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RESEARCH ARTICLE

Normalized duality mappings on linear 2-normed spaces and smoothness

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Abstract In this paper we introduce the concept of smoothness of the space $X \times V(z)$, where $(X, \|.,.\|)$ is a linear 2-normed space and $V(z)$ is the space spanned by some $z \in X$, in terms of the bilinear functionals on $X \times V(z)$. We characterize smoothness in terms of the normalized duality mapping and in terms of the Gâteaux differentiability of the function $x \mapsto \|x, z\|$.

Key Words Linear 2-normed space, Normalized duality mapping, smoothness, Gâteaux differentiability

MSC 2010 46G25, 46T20

1 Introduction

After the introduction of the concept of linear 2-normed spaces and 2- metric spaces by S.Gähler through his papers [9, 10] in 1960's, the subject has got attention of many mathematicians like Y.J.Cho, C.R.Diminnie, R.W.Freese, A.White, S.S.Dragomir and they developed extensively the geometric structure of linear 2-normed spaces [1, 2, 3, 5, 6, 11]. Studies of linear 2-normed spaces are still intensive and updating the subject with new concepts. The smoothness related concepts are well known in normed linear spaces.

Our interest here is to study the concept of smoothness in the setting of linear 2-normed spaces. We introduced the smoothness in terms of the linear 2-functionals and we revealed that the concept is equivalent to the univocalness of the normalized duality mapping and the Gâteaux differentiability of the function which maps $x \mapsto \|x, z\|$.

2 Preliminary Notes

The concept of 2-norms on a real linear space X of dimension greater than one, is introduced in [9], as a 2 dimensional analogue of a norm, and is defined as a real valued function $\|.,.\|$, defined on $X \times X$ satisfying the following conditions:

For all $x, y, z \in X$ and $\alpha \in \mathbb{R}$,

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N1. $\|x, y\| = 0$ if and only if x and y are linearly dependent,
N2. $\|x, y\| = \|y, x\|$,
N3. $\|\alpha x, y\| = |\alpha|\|x, y\|$and
N4. $\|x, y + z\| \leqslant \|x, y\| + \|x, z\|$.

The pair $(X, \|., .\|)$ is called a linear 2-normed space.

A simple and standard example of a 2-norm is the 2-norm $\|., .\|$ on $\mathbb{R}^2$, defined by $\|a, b\| = |a_1b_2-a_2b_1|$ where $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$. Geometrically this is the area of the parallelogram determined by the vectors a and b as the adjacent sides.

If the limit $\lim_{t\to 0} \frac{\|x+ty,z\|-\|x,z\|}{t}$ exists, we say that the 2-norm $\|., .\|$ is Gâteaux differentiable at (x, z) in the direction y.

We recall the concept of normalized duality mapping in normed linear spaces and some of its properties. Let $(X, \|.\|)$ be a normed space and X^* be the dual space of X (the set of all bounded linear functionals on X). Then the notion of normalized duality mapping $J : X \to 2^{X^*}$ which is defined by $J(x) = \{x^* \in X^* : x^*(x) = \|x\|^2 \text{ and } \|x^*\| = \|x\|\}$, is well known. A section $\tilde{J}$ of J is a function from $X \to X^*$ satisfying the condition $\tilde{J}(x) \in J(x)\ \forall x \in X$ (see [4]). A normed space is smooth at $0 \neq x \in X$, if there exists a unique bounded linear functional f on X such that $f(x) = \|x\|$ and $\|f\| = 1$ and X is said to be smooth if it is smooth at every of its non zero point. The characterizations of smooth normed spaces are well known (see [4, 10]).

Theorem 2.1. *(cf.[8]) Let X be a normed space and $x_0 \in X$ with $\|x_0\| = 1$. Then the space X is smooth at x_0 if and only if the normalized duality mapping at x_0, $J(x_0)$ contains a unique element in X^*.*

Definition 2.2. Let $(X, \|., .\|)$ be a linear 2-normed space. A function $F : X \times X \to \mathbb{R}$ is called linear 2-functional (or bilinear functional) if for $a, b, c, d \in X$ and $\alpha, \beta \in \mathbb{R}$, we have
(i) $F(a + b, c + d) = F(a, c) + F(a, d) + F(b, c) + F(b, d)$,
(ii) $F(\alpha a, \beta b) = \alpha\beta F(a, b)$.

A bilinear functional F on $(X, \|., .\|)$ is called bounded if there exists $M > 0$ such that $|F(a, b)| \leqslant M\|a, b\|\ \forall\ a, b \in X$. If F is bounded we can define norm of F by $\|F\| = \inf\{M : |F(a, b)| \leqslant M\|a, b\|\ \forall\ a, b \in X\}$.

3 Normalized Duality Mapping and Some Basic Properties

We start with the definition of normalized duality mapping which is in fact derived from the duality mapping defined in [5], and discuss some of its basic properties, which enable us to define sections of a normalized duality mapping as bounded bilinear functionals on $X \times V(z)$.

Definition 3.1. Let $(X, \|., .\|)$ be a linear 2-normed space , $z \in X$. Suppose $X_z^\dagger$ denotes the set of all bounded linear 2-functionals on $X \times V(z)$. The mapping $J_z : X \times V(z) \to 2^{X_z^\dagger}$ defined by

$$J_z(x, z) = \{F \in X_z^\dagger : F(x, z) = \|x, z\|^2 \text{ and } \|F\| = \|x, z\|\}$$

is called the normalized duality 2-mapping on $X \times V(z)$.

Theorem 3.2. *Let $(X, \|.,.\|)$ be a linear 2-normed space J_z be the normalized duality mapping on $X \times V(z)$. Then $J_z(x,z)$ is a non empty convex subset of $X_z^\dagger$ for every $(x,z) \in X \times V(z)$.*

Proof. If $\|x,z\| = 0$ then $J_z(x,z) = \{0\}$, which is non empty and convex. Assume that $x \in X$ such that $\|x,z\| \neq 0$. Let $g : V(x) \times V(z) \to \mathbb{R}$ be defined by

$$g(\lambda x, \mu z) = \lambda\mu\|x,z\|^2$$

Then g is bilinear on $V(x)\times V(z)$ because, if $x_1 = \lambda_1 x, x_2 = \lambda_2 x \in V(x)$ and $z_1 = \mu_1 z,\ z_2 = \mu_2 z \in V(z)$. Then we have

$$\begin{aligned} g(x_1+x_2, z_1+z_2) &= g((\lambda_1+\lambda_2)x, (\mu_1+\mu_2)z) \\ &= (\lambda_1+\lambda_2)(\mu_1+\mu_2)\|x,z\|^2 \\ &= \lambda_1\mu_1\|x,z\|^2 + \lambda_1\mu_2\|x,z\|^2 + \lambda_2\mu_1\|x,z\|^2 + \lambda_2\mu_2\|x,z\|^2 \\ &= g(x_1,z_1) + g(x_1,z_2) + g(x_2,z_1) + g(x_2,z_2) \end{aligned}$$

Also

$$\begin{aligned} g(\alpha x_1, \beta z_1) &= g(\alpha\lambda_1 x, \beta\mu_1 z) \\ &= \alpha\lambda_1\beta\mu_1\ g(x,z) \\ &= \alpha\beta g(x_1,z_1) \end{aligned}$$

We have

$$\begin{aligned} |g(x_1,z_1)| &= |\lambda_1\mu_1|\|x,z\|^2 \\ &= \|\lambda_1 x, \mu_1 z\|\|x,z\| \\ &= \|x,z\|\|x_1,z_1\| \end{aligned}$$

So g is a bounded linear 2-functional on $V(x) \times V(z)$, with $|g| = \|x,z\|$. Applying the Hahn- Banach extension theorem for bounded linear 2-functionals we get a bounded linear 2-functional $F : X\times V(z) \to \mathbb{R}$ such that $F(x,z) = \|x,z\|^2$ and $\|F\| = \|x,z\|$. That is $F \in J_z(x,z)$ so that $J_z(x,z)$ is non empty.

To show $J_z(x,z)$ is convex, assume $F_1, F_2 \in J_z(x,z)$.
$\big(i.e, F_1(x,z) = \|x,z\|^2$ and $\|F_1\| = \|x,z\|;\ F_2(x,z) = \|x,z\|^2$ and $\|F_2\| = \|x,z\|.\big)$ Now for $0 < \lambda < 1$, we have $(\lambda F_1 + (1-\lambda)F_2)(x,z) = \|x,z\|^2$.

For $\|x,z\| \neq 0$,

$$\begin{aligned} 0 < \|x,z\| &= (\lambda F_1 + (1-\lambda)F_2)(\frac{x}{\|x,z\|}, z) \\ &\leqslant \|\lambda F_1 + (1-\lambda)F_2)\| \end{aligned} \tag{1}$$

$$(\text{Since}\|F\| = \sup\{|F(x,z)|; \|x,z\| = 1\})$$

Also

$$\begin{aligned} \|\lambda F_1 + (1-\lambda)F_2\| &\leqslant \lambda\|F_1\| + (1-\lambda)\|F_2\| \\ &= \lambda\|x,y\| + (1-\lambda)\|x,z\| \\ &= \|x,z\| \end{aligned} \tag{2}$$

From (1) and (2) we have $\|\lambda F_1 + (1-\lambda)F_2\| = \|x,z\|$. Thus $\lambda F_1 + (1-\lambda)F_2 \in J_z(x,z)$. Therefore $J_z(x,z)$ is a non empty convex subset of $X_z^\dagger$. $\square$

Theorem 3.3. *Let $(X, \|.,.\|)$ be a linear 2-normed space and $z \in X$. If J_z is the normalized duality mapping on $X \times V(z)$, then $J_z(\lambda x, z) = \lambda J_z(x,z)\ \forall \lambda \in \mathbb{R}$.*

Proof. If $\lambda = 0$ or $\|x,z\| = 0$ the result is trivially true. Assume $\lambda \neq 0$ and $\|x,z\| \neq 0$. We know if $F \in X_z^\dagger$ then $\frac{1}{\lambda}F \in X_z^\dagger$.

Suppose $F \in X_z^\dagger$ then

$$\begin{aligned} F \in J_z(\lambda x, z) &\iff F(\lambda x, z) = \|\lambda x, z\|^2 \quad \text{and } \|F\| = \|\lambda x, z\| \\ &\iff \lambda F(x,z) = \lambda^2 \|x,z\|^2 \quad \text{and } \|F\| = |\lambda|\|x,z\| \\ &\iff \frac{1}{\lambda}F(x,z) = \|x,z\|^2 \quad \text{and } \|\frac{1}{\lambda}F\| = \|x,z\| \end{aligned}$$

$$\iff \frac{1}{\lambda}F \in J_z(x,z) \qquad \iff F \in \lambda J_z(x,z).$$

Therefore $J_z(\lambda x, y) = \lambda J_z(x,y)$ $\square$

A similar argument above would yield that $J_z(x, \lambda z) = \lambda J_z(x,z)$ for $\lambda \in \mathbb{R}$.

Definition 3.4. (Section of normalized duality mapping) Let $(X, \|.,.\|)$ be a linear 2-normed space and J_z be the normalized duality mapping on $X \times V(z)$. Then a section $\tilde{J}_z$ of J_z is a map from $X \times V(z) \to X_z^\dagger$ satisfying $\tilde{J}_z(x,z) \in J_z(x,z)\ \forall x \in X$.

Theorem 3.5. *If $S_z(X)$ denotes the unit sphere in $X \times V(z)$ given by $S_z(X) = \{(x,z) : \|x,z\| = 1\}$. Then the following are equivalent:*

(1) Every bounded linear 2-functional attains its norm on $S_z(X)$. i.e, for $F \in X_z^\dagger$ there exists $(x_0, z) \in S_z(X)$ such that $F(x_0, z) = \|F\|$

(2) Given $F \in X_z^\dagger$ there exists $(x,z) \in X \times V(z)$ and a section $\tilde{J}_z$ of the normalized duality mapping J_z such that $F = \tilde{J}_z(x,z)$.

Proof. Assume (1) and let $F \in X_z^\dagger$. Then there exists $(x_0, z) \in X \times V(z)$ with $\|x_0, z\| = 1$ and $F(x_0, z) = \|F\|$ (1)

Let $(x,z) = (\|F\|x_0, z)$. We claim that $F \in J_z(x,z)$.

Now

$$\begin{aligned} F(x,z) &= \|F\|F(x_0,z) \\ &= \|F\|^2 = \|x,z\|^2, \quad \text{from(1) and by the choice of } (x,z) \\ \text{and} \quad \|F\| &= \|x,z\| \end{aligned}$$

Therefore there exists a section $\tilde{J}_z$ of J_z such that $F = \tilde{J}_z(x,z)$.

Next we prove the converse part. If $F = 0$ then for any point $(x,z) \in S_z(X)$ we have $F(x,z) = \|F\|$. Assume that for every $0 \neq F \in X^\dagger$ there exits $(x,z) \in X \times V(z)$ with $\|x,z\| \neq 0$ such that $F = \tilde{J}_z(x,z)$. Then for $(u,z) = (\frac{x}{\|x,z\|}, z) \in S_z(X)$,

$$F(u,z) = \frac{1}{\|x,z\|}F(x,z) = \frac{1}{\|x,z\|}\|x,z\|^2$$

$$= \|x, z\| = \|F\|.$$

This proves the theorem. □

4 Normalized Duality Mapping and Smoothness

In this section our objective is to define the smoothness of the space $X \times V(z)$ and to establish the equivalency of the definition with the Gâteaux differentiability of the semi norm $p_z(x) = \|x, z\|$ and uniqueness of section of the normalized duality mapping. At the end we characterize the smoothness of a linear 2-normed space X in terms of the differentiability of the semi norms $p_z(x) = \|x, z\|$ for all $z \in X$.

Definition 4.1. Let $(X, \|.,.\|)$ be a linear 2-normed space and let $(x, z) \in X \times V(z)$ such that $\|x, z\| \neq 0$. We say that $X \times V(z)$ is smooth at the point (x, z) if there is a unique bounded linear 2-functional $F \in X_z^\dagger$ such that $F(x, z) = \|x, z\|$ and $\|F\| = 1$. The space $X \times V(z)$ is said to be smooth if it is smooth at every $(x, z) \in X \times V(z)$ for which $\|x, z\| \neq 0$.

Theorem 4.2. *Let $(X, \|.,.\|)$ be a linear 2-normed space and $z \in X$. Suppose $x_0 \in X$ be such that $\|x_0, z\| \neq 0$. Then the following are equivalent:*

1. *$X \times V(z)$ is smooth at (x_0, z).*
2. *$J_z(x_0, z)$ contains a unique element. In other words any section of the normalized duality mapping J_z assumes the same 2-functional at (x_0, z).*

Proof. Let $X \times V(z)$ be smooth at (x_0, z). Let $F_1 \neq F_2 \in J_z(x_0, z)$. That is $F_1(x_0, z) = \|(x_0, z)\|^2$ and $\|F_1\| = \|(x_0, z)\|$; $F_2(x_0, z) = \|(x_0, z)\|^2$ and $\|F_2\| = \|(x_0, z)\|$.

Then $f_1 = \frac{1}{\|F_1\|} F_1$ and $f_2 = \frac{1}{\|F_2\|} F_2$ are in $X_z^\dagger$ such that $f_1(x_0, z) = \frac{1}{\|F_1\|} F_1(x_0, z) = \frac{\|(x_0,z)\|^2}{\|(x_0,z)\|} = \|(x_0, z)\|$ and $\|f_1\| = 1$. Similarly $f_2(x_0, z) = \|(x_0, z)\|$ and $\|f_2\| = 1$, which contradicts the smoothness of $X \times V(z)$ at (x_0, z). Therefore (1) $\Rightarrow$ (2).

If $X \times V(z)$ is not smooth at (x_0, z) then there exist two bounded linear 2-functional f_1 and f_2 such that $f_1 \neq f_2$ and $\|f_1\| = \|f_2\| = 1$ and $f_1(x_0, z) = f_2(x_0, z) = \|x_0, z\|$. Take $F_1 = \|x_0, z\| f_1$ and $F_2 = \|x_0, z\| f_2$. Then $F_1(x_0, z) = \|x_0, z\|^2$; $\|F_1\| = \|x_0, z\|$ and $F_2(x_0, z) = \|x_0, z\|^2$; $\|F_2\| = \|x_0, z\|$, implying that $J_z(x_0, z)$ is not singleton. So (2) $\Rightarrow$ (1). □

Lemma 4.3. *Let $(X, \|.,.\|)$ be a linear 2-normed space and $F : X \times X \to \mathbb{R}$ be a bounded linear 2-functional on X. Let $c \neq 0$ be in X. Consider the quotient space $X_c = X/V(c)$ with norm given by $\|(x)_c\| = \|x, c\|$, where $(x)_c = x + V(c) \in X_c$. Define $F_c : X_c \to \mathbb{R}$ by $F_c((x)_c) = F(x, c)$. Then F_c is a bounded linear functional on X_c.*

Proof. First of all we show that F_c is well defined. Let $(x)_c = (y)_c$. Then

$$\begin{aligned} x - y \in V(c) &\Rightarrow F(x - y, c) = F(\alpha c, c) \text{ for some } \alpha \in \mathbb{R} \\ &\Rightarrow \|F(x - y, c)\| \leqslant \|F\| \|\alpha c, c\| = 0 \\ &\Rightarrow F(x, c) = F(y, c) \\ &\Rightarrow F_c((x)_c) = F_c((y)_c). \end{aligned}$$

F_c is linear because:

$$\begin{aligned} F_c((x)_c + (y)_c) &= F_c((x+y)_c) = F(x+y,c) \\ &= F(x,c) + F(y,c) = F_c((x)_c) + F_c((y)_c) \\ \text{and } \; F_(\alpha(x)_c) &= F_c((\alpha x)_c) = F(\alpha x, c) \\ &= \alpha F(x,c) \quad\; = \alpha F_c((x)_c). \end{aligned}$$

F_c is bounded for:

$$\begin{aligned} \|F_c\| &= \sup\{|F_c((x)_c)| : \|(x)_c)\| \leqslant 1\} \\ &= \sup\{|F(x,c)| : \|x,c\| \leqslant 1\} \\ &\leqslant \|F\|. \end{aligned}$$

□

Lemma 4.4. *Let $(X, \|.,.\|)$ be a linear 2-normed space and $0 \neq c \in X$. Suppose $x \in X$ is such that $\|x,c\| \neq 0$. If $X \times V(c)$ is smooth at (x,c) then the normed space X_c is smooth at $(x)_c$.*

Proof. Suppose X_c is not smooth at $(x)_c$. Then there exists F_c and G_c in X_c^* such that $F_c \neq G_c$; $\quad F_c((x)_c) = \|(x)_c)\|, \|F_c\| = 1$ and $G_c((x)_c) = \|(x)_c)\|, \|G_c\| = 1$ with $\|x,c\| \neq 0$.

Define a bounded linear 2-functional F on $X \times V(c)$ by

$$F(x,c) = F_c((x)_c)$$

$$\text{Then} \quad F(x,c) = \|(x)_c\| = \|x,c\|.$$

Hence $\|F\| = \inf\{k : \frac{|F(x,c)|}{\|x,c\|} \leqslant k, \ \forall x \in X\} \leqslant 1$. As in the proof of lemma 4.3 we can prove that $\|F_c\| \leqslant \|F\|$. Thus we have $\|F\| = 1$.

Similarly we can find a $G \in X_c^{\dagger}$ corresponding to G_c satisfying

$$G(x,c) = \|(x)_c\| \text{ and } \|G\| = 1.$$

Which contradicts the smoothness of $X \times V(c)$ at (x,c). Therefore X_c is smooth at $(x)_c$. □

The lemma follows, justifies the convergence of sequence of certain sections of normalized duality mapping, if the underlined space is smooth.The proof goes in line with the proof provided in [8].

Lemma 4.5. *Let $(X, \|.,.\|)$ be a linear 2-normed space and $0 \neq c \in X$. Let $X \times V(c)$ be smooth and (x_n) be a sequence in X such that $\|x_n, c\| = 1 \ \forall n \in \mathbb{N}$. Let (F_n) be a sequence in $X_c^{\dagger}$, with $F_n(x_n,c) = \|(x_n,c)\|^2 = 1$ and $\|F_n\| = 1$. If $\|x_n - x_0, c\| \to 0$ as $n \to \infty$ then for all $x \in X$, $(F_n(x,c)) \to F(x,c)$ as $n \to \infty$, where $\|x_0, c\| = 1$ and $F \in X_c^{\dagger}$ which satisfies $F(x_0,c) = \|x_0,c\|^2 = 1$ and $\|F\| = 1$.*

Proof. Consider $X_c = X/V(c)$. As $X \times V(c)$ is smooth we have by lemma 4.4, X_c is smooth. Define $F_{n_c} : X_c \to \mathbb{R}$ by $F_{n_c}((x)_c) = F_n(x,c) \ \forall n \in \mathbb{N}$. Then by lemma 4.3, (F_{n_c}) is a sequence of bounded linear functionals on the smooth normed space X_c satisfying $F_{n_c}((x_n)_c) = \|(x_n)_c\|^2 = 1$ and $\|F_{n_c}\| = \|(x_n)_c\| = 1$ and $((x_n)_c)$ is sequence in X_c such that $(x_n)_c \xrightarrow{\|.\|_c} (x_0)_c$. We claim that $F_{n_c} \to F_c$ in the weak-star topology $\sigma(X_c^*, X_c)$ of X_c^*, where $F_c \in X_c^*$ satisfies $F_c((x_0)_c) = \|(x_0)_c\|^2 = 1$ and $\|F_c\| = \|(x_0)_c\| = 1$.

We assume the contrary that (F_{n_c}) does not converge to F_c in $\sigma(X_c^*, X_c)$. Then there exists a neighborhood U of F_c in $\sigma(X_c^*, X_c)$ such that the exterior of U contains infinite number of terms of the sequence (F_{n_c}). We denote these terms by $F_{n_{k_c}},\ k \in \mathbb{N}$. Since the unit ball in the dual space X_c^* is $\sigma(X_c^*, X_c)$-compact (by the Banach- Alaoglu theorem), $(F_{n_{k_c}})$ has a sub sequence $(F_{n_{q_c}})$ which converges to linear functional G_c in the topology $\sigma(X_c^*, X_c)$, with $\|G_c\| \leqslant 1$. Also we have,

$$\begin{aligned} |G_c((x_0)_c) - 1| &= |G_c((x_0)_c) - F_{n_{q_c}}((x_{n_q})_c)| \\ &\leqslant |G_c((x_0)_c) - F_{n_{q_c}}((x_0)_c)| + |F_{n_{q_c}}((x_0)_c) - F_{n_{q_c}}((x_{n_q})_c)| \\ &\leqslant |G_c((x_0)_c) - F_{n_{q_c}}((x_0)_c)| + \|F_{n_{q_c}}\|\|(x_0)_c - (x_{n_q})_c\| \\ &= |G_c((x_0)_c) - F_{n_{q_c}}((x_0)_c)| + \|(x_0)_c - (x_{n_q})_c\|. \end{aligned}$$

As $q \to 0$ the right side tends to 0, and so $G_c((x_0)_c) = 1$. Therefore $\|G_c\| = 1$. Thus G_c and F_c are two bounded functionals on X_c satisfying the conditions $F_c((x_0)_c) = \|(x_0)_c\|^2$; $\|F_c\| = \|(x_0)_c\|$ and $G_c((x_0)_c) = \|(x_0)_c\|^2$; $\|G_c\| = \|(x_0)_c\|$, we infer that they are in $J((x_0)_c)$. As X_c is smooth by theorem 2.1, $J((x_0)_c)$ is singleton and so $F_c = G_c$. Which contradicts the existence of the neighborhood U. Hence $(F_{n_c}) \to F_c$ in $\sigma(X_c^*, X_c)$.

$$\begin{aligned} &i.e, \quad F_{n_c}((x)_c) \to F_c((x)_c) \quad \forall\ (x)_c \in X_c \\ &\text{or} \quad F_n(x,c) \to F(x,c) \quad \forall x \in X \end{aligned}$$

Where $F \in X_c^\dagger$ is such that$F((x_0, c) = \|(x_0, c\|^2 = 1$ and $\|F\| = \|(x_0, c\| = 1$. □

The following remark is an immediate consequence of the above theorem.

Remark 4.6. Let$(X, \|.,.\|)$ be a linear 2-normed space and $X \times V(z)$ be smooth. If (x_n) is a sequence in X with the property that $\|x_n, z\| = 1$ and $\|x_n - x_0, z\| \to 0$, then we have $\tilde{J}_z(x_n, z)(x, z) \to \tilde{J}_z(x_0, z)(x, z)$ for all $x \in X$.

Theorem 4.7. *Let $(X, \|.,.\|)$ be a linear 2-normed space and $(x_0, z) \in S_z(X)$. If $X \times V(z)$ is smooth, then the Gâteaux derivative of the function $x \mapsto \|x, z\|$ at x_0 exists in any direction $y \in X$.*

Proof. Assume that $X \times V(z)$ is smooth. Let (x_n) be a sequence in X such that $\|x_n, z\| = 1\ \forall n \in \mathbb{N}$ and $\|x_n - x_0, z\| \to 0$ as $n \to \infty$. Let $\tilde{J}_z$ be a section of J_z. Then we have $\tilde{J}_z(x_n, z)(x_n, z) = \|x_n, z\|^2 = 1$ and $\|\tilde{J}_z(x_n, z)\| = \|x_n, z\| = 1$.

By remark 4.6 for all $x \in X$, we have

$$\tilde{J}_z(x_n, z)(x, z) \to \tilde{J}_z(x_0, z)(x, z) \quad \text{as } \|x_n - x_0, z\| \to 0 \tag{1}$$

For $t \neq 0$ and $y \in X$ we have,

$$\begin{aligned} \|x_0 + ty, z\| - \|x_0, z\| &= \frac{1}{\|x_0, z\|}\left(\|x_0, z\|\|x_0 + ty, z\| - \|x_0, z\|^2\right) \\ &\geqslant \frac{1}{\|x_0, z\|}\left(\tilde{J}_z(x_0, z)(x_0 + ty, z) - \|x_0, z\|^2\right) \\ &(\because\ |\tilde{J}_z(x_0, z)(x_0 + ty, z)| \leqslant \|\tilde{J}_z(x_0, z)\|\|x_0 + ty, z\|) \\ &= \frac{t}{\|x_0, z\|}\tilde{J}_z(x_0, z)(y, z) \end{aligned}$$

$$\text{Then for } t > 0, \quad \frac{1}{\|x_0, z\|}\tilde{J}_z(x_0, z)(y, z) \leqslant \frac{\|x_0 + ty, z\| - \|x_0, z\|}{t} \tag{2}$$

On the other hand we have,

$$\begin{aligned}
\frac{\|x_0 + ty, z\| - \|x_0, z\|}{t} &= \frac{1}{t\|x_0 + ty, z\|}\left(\|x_0 + ty, z\|^2 - \|x_0 + ty, z\|\|x_0, z\|\right) \\
&= \frac{1}{t\|x_0 + ty, z\|}\left(\tilde{J}_z(x_0 + ty, z)(x_0 + ty, z) - \|x_0 + ty, z\|\|x_0, z\|\right) \\
&= \frac{1}{t\|x_0 + ty, z\|}\left(\tilde{J}_z(x_0 + ty, z)(x_0, z) + t\tilde{J}_z(x_0 + ty, z)(y, z) - \|x_0 + ty, z\|\|x_0, z\|\right) \\
&\leqslant \frac{1}{t\|x_0 + ty, z\|}t\tilde{J}_z(x_0 + ty, z)(y, z) \\
&(\because \ |\tilde{J}_z(x_0 + ty, z)(x_0, z)| \leqslant \|\tilde{J}_z(x_0 + ty, z)\|\|x_0, z\|)
\end{aligned}$$

$$i.e, \ \frac{\|x_0 + ty, z\| - \|x_0, z\|}{t} \leqslant \tilde{J}_z(\frac{x_0 + ty}{\|x_0 + ty, z\|}, z)(y, z) \text{ (By theorem 3.3)} \tag{3}$$

From (2) and (3) we have

$$\tilde{J}_z(x_0, z)(y, z) \leqslant \frac{\|x_0 + ty, z\| - \|x_0, z\|}{t} \leqslant \tilde{J}_z(\frac{x_0 + ty}{\|x_0 + ty, z\|}, z)(y, z) \tag{4}$$

Let $x_t = \frac{x_0+ty}{\|x_0+ty,z\|}$. Then $\|x_t, z\| = 1$ and $x_t \to x_0$ as $t \to 0$. By taking $t \to 0$ and using (1), (4) becomes,

$$\tilde{J}_z(x_0, z)(y, z) \leqslant \lim_{t\to 0^+} \frac{\|x_0 + ty, z\| - \|x_0, z\|}{t} \leqslant \tilde{J}_z(x_0, z)(y, z)$$

The right Gâteaux derivative of the function $x \mapsto \|x, z\|$, at x_0 in the direction of y is given by

$$D^1_{+[x_0,z]}(y) = \lim_{t\to 0^+} \frac{\|x_0 + ty, z\| - \|x_0, z\|}{t} = \tilde{J}_z(x_0, z)(y, z)$$

The left Gâteaux derivative of the function $x \mapsto \|x, z\|$, at x_0 in the direction of y is then given by,

$$\begin{aligned}
D^1_{-[x_0,z]}(y) &= \lim_{t\to 0^-} \frac{\|x_0 + ty, z\| - \|x_0, z\|}{t} \\
&= -\lim_{t\to 0+} \frac{\|x_0 - ty, z\| - \|x_0, z\|}{t} \\
&= -D^1_{+[x_0,z]}(-y) \\
&= -\tilde{J}_z(x_0, z)(-y, z) \\
&= \tilde{J}_z(x_0, z)(y, z) \\
&= D^1_{+[x_0,z]}(y).
\end{aligned}$$

Therefore the Gâteaux derivative of the function $x \mapsto \|x, z\|$ at x_0 in any direction y exists and the derivative is given by

$$D^1_{[x_0,z]}(y) = \tilde{J}_z(x_0, z)(y, z).$$

This complets the proof. $\square$

Theorem 4.8. *Let $(X, \|.,.\|)$ be a linear 2-normed space and $0 \neq z \in X$. If the Gâteaux derivative of the function $x \mapsto \|x,z\|$ at $x \in X$ with $\|x,z\| \neq 0$, in any direction exists, then the normalized duality mapping J_z at (x,z) is singleton.*

Proof. For $\|x,z\| \neq 0$, consider $(x_0, z) = (\frac{x}{\|x,z\|}, z)$. Then we have $\|x_0, z\| = 1$. Assume that the Gâteaux derivative of the function $x \mapsto \|x,z\|$ at x_0 exist, in any direction $y \in X$. For $t > 0$ and for every section $\tilde{J}_z$ of the normalized duality mapping J_z, we have from the inequality (2) in the proof of the theorem 4.7 that

$$\tilde{J}_z(x_0,z)(y,z) \leqslant \frac{\|x_0 + ty, z\| - \|x_0, z\|}{t}$$

Which implies for $s < 0$,

$$\frac{\|x_0 + sy, z\| - \|x_0, z\|}{s} \leqslant \tilde{J}_z(x_0,z)(y,z)$$

As the Gâteaux derivative of the function $x \mapsto \|x,z\|$ exists at x_0, from the above two inequalities we get the Gâteaux derivative of the function $x \mapsto \|x,z\|$ at x_0, in the direction of y,

$$D^1_{[x_0,z]}(y) = \tilde{J}_z(x_0,z)(y,z). \tag{4}$$

Suppose $J_z(x_0,z)$ contains two distinct functionals F_1 and F_2 and $\tilde{J}^1_z$ and $\tilde{J}^2_z$ be the sections of J_z such that $\tilde{J}^1_z(x_0,z) = F_1$ and $\tilde{J}^2_z(x_0,z) = F_2$. Then from the relation (4) above we have $\tilde{J}^1_z(x_0,z)(y,z) = \tilde{J}^2_z(x_0,z)(y,z)$

$$i.e, \quad F_1(y,z) = F_2(y,z) \quad \forall y \in X. \tag{5}$$

As $F_1, F_2 \in X^{\dagger}_z$, the above equation implies $F_1 = F_2$. Thus $J_z(x_0,z)$ is singleton. □

Corollary 4.9. *Let $(X, \|.,.\|)$ be a linear 2-normed space and $z \in X$. Then the following are equivalent:*
(i). $X \times V(z)$ is smooth.
(ii). The normalized duality mapping J_z at (x,z) is singleton for each $x \in X$.
(iii). The function $x \mapsto \|x,z\|$ Gâteaux differentiable at any point $x \in X$ with $\|x,z\| \neq 0$, in any direction.

Proof. Let $X \times V(z)$ be smooth and suppose $x \in X$ such that $\|x,z\| \neq 0$. Consider $(x_0,z) = (\frac{x}{\|x,z\|}, z)$. Then we have $\|x_0, z\| = 1$. By theorem 4.2, we have the following are equivalent:
(1) $X \times X$ is smooth at (x_0,z);
(2) the normalized duality mapping J_z at(x_0,z) is singleton.

By theorem 4.7, (2) implies that the Gâteaux derivative of the function $x \mapsto \|x,z\|$ at x_0 exist, in any direction.

The implication of (ii) by (iii) follows from theorem 4.8.

The equivalence of (i), (ii) and (iii) are then followed from the facts that $x \in X$ with $\|x,z\| \neq 0$ is arbitrary, $(x_0,z) = (\frac{x}{\|x,y\|}, y)$ and $\|x,z\| J_z(x_0,z) = J_z(x,z)$. □

Theorem 4.10. *Assume $(X, \|.,.\|)$ be a linear 2-normed space and $z \in X$. Then the following are equivalent:*
(i). $X \times V(z)$ is smooth.

(ii). For any section $\tilde{J}_z$ of the normalized duality mapping J_z on $X \times V(z)$ and $\forall x \in X$ with $\|x, z\| \neq 0$, we have

$$\lim_{t \to 0} \tilde{J}_z(x + ty, z)(y, z) = \tilde{J}_z(x, z)(y, z) \quad \forall y \in X \tag{1}$$

(iii). For all $x \in X$ with $\|x, z\| \neq 0$, we have,

$$\lim_{t \to 0} \frac{\tilde{J}_z(x + ty, z)(x, z) - \tilde{J}_z(x, z)(x, z)}{t} = \tilde{J}_z(x, z)(y, z) \tag{2}$$

Proof. Using the inequalities (4) in the proof of the theorem 4.7, for all $x \in X$ with $\|x, z\| \neq 0$ and $t > 0$ we have,

$$\tilde{J}_z(\frac{x}{\|x, z\|}, z)(y, z) \leqslant \frac{\|x + ty, z\| - \|x, z\|}{t} \leqslant \tilde{J}_z(\frac{x + ty}{\|x + ty, z\|}, z)(y, z) \; \forall y \in X \tag{3}$$

In the inequality (2) in the proof of the theorem 4.7, changing x_0 by $x + ty$ we get the inequality

$$\tilde{J}_z(\frac{x + ty}{\|x + ty, z\|}, z)(y, z) \leqslant \frac{\|x + 2ty, z\| - \|x + ty, z\|}{t} \tag{4}$$

From (3) and (4) we have,

$$\begin{aligned}
\lim_{t \to 0^+} \frac{\|x + ty, z\| - \|x, z\|}{t} &\leqslant \lim_{t \to 0^+} \tilde{J}_z(\frac{x + ty}{\|x + ty, z\|}, z)(y, z) \\
&\leqslant \lim_{t \to 0^+} \frac{\|x + 2ty, z\| - \|x + ty, z\|}{t} \\
&= \lim_{t \to 0^+} 2.\frac{\|x + 2ty, z\| - \|x, z\|}{2t} \\
&\quad - \lim_{t \to 0^+} \frac{\|x + ty, z\| - \|x, z\|}{t} \\
&= \lim_{s \to 0^+} 2.\frac{\|x + sy, z\| - \|x, z\|}{s} \\
&\quad - \lim_{t \to 0^+} \frac{\|x + ty, z\| - \|x, z\|}{t} \\
&= \lim_{t \to 0^+} \frac{\|x + ty, z\| - \|x, z\|}{t}.
\end{aligned}$$

Therefore we get,

$$\lim_{t \to 0^+} \frac{\|x + ty, z\| - \|x, z\|}{t} = \lim_{t \to 0^+} \tilde{J}_z(\frac{x + ty}{\|x + ty, z\|}, z)(y, z) \tag{5}$$

Assume (i). Then the Gâteaux derivative of the function $x \mapsto \|x, z\|$ at x exists in any direction $y \in X$ and so (5) becomes,

$$\begin{aligned}
D^1_{[x,z]}(y) &= \lim_{t \to 0^+} \tilde{J}_z(\frac{x + ty}{\|x + ty, z\|}, z)(y, z) \\
i.e, \quad \|x, z\| \; D^1_{[x,z]}(y) &= \lim_{t \to 0^+} \tilde{J}_z(x + ty, z)(y, z)
\end{aligned} \tag{6}$$

Replacing y by $-y$ in (6), we get

$$\|x, z\| \; D^1_{[x,z]}(-y) = \lim_{t \to 0^+} \tilde{J}_z(x - ty, z)(-y, z)$$

$$= -\lim_{t\to 0^+} \tilde{J}_z(x-ty,z)(y,z)$$

$$\therefore \lim_{t\to 0^+} \tilde{J}_z(x-ty,z)(y,z) = \|x,z\|\ D^1_{[x,z]}(y)$$

$$(\because D^1_{[x,z]}(-y) = -D^1_{[x,z]}(y))$$

$$i.e, \quad \lim_{s\to 0^-} \tilde{J}_z(x+sy,z)(y,z) = \|x,z\|\ D^1_{[x,z]}(y) \tag{7}$$

It follows from (6) and (7) that $\lim_{t\to 0} \tilde{J}_z(x+ty,z)(y,z)$ exists and that

$$\lim_{t\to 0} \tilde{J}_z(x+ty,z)(y,z) = \|x,z\|\ D^1_{[x,z]}(y) \quad \forall y \in X \tag{8}$$

Also from the inequalities (4) in the proof of the theorem 4.7, for all $x \in X$ with $\|x,z\| \neq 0$ and $s < 0$ and $t > 0$ we have,

$$\frac{\|x+ty,z\| - \|x,z\|}{s} \leqslant \tilde{J}_z(\frac{x}{\|x,z\|},z)(y,z) \leqslant \frac{\|x+ty,z\| - \|x,z\|}{t} \tag{9}$$

Letting $s \to 0^-$ and $t \to 0^+$ in (9) and using the smoothness of $X \times V(z)$ at (x,z) we have,

$$\|x,z\|\ D^1_{[x,z]}(y) = \tilde{J}_z(x,z)(y,z) \tag{10}$$

From (8) and (10)we get

$$\lim_{t\to 0} \tilde{J}_z(x+ty,z)(y,z) = \tilde{J}_z(x,z)(y,z) \quad \forall y \in X$$

Thus (i) implies (ii).

Next we prove (ii) implies (i). From (3) we have,

$$\begin{aligned}\lim_{t\to 0^+} \tilde{J}_z(\frac{x}{\|x,z\|},z)(y,z) &\leqslant \lim_{t\to 0^+} \frac{\|x+ty,z\| - \|x,z\|}{t} \\ &\leqslant \lim_{t\to 0^+} \tilde{J}_z(\frac{x+ty}{\|x+ty,z\|},z)(y,z),\ \forall y \in X \end{aligned} \tag{11}$$

Using (1), (11) becomes

$$\begin{aligned}\tilde{J}_z(\frac{x}{\|x,z\|},z)(y,z) &\leqslant D^1_{+_{[x,z]}}(y) \\ &\leqslant \tilde{J}_z(\frac{x}{\|x,z\|},z)(y,z) \\ \Rightarrow \quad D^1_{+_{[x,z]}}(y) &= \tilde{J}_z(\frac{x}{\|x,z\|},z)(y,z) \quad \forall y \in X \end{aligned} \tag{12}$$

Also using (12) and bilinearity of $\tilde{J}_z$ we have,

$$\begin{aligned}D^1_{-_{[x,z]}}(y) &= -D^1_{+_{[x,z]}}(-y) \\ &= -\tilde{J}_z(\frac{x}{\|x,z\|},z)(-y,z) \\ &= \tilde{J}_z(\frac{x}{\|x,z\|},z)(y,z) \\ &= D^1_{+_{[x,z]}}(y)\end{aligned}$$

That is $D^1_{[x,z]}(y)$ exists for all $y \in X$ and

$$D^1_{[x,z]}(y) = \tilde{J}_z(\frac{x}{\|x,z\|}, z)(y,z) \tag{13}$$

Since $x \in X$ is arbitrary with $\|x,z\| \neq 0$ by the corollary 4.9, $X \times V(z)$ is smooth.

Next we claim that (i) implies (iii). We have, for $x \in X$ and $t \neq 0$

$$\begin{aligned}\frac{\|x+ty,z\|^2 - \|x,z\|^2}{t} &= \frac{\tilde{J}_z(x+ty,z)(x+ty,z) - \tilde{J}_z(x,z)(x,z)}{t}\\ &= (\frac{\tilde{J}_z(x+ty,z) - \tilde{J}_z(x,z)}{t})(x,z) + \tilde{J}_z(x+ty,z)(y,z) \end{aligned}\tag{14}$$

Assume that $X \times V(z)$ is smooth at (x,z). Then

$$\begin{aligned}\lim_{t\to 0}(\frac{\tilde{J}_z(x+ty,z) - \tilde{J}_z(x,z)}{t})(x,z) &= \lim_{t\to 0}\frac{\|x+ty,z\|^2 - \|x,z\|^2}{t}\\ &\quad - \lim_{t\to 0}\tilde{J}_z(x+ty,z)(y,z)\\ = \lim_{t\to 0}(\|x+ty,z\| + \|x,z\|).\lim_{t\to 0}&\frac{\|x+ty,z\| - \|x,z\|}{t} - \tilde{J}_z(x,z)(y,z)\\ &= 2\|x,z\|D^1_{[x,z]}(y) - \tilde{J}_z(x,z)(y,z)\\ &= \tilde{J}_z(x,z)(y,z) \quad \text{from(13)}.\end{aligned}$$

Thus (iii) holds. Conversely assume that (iii) holds. Now

$$\begin{aligned}2\|x,z\|D^1_{+[x,z]}(y) &= \lim_{t\to 0^+}\frac{\|x+ty,z\|^2 - \|x,z\|^2}{t}\\ &= \lim_{t\to 0}(\frac{\tilde{J}_z(x+ty,z) - \tilde{J}_z(x,z)}{t})(x,z)\\ &\quad + \lim_{t\to 0^+}\tilde{J}_z(x+ty,z)(y,z)\\ &= \tilde{J}_z(x,z)(y,z) + \lim_{t\to 0^+}\tilde{J}_z(x+ty,z)(y,z)\\ &= \tilde{J}_z(x,z)(y,z) + \|x,z\|D^1_{+[x,z]}(y), \quad \text{using (6)}.\\ \therefore \quad \|x,z\|D^1_{+[x,z]}(y) &= \tilde{J}_z(x,z)(y,z).\end{aligned}$$

Also

$$\begin{aligned}\|x,z\|D^1_{-[x,z]}(y) = \ & -\|x,z\|D^1_{+[x,z]}(-y)\\ &= -\tilde{J}_z(x,z)(-y,z)\\ &= \tilde{J}_z(x,z)(y,z)\\ &= \|x,z\|D^1_{+[x,z]}(y)\\ \therefore \quad \|x,z\|D^1_{-[x,z]}(y) &= \|x,z\|D^1_{+[x,z]}(y) \quad \forall y \in X.\end{aligned}$$

Thus the 2-norm is Gâteaux derivative of the function $x \mapsto \|x,z\|$ at x, in any direction $y \in X$. Invoking the corollary 4.9 again, we could see that $X \times V(z)$ is smooth. $\square$

We follow the definitions of smoothness of a linear 2-normed space, as given in [11] and characterize the smoothness in terms of normalized duality mapping on $X \times V(z)$ and Gâteaux differentiability of the semi norm $x \mapsto \|x, z\|$.

Definition 4.11. (see [11]) A linear 2-normed space $(X, \|.,.\|)$ is said to be smooth for $x \neq 0$ in X and $z \notin V(x)$ the 2-norm $\|.,.\|$ is Gâteaux differentiable at (x, z) in any direction of y.

Lemma 4.12. *(cf [11]) Let $(X, \|.,.\|)$ be a linear 2-normed space and $(X_z, \|.\|_z)$ for fixed non zero element z in X. Then the 2-norm is Gâteaux differentiable at (x, z) in the direction of y if the norm $\|.\|_z$ is Gâteaux differentiable at $(x)_z$ in any direction $(y)_z$.*

From the above lemma it follows that the 2-norm $\|.,.\|$ is Gâteaux differentiable at (x, z) in any direction of y if and only if the semi norm $x \mapsto \|x, z\|$ is Gâteaux differentiable at x in any direction y. Thus we have the following theorem:

Theorem 4.13. *Let $(X, \|.,.\|)$ be a linear 2-normed space. The following are equivalent:*

1. *X is smooth.*
2. *$J_z(x, z)$ is singleton for all $x, z \in X$.*
3. *The semi norm $x \mapsto \|x, z\|$ is differentiable in any direction $y \in X$ for all $z \notin V(x)$.*

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