}}(1+a-x)^{1-\lambda}\,(1+2a-(x+y))^{1-\lambda}}dx\right)^p dy\right)^{1/q}$$

which implies

$$\int_a^b \frac{1}{(y-a)^{(1-\alpha)}\,(b-y)^{(1+\alpha-\beta)}}\left(\int_a^b \frac{F\left(\lambda,\gamma,\delta,\frac{b-a}{1+a-y}\right)^{-1/q} f(x)}{(1+a-y)^{\frac{\lambda-1}{q}}\,(1+2a-(x+y))^{1-\lambda}}dx\right)^p dy$$

$$\leqslant K^p\int_a^b \frac{F\left(\lambda,\alpha,\beta,\frac{b-a}{1+a-x}\right) f^p(x)\,dx}{(x-a)^{(\gamma-1)\,(p-1)}(b-x)^{(\delta-\gamma-1)\,(p-1)}(1+a-x)^{1-\lambda}}.$$

□

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