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A polynomial algorithm to get convergent series with general term tending to zero

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Abstract In the computation fractal dimension of level sets of Rademacher series which is induced by sequences on complex plane with general term tending to zero, the convergence of the Rademacher series must be needed for some real number belongs to $[0,1]$. In this paper, a polynomial algorithm algorithm with the order of complexity $O(n)$ is established to get a convergent series from such a sequence on complex plane with each term multiplied by a random variable where all random variables are independent identically distributed.

Key Words complex sequence, Rademather series, convegence, symbol space

MSC 2010 28A80, 47D20

1 Introduction

The Rademacher functions $R_n(x)(n \geqslant 1)$ are defined by

$$R_n(x) = \operatorname{sgn} \sin(2^n \pi x),$$

where sgn$x = -1, 0$ or 1 according to $x < 0$, $x = 0$ or $x > 0$. And call the series $S(x) = \sum_{n=1}^{\infty} a_n R_n(x)$ the Rademacher series, which is a spacial Weisertrass-typed series. Weisertrass-typed series has a colorful history, known as a continuous non-differentiable function as well as a classical fractal function. Considering Hausdorff dimension of the level sets[1,2,5] of such series, we need the convergence of those series. Fortunately, a special case of a result of Kaczmarz and Steinhaus[3] shows that if $\{a_n\}$ $(n = 1, 2, \ldots)$ is a sequence of real numbers with

$$\sum_{n=1}^{\infty} |a_i| = +\infty, \text{ and } a_i \to 0, \tag{H}$$

then the Rademacher series assumes every preassigned real value c (cardinal number of the continuum) times for $x \in (0,1)$, i.e., $A_r = \{x \in (0,1) : S(x) = r\}$ has a cardinality c. However, if the real sequence

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$\{a_n\}$ is replaced by a complex sequence $\{c_n\}$, set A_r might be an empty set. So we ask naturally can we find some real number r such that A_r is a nonempty set.

There is another motivation for this paper. It is known that a conditional convergent series can converge to each real number if it is rearranged the summational order. When we don't change the summational order but make each summational term a_n multiplied by a random variable ξ_n with all variables being independent identically distributed, can we make sure that the series $\sum_{n=1}^{\infty} a_n \xi_n$ converge for some $\{\xi_n\} \in \{-1,1\}^{\mathbb{N}}$, where $\{a_n\}$ satisfies condition (H)?

Since the order of complexity for an algorithm is very important[4], to make sure that the Rademacher series converge we would like find a polynomial algorithm. In this paper, we will establish an algorithm by which, for any complex sequence $\{c_n\}$, one can find an according $x \in (0,1)$ such that the Rademacher series $\sum_{n=1}^{\infty} c_n R_n(x)$ converges. This algorithm is a polynomial algorithm with the order of complexity $O(n)$.

2 Notes and lemmas

Let Σ_2 be the symbolic system $\prod_{n=1}^{\infty}\{-1,1\}$, and let d be the usual metric on Σ_2 defined by $\mathrm{d}(x,y) = 2^{-m}$ with $m = \min\{n : x_n \neq y_n\}$ for $x \neq y$, where $x = (x_n)$ and $y = (y_n)$ belong to Σ_2.

This paper always assume that $x, x_n, y_n, w_n \in \{-1,1\}$ and that complex number c is with form $c = (a,b) = a + ib$, $a, b \in \mathbb{R}$. In term of $c \in \mathbb{C}$, its norm is defined by $\|c\| = \max\{|a|,|b|\}$. For a sequence $\{c_n\}$, its norm is defined by $\|\{c_n\}\|_s = \sup_n \|c_n\|$.

Definition 2.1. Let $\{c_n\}_{n=1}^{N}$ be a sequence in $\mathbb{C}$. Say it is of type one if $\|\{c_n\}\|_s < 1$ and there is some $1 \leqslant n < N$ such that either $\|c_n + c_{n+1}\| < 1$ or $\|c_n - c_{n+1}\| < 1$ and say it is of type two if $\|\{c_n\}\|_s < 1$ and it is not of type one.

Lemma 2.2. *Suppose $S = \{c_n = a_n + ib_n\}_{n=1}^{5}$ be a sequence of type two in the complex plane. Let*

$$x_{n,j} = (-1)^{t_n}\mathrm{sgn}(a_j a_{j+n-1}), n = 1,2,3,4,$$

and $C_j = \sum_{n=1}^{4} x_{n,j} c_{n+j-1}$, $j = 1,2$, where

$$(t_1,t_2,t_3,t_4) = (0,1,1,0).$$

Denote $P(S) = \{C_1, c_5\}$ or $\{c_1, C_2\}$ according to $\|C_1\| < 1$ or $\geqslant 1$. Then $\|P(S)\|_s < 1$.

Proof. Here we only show $\|C_2\| < 1$ when $\|C_1\| \geqslant 1$. For $j = 1,2$, denote

$$A_j = \sum_{n=1}^{4} x_{n,j} a_{n+j-1},\ B_j = \sum_{n=1}^{4} x_{n,j} b_{n+j-1}.$$

Then, by the type of S, we have

$$|B_j| = \big||b_j| + |b_{j+1}| - |b_{j+2}| - |b_{j+3}|\big| < 1$$

and $\mathrm{sgn}(a_j)A_j = |a_j| - |a_{j+1}| - |a_{j+2}| + |a_{j+3}| < 1$ for $j = 1, 2$. If $|A_1| \geqslant 1$, then $\mathrm{sgn}(a_1)A_1 < -1$. On the other hand, we have

$$\mathrm{sgn}(a_1)A_1 + \mathrm{sgn}(a_2)A_2 = |a_1| - 2|a_3| + |a_5| > -2,$$

implying $\mathrm{sgn}(a_2)A_2 > -1$ and so $|A_2| < 1$ since $A_2 < 1$. □

Let $S = \{c_1, c_2, \ldots, c_5\}$ with norm less than one. If S is of type one, denote

$$T(S) = \min\{n : \{c_n, c_{n+1}\} \text{ is of type one}\},$$
$$M(S) = \{w_n\}_{n=1}^5,$$
$$P(S) = \{c_1, \ldots, c_m + w_{m+1}c_{m+1}, \ldots, c_5\},\ m = T(S),$$

where $w_n = 1$ if $n \neq T(S)+1$ and $w_n = -1$ or 1 according to $\|c_n + c_{n+1}\| \geqslant 1$ or < 1 if $n = T(S)+1$. If S is of type two, denote

$$M(S) = \begin{cases} \{x_{1,1}, x_{2,1}, x_{3,1}, x_{4,1}, 1\}, & \text{if } P(S) = \{C_1, s_5\}; \\ \{1, x_{1,2}, x_{2,2}, x_{3,2}, x_{4,2}\}, & \text{if } P(S) = \{s_1, C_2\}, \end{cases}$$

where $x_{n,i}$ and C_j are given by Lemma 2.2. For stating our idea clearly, we give an example as follow:

Example 2.3. *Let $S_i = \{c_n^i\}_{n=1}^5$, $i = 1, 2$ be defined by*

$$S_1 = \{.5 + .7i, -.6 + .4i, .7 + .9i, .92 + .83i, .1 + .4i\},$$
$$S_2 = \{.02 - .6i, .99 + .5i, .99 - .7i, -.03 - .8i, .98 - .7i\}.$$

Then S_1 is of type one while S_2 is of type two, and

$$M(S_1) = \{1, 1, 1, -1, 1\},\ P(S_1) = \{c_1^1, c_2^1, c_3^1 - c_4^1, c_5^1\},$$
$$M(S_2) = \{1, 1, -1, 1, 1\},\ P(S_2) = \{c_1^2, c_2^2 - c_3^2 + c_4^2 + c_5^2\}.$$

3 Main results

Theorem 3.1. *Let $\{c_n\}_{n=1}^N$ be a sequence in $\mathbb{C}$ with norm less than one. Then there exists another sequence $\{x_n\}_{n=1}^N$ such that for any integer number $1 \leqslant n \leqslant N$, we have*

$$\Big\| \sum_{j=1}^n x_j c_j \Big\| < 5. \tag{3.1}$$

Proof. If $N \leqslant 4$, then $\{x_n = 1\}_{n=1}^N$ is a desired sequence. So we will show the conclusion is true when $N > 4$.

Let $S = \{s_n\}_{n=1}^5$ be the first five terms of $\{c_n\}$ and $\lambda_0 = 5$. Denote $\{w_n^0\}_{n=1}^5 = M(S)$ and $\lambda_1 = \min\{\lambda_0 + 1, N\}$, $\{c_n^1\}_{n=1}^4 = P(S)$ if S is of type one or $\lambda_1 = \min\{\lambda_0 + 3, N\}$, $\{c_n^1\}_{n=1}^2 = P(S)$ if S is of type two. Suppose that $\{w_n^{k-1}\}_{n=1}^5$, $\{c_n^k\}_{n=1}^{n_k}$ and λ_k are given. Let $S = \{s_n\}_{n=1}^5$ be the first five terms of the following sequence

$$\{c_1^k, \ldots, c_{n_k}^k, c_{\lambda_k+1}, c_{\lambda_k+2}, \ldots, c_N\}. \tag{3.2}$$

Put $\{w_n^k\}_{n=1}^5 = M(S)$ and $\lambda_{k+1} = \min\{\lambda_k + 1, N\}$, $\{c_n^{k+1}\}_{n=1}^4 = P(S)$ if S is of type one or $\lambda_{k+1} = \min\{\lambda_k + 3, N\}$, $\{c_n^{k+1}\}_{n=1}^2 = P(S)$ if S is of type two.

This definition process would be stopped if the number of terms of sequence (3.2) (denoted by N_k) is no more than four for some k and must be stopped for some k since $\{N_0 = N, N_1, \ldots, N_k\}$ is a decreasing sequence and $N < +\infty$.

Now we assume k is the desired number, that's $N_k < 5$ and $N_{k-1} \geqslant 5$, while $\lambda_k = N$. Denote the sequence defined in (3.2) by $\{c_n^k\}_{n=1}^{N_k}$ and let $w_n^k = 1$ for $1 \leqslant n \leqslant N_k$. Fixed $1 \leqslant n \leqslant N$, we will define the coefficient x_n of c_n. If $n > \lambda_{k-1}$, let $x_n = 1$. Now we consider the case $n \leqslant \lambda_{k-1}$. From the definition of c_j^i, $j \leqslant n_i$, we know there are two integer number $m_1 = m_1(j,i), m_2 = m_2(j,i)$ and $w_{m_1}, w_{m_1+1}, \ldots, w_{m_2} \in \{-1, 1\}$ such that

$$c_j^i = \sum_{l=m_1}^{m_2} w_l c_l.$$

Denote $\Gamma_j^i = \{l : m_1(j,i) \leqslant l \leqslant m_2(j,i)\}$, where $j \leqslant n_i$. Let

$$i_0 := i_0(n) = \min\{i : n \in \Gamma_j^i \text{ for some } j\}.$$

By the definitions of Γ_j^i's and c_j^i's, there is a sequence $\{k_i\}_{i=i_0}^k$ such that $n \in \cap_{i=i_0}^k \Gamma_{k_i}^i$. Let $x_n = \prod_{i=i_0}^k w_{k_i}^i$.

From the following two facts that $w_1^i = 1$ for any $1 \leqslant i \leqslant k$ by the construction of w_1^i and

$$\Big\| \sum_{j=1}^n x_j c_j \Big\| \leqslant \sum_{j=1}^{n_{i_0}} \|c_j^{i_0}\| + \sum_{j=\lambda_{i_0-1}}^n \|x_j c_j\| < 5.$$

such a sequence $\{x_n\}_{n=1}^N$ can make the result to be true. □

Theorem 3.2. *Let $S = \{c_n\}$ be a sequence in $\mathbb{C}$ with $c_n \to 0$. Then there is an $x = (x_n) \in \Sigma_2$ such that*

$$\Big\| \sum_{n=1}^\infty x_n c_n \Big\| < 10 \|\{c_n\}\|_s.$$

Proof. Denote $s = \|\{c_n\}\|_s$. There is an increasing integer sequence $\{N_0 = 0, N_1, \ldots\}$ such that $\|c_n\| < 2^{-k}s$ for $N_k < n \leqslant N_{k+1}$. For each $k \geqslant 1$, let $\{x_n\}_{n=N_{k-1}+1}^{N_k}$ be given by Theorem 3.1 for the sequence $\{2^k s^{-1} c_n\}_{n=N_{k-1}+1}^{N_k}$. Fix $0 < \varepsilon < 1$, set $k_0 = \min\{k : 2^{-k}s < \varepsilon\}$. Letting $n > N_{k_0}$ and $0 \leqslant p \in \mathbb{Z}$, there are two integers k_1, k_2 with $k_0 \leqslant k_1 \leqslant k_2$, $N_{k_1} < n \leqslant N_{k_1+1}$ and $N_{k_2} < n + p \leqslant N_{k_2+1}$. Thus

$$\begin{aligned} \Big\| \sum_{i=n}^{n+p} x_i c_i \Big\| &= \Big\| \sum_{i=N_{k_1}+1}^{N_{k_2}} x_i c_i + \sum_{i=N_{k_2}+1}^{n+p} x_i c_i - \sum_{i=N_{k_1}+1}^{n-1} x_i c_i \Big\| \\ &< 10 \cdot 2^{-k_0} s + 5 \cdot 2^{-k_0} s + 5 \cdot 2^{-k_0} s < 20\varepsilon, \end{aligned}$$

implying that $\sum_{n \geqslant 1} x_n c_n$ converges. Using Lemma 3.1 again,

$$\Big\| \sum_{n=1}^\infty x_n c_n \Big\| \leqslant \sum_{k=0}^\infty \Big\| \sum_{n=N_k+1}^{N_{k+1}} x_n c_n \Big\| < \sum_{k=0}^\infty 5 \cdot 2^{-k} s = 10s,$$

so we can end the proof here. □

4 The algorithm

Now, we give the algorithm by Theorem 3.1. Recall that $\{c_n\}_{n=1}^N$ is a sequence of complex numbers.

Step 1. Let $\{x_n = 1\}_{n=1}^N$. If $N \leqslant 4$, goto Step 7.

Step 2. Initializing. Let $i = 0$, $\lambda_i = 5$ and $S = \{s_k = c_k\}_{k=1}^5 = \{(a_k, b_k)\}_{k=1}^5$. Let $\Lambda_k = \{k\}$, $1 \leqslant k \leqslant 5$.

Step 3. If $\Lambda_5 = \emptyset$, goto Step 7. If S is of type two turn to Step 4. Let $\lambda_{i+1} \leftarrow \min\{\lambda_i + 1, N\}$. Denote

$$m = \min\{k : \|s_k + s_{k+1}\| < 1 \text{ or } \|s_k - s_{k+1}\| < 1\},$$

$$flag = \begin{cases} 1, & \|s_m + s_{m+1}\| < 1, \\ -1, & \text{otherwise}. \end{cases}$$

If $flag = -1$, set $x_n \leftarrow -x_n$ for each $n \in \Lambda_m$ and put

$$\Lambda_m \leftarrow \Lambda_m \bigcup \Lambda_{m+1},$$

$$\Lambda_{m+k} \leftarrow \Lambda_{m+k+1}, 1 \leqslant k \leqslant 4 - m,$$

$$\Lambda_5 \leftarrow \begin{cases} \{\lambda_{i+1}\}, & \lambda_{i+1} > \lambda_i \\ \emptyset, & \lambda_{i+1} = \lambda_i. \end{cases}$$

$$s_k \leftarrow \begin{cases} s_k, & k < m; \\ s_k + flag \cdot s_k; & k = m; \\ s_{k+1}, & m < k < 5; \\ c_{\lambda_{i+1}}, & k = 5. \end{cases}$$

Goto Step 6.

Step 4. Let $\lambda_{i+1} \leftarrow \min\{\lambda_i + 3, N\}$. Define

$$t = s_1 - \operatorname{sgn}(a_1 a_2) s_2 - \operatorname{sgn}(a_1 a_3) s_3 + \operatorname{sgn}(a_1 a_4) s_4.$$

If $\|t\| \geqslant 1$, goto Step 5. Let $w_2 = -\operatorname{sgn}(a_1 a_2), w_3 = -\operatorname{sgn}(a_1 a_3), w_4 = \operatorname{sgn}(a_1 a_4)$ and let $x_n \leftarrow x_n w_k$ if $n \in \Lambda_k$, $k = 2, 3, 4$. Let

$$\Lambda_k \leftarrow \begin{cases} \Lambda_5, & k = 2; \\ \{\lambda_i + k - 2\}, & 2 < k \leqslant \lambda_{i+1} - \lambda_i + 2; \\ \emptyset, & \lambda_{i+1} - \lambda_i < k \leqslant 5. \end{cases}$$

$$s_k \leftarrow \begin{cases} t, & k = 1; \\ s_5, & k = 2; \\ c_{\lambda_i + k - 2}, & 2 < k \leqslant \lambda_{i+1} - \lambda_i + 2; \\ c_N, & \lambda_{i+1} - \lambda_i + 2 < k \leqslant 5. \end{cases}$$

Step 5. Let $w_3 = -\text{sgn}(a_2a_3), w_4 = -\text{sgn}(a_2a_4), w_5 = \text{sgn}(a_2a_5)$ and let $x_n \leftarrow x_nw_k$ if $n \in \Lambda_k$, $k = 3, 4, 5$. Let

$$\Lambda_k \leftarrow \begin{cases} \cup_{m=2}^{5}\Lambda_m, & k = 2; \\ \{\lambda_i + k - 2\}, & 2 < k \leqslant \lambda_{i+1} - \lambda_i + 2; \\ \emptyset, & \lambda_{i+1} - \lambda_i < k \leqslant 5. \end{cases}$$

$$s_k \leftarrow \begin{cases} s_2 + \sum_{m=3}^{5} w_m s_m, & k = 2; \\ c_{\lambda_i + k - 2}, & 2 < k \leqslant \lambda_{i+1} - \lambda_i + 2; \\ c_N, & \lambda_{i+1} - \lambda_i + 2 < k \leqslant 5. \end{cases}$$

Step 6. Let $i \leftarrow i + 1$. Goto Step 3.

Step 7. Output $\{x_n\}$ and end.

5 Conclusion

The algorithm introduced in above section is a polynomial algorithm, of which the order of complexity is $O(N)$ for a complex sequence $\{c_n\}_{n=1}^{N}$.

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