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On various types of 3-GDDs

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Abstract In this paper, we define group divisible t-designs shortly t-GDDs of various types. We also construct 3-GDDs of various types and give some necessary conditions of them.

Key Words group divisible design, group type

MSC 2010 20D06

1 Introduction

Group divisible design or GDD has immense applications for constructing of new designs. It is also usefull in Statistics. So the research of constructing various types of GDDs is increasing day by day.

A group divisible design GDD(n, m, k; λ_1, λ) is a collection of $k-$subset, called blocks, of a set V of cardinality mn, where the set V is partioned into m groups of size n, such that each pair of elements from the same group occurs in exactly λ_1 blocks, and each pair of elements from the different groups occurs in exactly λ blocks. The symbol λ_1 occuring in the sane group is konown as first associate or first index and the second symbol λ is known as second associate or second index.

In 2002, Mohacsy, D.K. Ray-Chaudhuri [7] posed an existence theorem for group divisible design of large oder. In 2003, Mohacsy, D.K. Ray-Chaudhuri [6] constructed group divisible $t-$designs with strength $t \geqslant 2$ and index unity. In 2004, Hurd and Sarvate [10] defined and studied odd and even GDDs with two groups and block size four. In 2007, Henson, sarvate and Hurd [2] defined and studied odd, even and mixed GDDs with three groups and block size four. In 2010, Zhu, Ge [8] constructed odd, even and mixed GDDs with three groups and block size four. Later on, Mohacsy [5] shown the asymptotic existence of GDDs of large order with index one, also Hurd and Sarvate [9] constructed and discussed about the group divisible designs with three unequal groups and larger first index. Recently, Mohacsy [4] discussed about a large order asymptotic existence theorem for group divisible 3-designs with index one.

Mohacsy [5] define $t-$GDD with block sizes from a set of positive integers K. In this paper, we define new $t-$GDD and a mixed group divisible $t-$design($t-$MGDD) with fixed block size k and fixed group size n; also with block sizes from a set of positive integers K and group sizes from a set of positive integers N. Here we present constructions and results about $3-$GDD and a $3-$MGDD with m groups of size n and block size four. We also illustrate all the new designs with examples.

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Definition 1.1. *Let G and K be sets of positive integers and λ be a positive integer. A group divisible design(GDD) is an ordered triple $(X, \mathcal{G}, \mathcal{B})$, where X is a finite set of cardinality v, $\mathcal{G}$ is a partition of X into groups whose sizes lies in G, and $\mathcal{B}$ is a set of subsets of X, called blocks set such that*

1. *every pair of distinct elements of X from distinct groups occurs in exactly λ blocks, and*
2. *$|\mathcal{G}| > 1$.*

The GDD defined above is denoted by $(K, \lambda)-GDD$ if $\lambda > 1$ and $K-GDD$ if $\lambda = 1$. A group type in a GDD is of the form $g_1^{u_1} g_2^{u_2} \ldots g_i^{u_i}$ in which there are u_i gruops of size g_i, $i = 1, 2, 3, \ldots, s$.

Example 1.1.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16				

Let $X = \{1, 2, \ldots, 15, 16\}$ and $\mathcal{B}$ contains the following blocks:
$\{\{1,6,11,16\}, \{2,7,12,16\}, \{3,8,13,16\}, \{4,9,14,16\}, \{5,10,15,16\}, \{1,2,10\}, \{1,3,7\}, \{1,4,8\}, \{1,5,9\}, \{2,3,9\}, \{2,4,6\}, \{2,5,8\}, \{3,4,10\}, \{3,5,6\}, \{4,5,7\}, \{6,7,15\}, \{6,8,12\}, \{6,9,13\}, \{6,10,14\}, \{7,8,14\}, \{7,9,11\}, \{7,10,13\}, \{8,9,15\}, \{8,10,11\}, \{9,10,12\}, \{11,12,5\}, \{11,13,2\}, \{11,14,4\}, \{11,15,4\}, \{12,13,4\}, \{12,14,1\}, \{12,15,3\}, \{13,14,5\}, \{13,15,1\}, \{14,15,2\}\}$.
So $(X, \mathcal{B})$ is a $\{3,4\}-GDD$ of the type $1^1 5^3$.

2 Some New $t-$GDDs

Definition 2.1. *Let n, m, λ be positive integers. A $t-GDD(n,\ m,\ k;\ \lambda)$ is an ordered triple $(X, \mathcal{G}, \mathcal{B})$ for $0 < t \leqslant k$, $0 < t \leqslant m$ where X is a finite set of cardinality mn, $\mathcal{G}$ is a partition of X into m groups of size n, and a collection $\mathcal{B}$ of $k-$element subsets of X, called blocks, such that*

1. *every t distinct elements of X from t distinct groups occurs in exactly λ blocks, and*
2. *$|\mathcal{G}| > t - 1$.*

Definition 2.2. *In a $t-GDD(n,\ m,\ k;\ \lambda)$ if $\lambda = 1$ then it is called Steiner GDD and is denoted by SGDD(t,k,n,m).*

2.1 Construction of $3-$GDD$(n,\ m,\ 4;\ \lambda)$:

Let X is a finite set of cardinality mn, $\mathcal{G}$ is a partition of X into m groups of size n, and a collection $\mathcal{B}$ of $4-$element subsets of X, called blocks, such that $\mathcal{B} = \{(i,p), (j,x), (l,q), (s,y)\}$, where vth element of uth group is denoted by (u,v), $1 \leqslant i < j < l < s \leqslant m$, $1 \leqslant p, q \leqslant n$, $0 \leqslant r \leqslant n-1$ and

$$\begin{cases} \text{x} = & \text{(p+r)(mod n)}, \quad \text{if } (p+r) \neq n; \\ \quad = & \text{n}, \quad \text{if } (p+r) = n, \end{cases}$$

and

$$\begin{cases} \text{y} = & \text{(q+r)(mod n)}, \quad \text{if } (q+r) \neq n; \\ = & \text{n}, \quad \text{if } (q+r) = n. \end{cases}$$

Theorem 2.1. $3-GDD(n,\ m,\ 4;\ \lambda)$ *satiesfies the necessary conditions*

1. $bk = vr$, *and*

2. $\lambda = \lceil \frac{b\ {}^{k}C_t}{{}^{v}C_t} \rceil$, *where the number of blocks* $b = n^3.\ {}^{m}C_k$ *and replication number* $r = n^2\ {}^{m-1}C_{k-1}$.

Theorem 2.2. *If* $k = m = 4$ *then it is* $SGDD(3,\ 4,\ n,\ m)$.

Example 2.1. *Let* $X = \{1, 2, \ldots, 20\}$, *and the groups of* $\mathcal{G}$ *are as follows*

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20.

The block set $\mathcal{B} = \{\{1,5,9,13\}, \{1,5,10,14\}, \{1,5,11,15\}, \{1,5,12,16\}, \{1,6,9,14\}, \{1,6,10,15\},$ $\{1,6,11,16\}, \{1,6,12,13\}, \{1,7,9,15\}, \{1,7,10,16\}, \{1,7,11,13\}, \{1,7,12,14\}, \{1,8,9,16\}, \{1,8,10,13\},$ $\{1,8,11,14\}, \{1,8,12,15\}, \{1,5,9,17\}, \{1,5,10,18\}, \{1,5,11,19\}, \{1,5,12,20\}, \{1,6,9,18\}, \{1,6,10,19\},$ $\{1,6,11,20\}, \{1,6,12,17\}, \{1,7,9,19\}, \{1,7,10,20\}, \{1,7,11,17\}, \{1,7,12,18\}, \{1,8,9,20\}, \{1,8,10,17\},$ $\{1,8,11,18\}, \{1,8,12,19\}, \{1,5,13,17\}, \{1,5,14,18\}, \{1,5,15,19\}, \{1,5,16,20\}, \{1,6,13,18\}, \{1,6,14,19\},$ $\{1,6,15,20\}, \{1,6,16,17\}, \{1,7,13,19\}, \{1,7,14,20\}, \{1,7,15,17\}, \{1,7,16,18\}, \{1,8,13,20\}, \{1,8,14,17\},$ $\{1,8,15,18\}, \{1,8,16,19\}, \{1,9,13,17\}, \{1,9,14,18\}, \{1,9,15,19\}, \{1,9,16,20\}, \{1,10,13,18\},$ $\{1,10,14,19\}, \{1,10,15,20\}, \{1,10,16,17\}, \{1,11,13,19\}, \{1,11,14,20\}, \{1,11,15,17\}, \{1,11,16,18\},$ $\{1,12,13,20\}, \{1,12,14,17\}, \{1,12,15,18\}, \{1,12,16,19\}, \{5,9,13,17\}, \{5,9,14,18\}, \{5,9,15,19\}, \{5,9,16,20\},$ $\{5,10,13,18\}, \{5,10,14,19\}, \{5,10,15,20\}, \{5,10,16,17\}, \{5,11,13,19\}, \{5,11,14,20\}, \{5,11,15,17\},$ $\{5,11,16,18\}, \{5,12,13,20\}, \{5,12,14,17\}, \{5,12,15,18\}, \{5,12,16,19\}, \{2,6,9,13\}, \{2,6,10,14\},$ $\{2,6,11,15\}, \{2,6,12,16\}, \{2,7,9,14\}, \{2,7,10,15\}, \{2,7,11,16\}, \{2,7,12,13\}, \{2,8,9,15\}, \{2,8,10,16\},$ $\{2,8,11,13\}, \{2,8,12,14\}, \{2,5,9,16\}, \{2,5,10,13\}, \{2,5,11,14\}, \{2,5,12,15\}, \{2,6,9,17\}, \{2,6,10,18\},$ $\{2,6,11,19\}, \{2,6,12,20\}, \{2,7,9,18\}, \{2,7,10,19\}, \{2,7,11,20\}, \{2,7,12,17\}, \{2,8,9,19\}, \{2,8,10,20\},$ $\{2,8,11,17\}, \{2,8,12,18\}, \{2,5,9,20\}, \{2,5,10,17\}, \{2,5,11,18\}, \{2,5,12,19\}, \{2,6,13,17\}, \{2,6,14,18\},$ $\{2,6,15,19\}, \{2,6,16,20\}, \{2,7,13,18\}, \{2,7,14,19\}, \{2,7,15,20\}, \{2,7,16,17\}, \{2,8,13,19\}, \{2,8,14,20\},$ $\{2,8,15,17\}, \{2,8,16,18\}, \{2,5,13,20\}, \{2,5,14,17\}, \{2,5,15,18\}, \{2,5,16,19\}, \{2,10,13,17\}, \{2,10,14,18\},$ $\{2,10,15,19\}, \{2,10,16,20\}, \{2,11,13,18\}, \{2,11,14,19\}, \{2,11,15,20\}, \{2,11,16,17\}, \{2,12,13,19\},$ $\{2,12,14,20\}, \{2,12,15,17\}, \{2,12,16,18\}, \{2,9,13,20\}, \{2,9,14,17\}, \{2,9,15,18\}, \{2,9,16,19\}, \{6,10,13,17\},$ $\{6,10,14,18\}, \{6,10,15,19\}, \{6,10,16,20\}, \{6,11,13,18\}, \{6,11,14,19\}, \{6,11,15,20\}, \{6,11,16,17\},$ $\{6,12,13,19\}, \{6,12,14,20\}, \{6,12,15,17\}, \{6,12,16,18\}, \{6,9,13,20\}, \{6,9,14,17\}, \{6,9,15,18\}, \{6,9,16,19\},$ $\{3,7,9,13\}, \{3,7,10,14\}, \{3,7,11,15\}, \{3,7,12,16\}, \{3,8,9,14\}, \{3,8,10,15\}, \{3,8,11,16\}, \{3,8,12,13\},$ $\{3,5,9,15\}, \{3,5,10,16\}, \{3,5,11,13\}, \{3,5,12,14\}, \{3,6,9,16\}, \{3,6,10,13\}, \{3,6,11,14\}, \{3,6,12,15\},$

$\{3,7,9,17\}$, $\{3,7,10,18\}$, $\{3,7,11,19\}$, $\{3,7,12,20\}$, $\{3,8,9,18\}$, $\{3,8,10,19\}$, $\{3,8,11,20\}$, $\{3,8,12,17\}$, $\{3,5,9,19\}$, $\{3,5,10,20\}$, $\{3,5,11,17\}$, $\{3,5,12,18\}$, $\{3,6,9,20\}$, $\{3,6,10,17\}$, $\{3,6,11,18\}$, $\{3,6,12,19\}$, $\{3,7,13,17\}$, $\{3,7,14,18\}$, $\{3,7,15,19\}$, $\{3,7,16,20\}$, $\{3,8,13,18\}$, $\{3,8,14,19\}$, $\{3,8,15,20\}$, $\{3,8,16,17\}$, $\{3,5,13,19\}$, $\{3,5,14,20\}$, $\{3,5,15,17\}$, $\{3,5,16,18\}$, $\{3,6,13,20\}$, $\{3,6,14,17\}$, $\{3,6,15,18\}$, $\{3,6,16,19\}$, $\{3,11,13,17\}$, $\{3,11,14,18\}$, $\{3,11,15,19\}$, $\{3,11,16,20\}$, $\{3,12,13,18\}$, $\{3,12,14,19\}$, $\{3,12,15,20\}$, $\{3,12,16,17\}$, $\{3,9,13,19\}$, $\{3,9,14,20\}$, $\{3,9,15,17\}$, $\{3,9,16,18\}$, $\{3,10,13,20\}$, $\{3,10,14,17\}$, $\{3,10,15,18\}$, $\{3,10,16,19\}$, $\{7,11,13,17\}$, $\{7,11,14,18\}$, $\{7,11,15,19\}$, $\{7,11,16,20\}$, $\{7,12,13,18\}$, $\{7,12,14,19\}$, $\{7,12,15,20\}$, $\{7,12,16,17\}$, $\{7,9,13,19\}$, $\{7,9,14,20\}$, $\{7,9,15,17\}$, $\{7,9,16,18\}$, $\{7,10,13,20\}$, $\{7,10,14,17\}$, $\{7,10,15,18\}$, $\{7,10,16,19\}$, $\{4,8,9,13\}$, $\{4,8,10,14\}$, $\{4,8,11,15\}$, $\{4,8,12,16\}$, $\{4,5,9,14\}$, $\{4,5,10,15\}$, $\{4,5,11,16\}$, $\{4,5,12,13\}$, $\{4,6,9,15\}$, $\{4,6,10,16\}$, $\{4,6,11,13\}$, $\{4,6,12,14\}$, $\{4,7,9,16\}$, $\{4,7,10,13\}$, $\{4,7,11,14\}$, $\{4,7,12,15\}$, $\{4,8,9,17\}$, $\{4,8,10,18\}$, $\{4,8,11,19\}$, $\{4,8,12,20\}$, $\{4,5,9,18\}$, $\{4,5,10,19\}$, $\{4,5,11,20\}$, $\{4,5,12,17\}$, $\{4,6,9,19\}$, $\{4,6,10,20\}$, $\{4,6,11,17\}$, $\{4,6,12,18\}$, $\{4,7,9,20\}$, $\{4,7,10,17\}$, $\{4,7,11,18\}$, $\{4,7,12,19\}$, $\{4,8,13,17\}$, $\{4,8,14,18\}$, $\{4,8,15,19\}$, $\{4,8,16,20\}$, $\{4,5,13,18\}$, $\{4,5,14,19\}$, $\{4,5,15,20\}$, $\{4,5,16,17\}$, $\{4,6,13,19\}$, $\{4,6,14,20\}$, $\{4,6,15,17\}$, $\{4,6,16,18\}$, $\{4,7,13,20\}$, $\{4,7,14,17\}$, $\{4,7,15,18\}$, $\{4,7,16,19\}$, $\{4,12,13,17\}$, $\{4,12,14,18\}$, $\{4,12,15,19\}$, $\{4,12,16,20\}$, $\{4,9,13,18\}$, $\{4,9,14,19\}$, $\{4,9,15,20\}$, $\{4,9,16,17\}$, $\{4,10,13,19\}$, $\{4,10,14,20\}$, $\{4,10,15,17\}$, $\{4,10,16,18\}$, $\{4,11,13,20\}$, $\{4,11,14,17\}$, $\{4,11,15,18\}$, $\{4,11,16,19\}$, $\{8,12,13,17\}$, $\{8,12,14,18\}$, $\{8,12,15,19\}$, $\{8,12,16,20\}$, $\{8,9,13,18\}$, $\{8,9,14,19\}$, $\{8,9,15,20\}$, $\{8,9,16,17\}$, $\{8,10,13,19\}$, $\{8,10,14,20\}$, $\{8,10,15,17\}$, $\{8,10,16,18\}$, $\{8,11,13,20\}$, $\{8,11,14,17\}$, $\{8,11,15,18\}$, $\{8,11,16,19\}\}$.

Hence $(X,\mathcal{G},\mathcal{B})$ *is a* $3-GDD(4,5,4;2)$ *which satisfies that* $b = n^3.\ {}^{m}C_k = 4^3.\ {}^{5}C_4 = 4^3.5 = 320$, $r = n^2\ {}^{m-1}C_{k-1} = 4^2\ {}^{5-1}C_{4-1} = 4^2\ {}^{4}C_3 = 4^2.4 = 64$ *and* $\lambda = \lceil \frac{b\ {}^{k}C_t}{{}^{v}C_t} \rceil = \lceil \frac{320\ {}^{4}C_3}{{}^{2}0C_3} \rceil = \lceil \frac{320.4.3!.17!}{20!} \rceil = \lceil \frac{64}{57} \rceil = 2$.

Definition 2.3. *Let* λ_1, λ *be two positive integers. A* $t-GDD(n,\ m,\ k;\ \lambda_1,\ \lambda)$ *is an ordered triple* $(X,\mathcal{G},\mathcal{B})$ *for* $0 < t \leqslant k$, $0 < t \leqslant n$, *where* X *is a finite set of cardinality* mn, $\mathcal{G}$ *is a partition of* X *into* m *groups of size* n, *and a collection* $\mathcal{B}$ *of* $k-$*element subsets of* X, *called blocks, such that*

1. *every* $t-$*distinct elements of* X *form same group of* $\mathcal{G}$ *occurs in exactly* λ_1 *blocks or every* $t-$*distinct elements of* X *from* $t-$*distinct groups occurs in exactly* λ *blocks, and*

2. $|\mathcal{G}| > t-1$.

2.2 Construction of $3-$GDD$(n,\ m,\ 4;\ \lambda_1,\ \lambda)$:

Let X is a finite set of cardinality mn, $\mathcal{G}$ is a partition of X into m groups of size $n(\geqslant 4)$, and a collection $\mathcal{B}$ of $4-$element subsets of X, called blocks, is of the following steps:

Step I : Taking all $4-$subset from each group.

Step II : Taking blocks are of the type $\{(i,p),(j,x),(l,q),(s,y)\}$, where vth element of uth group is denoted by (u,v), $1 \leqslant i < j < l < s \leqslant m$, $1 \leqslant p,q \leqslant n$, $0 \leqslant r \leqslant n-1$ and

$$\begin{cases} \text{x} = & \text{(p+r)(mod n)}, \quad \text{if } (p+r) \neq n; \\ \quad = & \text{n}, \quad \text{if } (p+r) = n, \end{cases}$$

and

$$\begin{cases} \text{y} = & \text{(q+r)(mod n)}, & \text{if } (q+r) \neq n; \\ = & \text{n}, & \text{if } (q+r) = n. \end{cases}$$

Theorem 2.3. *A* $3-GDD(n,\ m,\ 4;\ \lambda)$ *satiesfies the necessary conditions*

1. $bk = vr$, *and*
2. $\lambda = \lceil \frac{b\ {}^kC_t}{{}^vC_t} \rceil$, *where the number of blocks* $b = n^3.\ {}^mC_k + m\ {}^nC_4$ *and replication number* $r = n^2\ {}^{m-1}C_{k-1} + {}^{n-1}C_{k-1}$.

Example 2.2. *Let* $X = \{1, 2, \ldots, 20\}$, *and the groups of* $\mathcal{G}$ *are as follows*

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20.

The block set $\mathcal{B}_1 = \mathcal{B} \cup \mathcal{B}'$, *where we* $\mathcal{B}$ *from example 2.1 and* $\mathcal{B}' = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}, \{13, 14, 15, 16\}\}$.

Hence $(X, \mathcal{G}, \mathcal{B}_1)$ *is a* $3-GDD(4, 5, 4; 1, 2)$ *which satiesfies that* $b = n^3.\ {}^mC_k + m\ {}^nC_k = 4^3.{}^5C_4 + 5^4C_4 = 4^3.5 + 5 = 325$, *and* $r = n^2.\ {}^{m-1}C_{K-1} + {}^{n-1}C_{k-1} = 4^2.\ {}^{5-1}C_{4-1} + {}^{4-1}C_{4-1} = 16.4 + 1 = 65$ *and* $\lambda = \lceil \frac{b\ {}^kC_t}{{}^vC_t} \rceil = \lceil \frac{325\ {}^4C_4}{{}^{20}C_3} \rceil = \lceil \frac{325.4.3!.17!}{20!} \rceil = \lceil \frac{65}{57} \rceil = 2$.

Definition 2.4. *Let* $\lambda_1^{t_1}, \lambda_2^{t_2}, \ldots, \lambda_s^{t_s}, \lambda$ *be positive integers. A* $t-MGDD(n,\ m,\ k;\ \lambda_1^{t_1},\ \lambda_2^{t_2},\ \ldots,\ \lambda_s^{t_s},\ \lambda)$ *is an ordered triple* $(X, \mathcal{G}, \mathcal{B})$ *for* $0 < t \leqslant k$, $t < s$, $0 < t \leqslant n$, *where* X *is a finite set of cardinality* mn, $\mathcal{G}$ *is a partition of* X *into* m *groups of size* n, *and a collection* $\mathcal{B}$ *of* $k-$*element subsets of* X, *called blocks, such that*

1. *among every* $t-$*distinct elements of* X, $t_1,\ t_2,\ \ldots,\ t_s$ *elements form* s *groups occurs in exactly* $\lambda_1^{t_1}, \lambda_2^{t_2}, \ldots, \lambda_s^{t_s}$ *blocks respectively where* $t_1 + t_2 + \ldots + t_s = t$ *and* $\lambda_a^{t_p} = \lambda_b^{t_q}$ *if* $t_p = t_q$, a, b, p, q *are all distinct positive integers or every* $t-$*distinct elements of* X *from* $t-$*distinct groups of* $\mathcal{G}$ *occurs in exactly* λ *blocks, and*
2. $|\mathcal{G}| > t - 1$.

2.3 Construction of $3-$MGDD$(n,\ m,\ 4;\ \lambda_1^{t_1}, \lambda_2^{t_2},\ \ldots,\ \lambda_s^{t_s},\ \lambda)$:

Let X is a finite set of cardinality mn, $\mathcal{G}$ is a partition of X into m groups of size n, and a collection $\mathcal{B}$ of $4-$element subsets of X, called blocks, is of the following steps:

Step I : For m is an even number we take blocks are of the type $\{(x, p), (x, q), (x+1, p), (x+1, q)\}$ where x is odd positive integer and $1 \leqslant p < q \leqslant n$. Also for m is an odd number proceeding as even m we get last group is lone then we take the pair of groups as last-first, 2nd-3rd, $\ldots$, $(m-1)$th-mth.

Step II : Taking blocks are of the type $\{(i, p), (j, x), (l, q), (s, y)\}$, where vth element of uth group is denoted by (u, v), $1 \leqslant i < j < l < s \leqslant m$, $1 \leqslant p, q \leqslant n$, $0 \leqslant r \leqslant n-1$ and

$$\begin{cases} \text{x} = & \text{(p+r)(mod n)}, & \text{if } (p+r) \neq n; \\ = & \text{n}, & \text{if } (p+r) = n, \end{cases}$$

and

$$\begin{cases} \text{y} = & \text{(q+r)(mod n)}, & \text{if } (q+r) \neq n; \\ = & \text{n}, & \text{if } (q+r) = n. \end{cases}$$

Theorem 2.4. *A* $3-GDD(n,\ m,\ 4;\ \lambda)$ *satiesfies the necessary conditions*

1. $bk = vr$, *and*
2. $\lambda = \lceil \frac{b\ {}^kC_t}{{}^vC_t} \rceil$, *where the number of blocks* $b = n^3.\ {}^mC_k + m/2\ {}^nC_2$, *when* m *is even;* $b = n^3.\ {}^mC_k + m\ {}^nC_2$, *when* m *is odd and the replication number* $r = n^2.\ {}^{m-1}C_{k-1} + (n-1)k/4$, *when* m *is even;* $r = n^2.\ {}^{m-1}C_{k-1} + (n-1)k/2$, *when* m *is odd.*

Example 2.3. *Let* $X = \{1, 2, \ldots, 20\}$, *and the groups of* $\mathcal{G}$ *are as follows*

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20.

The block set $\mathcal{B}_2 = \mathcal{B} \cup \mathcal{B}''$, *where we get* $\mathcal{B}$ *from example 2.1 and* $\mathcal{B}'' = \{\{1,2,5,6\}, \{1,3,5,7\}, \{1,4,5,8\}, \{2,3,6,7\}, \{2,4,6,8\}, \{3,4,7,8\}, \{9,10,13,14\}, \{9,11,13,15\}, \{9,12,13,16\}, \{10,11,14,15\}, \{10,12,14,16\}, \{11,12,15,16\}, \{1,2,17,18\}, \{1,3,17,19\}, \{1,4,17,20\}, \{2,3,18,19\}, \{2,4,18,20\}, \{3,4,19,20\}, \{5,6,9,10\}, \{5,7,9,11\}, \{5,8,9,12\}, \{6,7,10,11\}, \{6,8,10,12\}, \{7,8,11,12\}, \{13,14,17,18\}, \{13,15,17,19\}, \{13,16,17,20\}, \{14,15,18,19\}, \{14,16,18,20\}, \{15,16,19,20\}\}$.

Hence $(X, \mathcal{G}, \mathcal{B}_2)$ *is a* $3-MGDD(4,5,4;70,2,2)$ *which satisfies that* $b = n^3.\ {}^mC_k + m.\ {}^nC_2 = 4^3.{}^5C_4 + 5^4C_2 = 4^3.5 + 5.6 = 320 + 30 = 350$, $r = n^2.\ {}^{m-1}C_{k-1} + (n-1)k/2 = 4^2.\ {}^{5-1}C_{4-1} + (4-1)4/2 = 4^2.4 + 3.2 = 70$ *and* $\lambda = \lceil \frac{b\ {}^kC_t}{{}^vC_t} \rceil = \lceil \frac{350\ {}^4C_3}{{}^20C_3} \rceil = \lceil \frac{350.4.3!.17!}{20!} \rceil = \lceil \frac{70}{57} \rceil = 2$.

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