South Asian Journal of Mathematics 2012, Vol.3 (1): 1 ∼ 7
www.sajm-online.com ISSN 2251-1512 RESEARCH ARTICLE

Common fixed point theorems on 2-Merger spaces for integral type mappings

Kamal Wadhwa ①, Jyoti Panthi ①, Ramakant Bhardwaj ②*

① Govt. Narmada Mahavidyalaya , Hoshangabad, India

② Truba Institute of Engineering and Information Technology, Bhopal, India
E-mail: rkbhardwaj100@gmail.com

Received: 11-20-2012; Accepted: 1-1-2013 *Corresponding author
The author (R.Bhardwaj) is thankful to MPCOST for funding the project No.2556.

Abstract In this paper we prove some common fixed point theorem from a 2-menger space into itself using integral type. It is also shown that the two composition of two such mapping is a constant mapping under certain conditions.

Key Words fixed point, common fixed point, menger space, contractive mapping, constant mapping

MSC 2010 46G25, 46T20

1 Introduction

Sehgal [13] initiated the study of fixed points of contraction mappings on a PM-space. He introduced the notion of contraction mapping on a PM-Space as a generalization of the Bop, and proved some fixed point theorems for such mappings. These results were first published in 1972 by Sehgal and Bharucha-Reid[14], subsequently, several generalizations of Sehgal's results were obtained on PM-Spaces.

On the other hand Jungck[5] generalized the Bop by introducing a contraction condition for a pair of commuting self mappings on a metric space and pointed out the potential of commuting mappings for generalizing fixed point theorems in metric space. Jungck's result has been further generalized by considering general type of contractive on functional conditions on the pair of mappings. Further generalizations have been obtained by taking contractive type or functional conditions for three self-mappings on a Metric space one of the mappings commuting with other two.

In fact, the theory of contractive principles got a spurt due to Jungck s result [5] and its first generalization due to Singh[10], Jungck s result proved in [6] and improved in [10] states that pair of commuting mappings P, T from a Metric space (M, d) into itself and satisfying.

$$P(M) \subseteq T(M), d(Pu, Pv) \leqslant h.d(Tu, Tv).$$

For every $u, v \in M$ and some $h \in (0, 1)$ posses a unique fixed point in the complete subspsce $T(M)$ of M. The contractive principle has recently been generalized for four mappings as well and some common

Citation: K. Wadhwa, J. Panthi, R. Bhardwaj, Common fixed point theorems on 2-Merger spaces for integral type mappings, South Asian J Math, 2013, 3(1), 1-7.

fixed point theorems have been proved by Fisher [3] and Singh and Kasahara [11] for four mappings on a metric space.

Theorem 1.1 (Branciari). *Let (X, d) be a complete metric space, $c \in (0, 1)$ and let $f : X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \xi(t)dt \leqslant c \int_0^{d(x,y)} \xi(t)dt$$

where $\xi : [0, +\infty) \to [0, +\infty)$ is a lebesgue integrable mapping which is summable on each compact subset of $[0, +\infty)$, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \xi(t)dt$, then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

After the paper of Branciari, a lot of research works have been carried out on generalizing contractive condition of integral type for different contractive mappings satisfying various known properties.

Theorem 1.2. *Let (X, d) be a complete metric space and $f : X \to X$ such that*

$$\int_0^{d(fx,fy)} u(t)dt \leqslant \alpha \int_0^{d(x,fx)+d(y,fy)} u(t)dt + \beta \int_0^{d} (x, y)u(t)dt + \gamma \int_0^{maxd(x,fx)+d(y,fx)} u(t)dt.$$

For each $x, y \in X$ with non-negative reals α, β, γ such that $2\alpha + \beta + \gamma < 1$, where $u : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping which is summable , non-negative and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} u(t)dt > 0.$$

Then f has a unique fixed point in X.

There is a gap in the proof of theorem 1.2. In fact, the authors [4] used the inequality $\int_0^{a+b} u(t)dt \leqslant \int_0^{a} u(t)dt + \int_0^{b} u(t)dt$ for $0 \leqslant a < b$, which is not true in general. The aim research paper of H. Aydi [8] is to present in the presence of this inequality an extension of theorem 1.2 using altering distance functions.

2 Main Results

Now, we prove the following theorem

Theorem 2.1. *If S and T are mappings of a 2-Menger space (X, F, t) into itself the inequality:*

$$\int_0^{F_{STx,TSy,w}(p)} \xi(t)dt < \int_0^{t(F_{Sy,STx,w}(p),F_{Sy,TSy,w}(p),1)} \xi(t)dt$$

for all x, y, w in X with $STx \neq TSy$, where $\xi : \mathfrak{R}^+ \to \mathfrak{R}^+$ is a lesbesgue- integrable mapping which is summable on each compact subset of $\mathfrak{R}^+$, non negative, and such that for each $\varepsilon > 0$, $\int \xi(t)dt \in 0$. Then S and T have a unique common fixed point z. Further ST is a constant mapping with $STx = z$ for all x in X.

Proof. Let x be an arbitrary point in X, and suppose that $TSTx \neq STx$ Then,

$$\int_0^{F_{STx,TSTx,w}(p)} \xi(t)dt < \int_0^{t(F_{STx,STx,w}(p),F_{STx,TSTx,w}(p),1)} \xi(t)dt$$

$$= \int_0^{t(1,F_{STx,TSTx,w}(p),1)} \xi(t)dt = \int_0^{F_{STx,TSTx,w}(p)} \xi(t)dt$$

i.e.

$$\int_0^{F_{STx,TSTx,w}(p)} \xi(t)dt < \int_0^{F_{STx,TSTx,w}(p))} \xi(t)dt.$$

which is a contradiction. Therefore $STx = z$ is a fixed point of T. Now, suppose that $Sz \neq TSz$. Then,

$$\int_0^{F_{Sz,TSz,w}(p)} \xi(t)dt = \int_0^{F_(STz,TSz,w(p)} \xi(t)dt = \int_0^{t(F_(Sz,STz,w(p),F_{Sz,TSz,w}(p),1)} \xi(t)dt$$

$$= \int_0^{t(F_{Sz,Sz,w}(p),F_{Sz,TSz,w}(p),1)} \xi(t)dt = \int_0^{t(1,F_{Sz,TSz,w}(p),1)} \xi(t)dt = \int_0^{F_{Sz,TSz,w}(p)} \xi(t)dt,$$

$$\int_0^{F_(Sz,TSz,w(p)} \xi(t)dt < \int_0^{F_{Sz,TSz,w}(p)} \xi(t)dt,$$

which is a contradiction. It follows that $STz = Sz = TSz$.

If $Sz \neq z$ then,

$$\int_0^{F_{Sz,z,w}(p)} \xi(t)dt = \int_0^{F_{TSz,TSx,w}(p)} \xi(t)dt < \int_0^{t(F_{Sz,STx,w}(p),F_{Sz,TSz,w}(p),1)} \xi(t)dt$$

$$= \int_0^{t(F_{Sz,z,w}(p),F_{Sz,Sz,w}(p),1)} \xi(t)dt = \int_0^{t(F_{Sz,z,w}(p),1,1)} \xi(t)dt = \int_0^{F_{Sz,z,w}(p)} \xi(t)dt$$

Therefore,

$$\int_0^{F_{Sz,z,w}(p)} \xi(t)dt < \int_0^{F_{Sz,z,w}(p)} \xi(t)dt$$

This contradiction proves that $Sz = z$.

Hence, it follows that z is a common fixed point of S and T. If there exists another fixed point r of S and T, then

$$\int_0^{F_{z,r,w}(p)} \xi(t)dt = \int_0^{F_{STz,TSr,w}(p)} \xi(t)dt < \int_0^{t(F_{Sr,STz,w}(p),F_{Sr,TSr,w}(p),1)} \xi(t)dt$$

$$= \int_0^{t(F_{r,z,w}(p),F_{r,r,w}(p),1)} \xi(t)dt = \int_0^{t(F_{r,z,w}(p),1,1)} \xi(t)dt = \int_0^{F_{z,r,w}(p)} \xi(t)dt$$

Therefore,

$$\int_0^{F_{z,r,w}(p)} \xi(t)dt < \int_0^{F_{z,r,w}(p)} \xi(t)dt$$

This contradiction proves that $z = r$. Hence, the common fixed point z must therefore be unique. Since we have proved that $STx = z$ for all $x \in X$, ST is a constant mapping. $\square$

Corollary 2.2. *If S and T are mappings of the Metric space (X, d) into itself satisfying the inequality:*

$$\int_0^{d(STx,TSy,w)} \xi(t)dt < a\int_0^{d(Sy,STx,w)} \xi(t)dt + b\int_0^{d(Sy,TSy,w)} \xi(t)dt.$$

For all x, y, w in X, with non-negative reals a, b, c, d such that $0 < a+b+c+d < 1$, where $\xi : \mathfrak{R}^+ \to \mathfrak{R}^+$ is a lebesgue-integrable mapping which is summable on each compact subset of $\mathfrak{R}^+$, non-negative and such *that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \xi(t)dt$ with $STx \neq TSy$. Then S and T have a unique fixed point z. Further, ST is a constant mapping with $STx = z$ for all x in X.*

Proof. The metric space (X, d) can be regarded as a 2-Menger space (X, F, t), where $Fp, q, r(x) = H(x - d(p, q, r))$,

$$H(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ 1 & \text{if } x > 0. \end{cases}$$

For $p, q, r \in X$ and $x \in R$ and $t(x, y, z) = \min(x, y, z)$ for $x, y, z \in [0, 1]$ If

$$\int_0^{F_{Sy,STx,w}(p)} \xi(t)dt = 0 \textit{ and } \int_0^{F_{Sy,TSy,w}(p)} \xi(t)dt = 0$$

then the inequality

$$\int_0^{F_{STx,TSy,w}(p)} \xi(t)dt < \int_0^{t(F_{Sy,STx,w}(p),F_{Sy,TSy,w}(p),1)} \xi(t)dt = \int_0^{t(0,0,1)} \xi(t)dt = 0$$

Therefore,

$$\int_0^{F_{Sy,STx,w}(p)} \xi(t)dt = 0 \textit{ and } \int_0^{F_{Sy,TSy,w}(p)} \xi(t)dt = 0$$

cannot happen. If

$$\int_0^{F_{Sy,STx,w}(p)} \xi(t)dt = 0 \textit{ and } \int_0^{F_{Sy,TSy,w}(p)} \xi(t)dt = 1$$

then the inequality

$$\int_0^{F_{STx,TSy,w}(p)} \xi(t)dt < \int_0^{t(F_{Sy,STx,w}(p),F_{Sy,TSy,w}(p),1)} \xi(t)dt = \int_0^{t(0,1,1)} \xi(t)dt$$

is trivially satisfied. If

$$\int_0^{F_{Sy,STx,w}(p)} \xi(t)dt = 1 \textit{ and } \int_0^{F_{Sy,TSy,w}(p)} \xi(t)dt = 1,$$

then

$$\int_0^{d(STx,TSy,w)} \xi(t)dt < p \textit{ and } \int_0^{d(Sy,TSy,w)} \xi(t)dt < p$$

Therefore by assumption

$$\int_0^{d(STx,TSy,w)} \xi(t)dt < a \int_0^{d(Sy,STx,w)} \xi(t)dt + b \int_0^{d(Sy,TSy,w)} \xi(t)dt, \quad \int_0^{d(STx,TSy,w)} \xi(t)dt < (a+b)p$$

as $a + b < 1$. This implies that

$$\int_0^{F_{STx,TSy,w}(p)} \xi(t)dt < 1$$

as F is non-decreasing and $STx \neq TSy$. Therefore, all the conditions of theorem 2.1 are satisfied. So, there exists a unique common fixed point for S and T and ST is a constant mapping. □

Corollary 2.3. *If S and T are mappings of a 2-Menger space (X, F, t) into itself satisfying the inequality:*

$$\int_0^{F_{STx,TSy,w}(p)} \xi(t)dt < \int_0^{t(F_{Sy,STx,w}(p),F_{Sy,TSy,w}(cp),1)} \xi(t)dt$$

for all x, y, w in X and $0 < \alpha < 1$. Then S and T have a unique common fixed point z. Further, ST is constant mapping with $STx = z$, for all x in X.

Proof. Now $p \in \Re^+$ and $\alpha \in (0,1)$ therefore, $\alpha_p < p$,

$$\int_0^{F_{STx,TSy,w}(p)} \xi(t)dt < \int_0^{t(F_{Sy,STx,w}(p),F_{Sy,TSy,w}(p),1)} \xi(t)dt$$

Since, $\alpha_p < p$. Therefore,

$$\int_0^{F_{STx,TSy,w}(p)} \xi(t)dt < \int_0^{t(F_{Sy,STx,w}(p),F_{Sy,TSy,w}(p),1)} \xi(t)dt$$

For all x, y, w in X , and $0 < \alpha < 1$. Therefore, all the conditions of theorem 2.1 are satisfied. So, there exists a unique common fixed point for S and T and ST is a constant mapping. □

Corollary 2.4. *If T is a mapping of a 2-Menger space (X, F, t) into itself satisfying the inequality:*

$$\int_0^{F_{T^2x,T^2y,w}(p)} \xi(t)dt < \int_0^{t(F_{Ty,T^2x,w}(p),F_{Ty,T^2y,w}(p),1)} \xi(t)dt$$

for all x, y, w in X with $T^2X \neq T^2y$, then T has a unique fixed point z. Further, T^2 is a constant mapping with $T^2x = z$ for all x in X.

Proof. From the inequality of theorem 2.1,

$$\int_0^{F_{STx,TSy,w}(p)} \xi(t)dt < \int_0^{t(F_{Sy,STx,w}(p),F_{Sy,TSy,w}(p),1)} \xi(t)dt$$

If we put $S = T$, then above inequality becomes,

$$\int_0^{F_{T^2x,T^2y,w}(p)} \xi(t)dt < \int_0^{t(F_{Ty,T^2x,w}(p),F_{Ty,T^2y,w}(p),1)} \xi(t)dt$$

For all x, y, w in X with $T^2x \neq T^2y$. Therefore, all the conditions of theorem 2.1 are satisfied. So, there exists a unique common fixed point for T and T^2 is a constant mapping. □

Theorem 2.5. If S and T are mappings of a 2-Menger space (X, F, t) into itself satisfying the inequality

$$\int_0^{F_{STx,TSy,w)}(cp)} \xi(t)dt < \int_0^{[t(F_{Sy,STx,w}(p),F_{Sy,TSy,w}(p),1)]^{1/2}} \xi(t)dt$$

for all x, y, w in X where $c > 1$. Then S and T have a unique common fixed point z. Further, ST is a constant mapping with $STx = z$ for all $x \in X$.

Proof. Let x be an arbitrary point in X. Then

$$\int_0^{F_{STx,TSTx,w}(cp)} \xi(t)dt < \int_0^{[t(F_{STx,STx,w}(p),F_{STx,TSTx,w}(p),1)]^{1/2}} \xi(t)dt$$

$$= \int_0^{[t(1,F_{STx,TSTx,w}(p),1)]^{1/2}} \xi(t)dt = \int_0^{(F_{STx,TSTx,w}(p))^{1/2}} \xi(t)dt.$$

Therefore
$$\int_0^{F_{STx,TSTx,w}(cp)} \xi(t)dt < \int_0^{(F_{STx,TSTx,w}(p))^{1/2}} \xi(t)dt$$
which is a contradiction to the non-decreasing nature of F. Therefore $STx = z$ is a fixed point of T. We now have
$$\int_0^{F_{Sz,TSz,w}(cp)} \xi(t)dt = \int_0^{F_{STz,TSz,w}(cp)} \xi(t)dt = \int_0^{[t(F_{Sz,STz,w}(p),F_{Sz,TSz,w}(p),1)]^{1/2}} \xi(t)dt$$
$$= \int_0^{[t(F_{Sz,Sz,w}(p),F_{Sz,TSz,w}(p),1)]^{1/2}} \xi(t)dt = \int_0^{[t(1,F_{Sz,TSz,w}(p),1)]^{1/2}} \xi(t)dt = \int_0^{(F_{Sz,TSz,w}(p))^{1/2}} \xi(t)dt$$
Therefore,
$$\int_0^{F_{Sz,TSz,w}(cp)} \xi(t)dt < \int_0^{(F_{Sz,TSz,w}(p))^{1/2}} \xi(t)dt$$
which is a contradiction to the non-decreasing nature of F. So, $Sz = TSz = STz$ and
$$\int_0^{F_{z,Sz,w}(cp)} \xi(t)dt = \int_0^{F_{STx,TSz,w}(cp)} \xi(t)dt < \int_0^{[t(F_{STx,Sz,w}(p),F_{Sz,TSz,w}(p),1)]^{1/2}} \xi(t)dt$$
$$= \int_0^{[t(F_{z,Sz,w}(p),F_{Sz,Sz,w}(p),1)]^{1/2}} \xi(t)dt = \int_0^{[t(F_{z,Sz,w}(p),1,1)]^{1/2}} \xi(t)dt = \int_0^{(F_{z,Sz,w}(p))^{1/2}} \xi(t)dt$$
Therefore,
$$\int_0^{F_{z,Sz,w}(cp)} \xi(t)dt < \int_0^{(F_{z,Sz,w}(p))^{1/2}} \xi(t)dt$$
This contradiction proves that $Sz = z$ and hence z is a common fixed point of S and T. Suppose that S and T have a second distinct common fixed point r. Then
$$\int_0^{F_{z,r,w}(cp)} \xi(t)dt = \int_0^{F_{STz,TSr,w}(cp)} \xi(t)dt < \int_0^{[t(F_{Sr,STz,w}(p),F_{Sr,TSr,w}(p),1)]^{1/2}} \xi(t)dt$$
$$= \int_0^{[t(F_{r,z,w}(p),F_{r,r,w}(p),1)]^{1/2}} \xi(t)dt = \int_0^{[t(F_{r,z,w}(p),1,1)]^{1/2}} \xi(t)dt = \int_0^{(F_{r,z,w}(p))^{1/2}} \xi(t)dt$$
Therefore,
$$\int_0^{F_{z,r,w}(cp)} \xi(t)dt < \int_0^{(F_{z,r,w}(p))^{1/2}} \xi(t)dt$$
This contradiction proves that $z = r$. Hence, the common fixed point must be therefore unique. Since we have proved that $STx = z$ for all $x \in X$, ST is a constant mapping. □

Corollary 2.6. *If S and T are mapping of the metric space X into itself satisfying the inequality*
$$\int_0^{[d(STx,TSy,w)]^2} \xi(t)dt < c\int_0^{d(Sy,STx,w)d(Sy,TSy,w)} \xi(t)dt$$
for all x, y, w in X, where $0 \leqslant c < 1$, then S and T have a unique common fixed point z. Further ST is a constant mapping with $STx = z$ for all x in X.

Proof. Given $\varepsilon > 0$. If $\int_0^{F_{Sy,STx,w}(c\varepsilon)} \xi(t)dt = 0$ and $\int_0^{F_{Sy,TSy,w}(c\varepsilon)} \xi(t)dt = 0$, then

$$\int_0^{F_{STx,TSy,w}(c\varepsilon)} \xi(t)dt < \int_0^{[t(F_{Sy,STx,w}(\varepsilon),F_{Sy,TSy,w}(\varepsilon),1)]^{1/2}} \xi(t)dt$$

is trivially satisfied. Whenever

$$\int_0^{F_{Sy,STx,w}(c\varepsilon)} \xi(t)dt = 1 and \int_0^{F_{Sy,TSy,w}(c\varepsilon)} \xi(t)dt = 1$$

$$\int_0^{d(Sy,STx,w)} \xi(t)dt < c\varepsilon and \int_0^{d(Sy,TSy,w)} \xi(t)dt < c\varepsilon$$

By assumption $\int_0^{d(STx,TSy,w)} \xi(t)dt < c.c\varepsilon.c\varepsilon = c^3\varepsilon^2 < c\varepsilon$ and consequently, $\int_0^{F_{STx,TSy,w}(c\varepsilon)} \xi(t)dt = 1$. Thus,

$$\int_0^{F_{STx,TSy,w}(c\varepsilon)} \xi(t)dt < \int_0^{[t(F_{Sy,STx,w}(\varepsilon),F_{Sy,TSy,w}(\varepsilon),1)]^{1/2}} \xi(t)dt$$

for each $\varepsilon > 0$ and all conditions of theorem 2.5 are satisfied. So, there exists a unique common fixed point of S and T and ST is a constant mapping. □

References

1 A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. Jour. Math. Sci. 29 (9) (2002) 531-536 .

2 B. E. Rhoades , Two fixed point theorems for mapping satisfying a general contractive condition of integral type , , Int. Jour. Math . Math Sci. 63 (2003) 4007-4013 .

3 B. Fisher, Common fixed point for four mappings. Bull. Inst. Math. Acad. Sinica. 11, 103-113, (1983).

4 D. Dey , A. Ganguly and M. Saha , Fixed point theorems for mappings under general contractive condition of integral type , Bulletin of Mathematical Analysis and Applications , 3 (1) (2011) 27-34 .

5 G. Jungck, Commuting mappings and fixed points. Amer. Math. Monthly 83, 261-263, (1976)

6 G. Jungck, Periodic and fixed points and commuting mappings. Proc. Amer. Math. Soc.76, 333-338,(1979)

7 H. Aydi, A common fixed point result by altering distances involving a contractive condition of integral type in partial metric spaces , Demonstratio Mathematica , 46 (1/2) (2003)

8 H. Aydi , A fixed point theorem for a contractive condition of integral type involving altering distances , Int. J. Non linear Appl. 3 (2012) No.1

9 S. Kumar , R . Chug and R. Kumar , Fixed point theorem for compatible mappings satisfying a contractive condition of integral type , Soochow Journal Math . 33 (2) (2007) 181-185.

10 S. L. Singh, A fixed point theorem. Indian J. Pure Appl. Math. 11, 1584-1586, (1980).

11 S. L. Singh and S. Kasahara, On some recent results on common fixed points. Indian J. Pure Appl. Math. 13, 757-761, (1982).

12 V. M. Sehgal, A fixed point theorem for mapping with a contractive iterate. Proc.Amer.Math.Soc.(3),23,631-634, (1969)

13 V. M. Sehgal, Some fixed point theorems in functional analysis and probability. Ph.D. Dissertation, Wayne State univ. Michigan (1996).

14 V. M. Sehgal and A. I. Bharucha-Reid, Fixed points of contraction mappings on probabilistic metric spaces. Math. System Theory,6,97-102, (1972)

15 U. C. Gairola and A. S. Rawat , A fixed point theorem for integral type inequality, Int. Journal Math. Analysis, 2 (15) (2008) 709-712.