South Asian Journal of Mathematics 2012, Vol. 2 (5): 508 ∼ 516
www.sajm-online.com ISSN 2251-1512 RESEARCH ARTICLE

The connected hull number of a graph

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Abstract For a connected graph $G = (V, E)$, a connected hull set of a graph G is a hull set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected hull set of G is the connected hull number of G and is denoted by $h_c(G)$. Connected graphs of order p with connected hull number 2 or p are characterized. It is shown that for any positive integers $2 \leqslant a < b \leqslant c$, there exists a connected graph G such that $h(G) = a$ and $h_c(G) = b$ and $g(G) = c$ where $g(G)$ is a geodetic number of a graph. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum connected hull set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing connected hull number of S, denoted by $f_{hc}(S)$, is the cardinality of a minimum forcing subset of S. The forcing connected hull number of G, denoted by $f_{hc}(G)$, is $f_{hc}(G) = \min\{f_{hc}(S)\}$, where the minimum is taken over all minimum connected hull sets S in G. It is shown that for every pair a, b of integers with $a \geqslant 0$ and $b > 2a+2$, there exists a connected graph G such that $f_{hc}(G) = a$ and $h_c(G) = b$.

Key Words hull number, connected hull number,forcing hull number, forcing connected hull number

MSC 2010 05C12

1 Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [1,7]. A convexity on a finite set V is a family C of subsets of V, convex sets which is closed under intersection and which contains both V and the empty set. The pair (V, E) is called a convexity space. A finite graph convexity space is a pair (V, E), formed by a finite connected graph $G = (V, E)$ and a convexity C on V such that (V, E) is a convexity space satisfying that every member of C induces a connected subgraph of G. Thus, classical convexity can be extended to graphs in a natural way. We know that a set X of R^n is convex if every segment joining two points of X is entirely contained in it. Similarly a vertex set W of a finite connected graph is said to be convex set of G if it contains all the vertices lying in a certain kind of path connecting vertices of W[8]. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. A vertex x is said to lie on a $u - v$ geodesic P

Citation: J. John, V. Mary Gleeta, The connected hull number of a graph, South Asian J Math, 2012, 2(5), 508-516.

if x is a vertex of P including the vertices u and v.For two vertices u and v, let $I[u, v]$ denotes the set of all vertices which lie on $u - v$ geodesic. For a set S of vertices, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. The set S is convex if $I[S] = S$. Clearly if $S = \{v\}$or $S = V$, then S is convex. The convexity number, denoted by $C(G)$, is the cardinality of a maximum proper convex subset of V. The smallest convex set containing S is denoted by $I_h(S)$ and called the convex hull of S. Since the intersection of two convex sets is convex, the convex hull is well defined. Note that $S \subseteq I[S] \subseteq I_h(S) \subseteq V$. A subset $S \subseteq V$ is called a geodetic set if $I[S] = V$ and a hull set if $I_h(S) = V$. The geodetic number $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a minimum geodetic set or simply a g- set of G. Similarly, the hull number $h(G)$ of G is the minimum order of its hull sets and any hull set of order $h(G)$ is a minimum hull set or simply a h- set of G. The geodetic number of a graph is studied in [1,2,3,5,9,10] and the hull number of a graph is studied in [1,4,6,10]. For the graph G given in Figure 1.1, $S = \{v_1, v_7\}$ is a h- set of G so that $h(G) = 2$ and also $S_1 = \{v_1, v_6, v_7\}$is a g-set of G so that $g(G) = 3$. A vertex v is an extreme vertex of a graph G if the subgraph induced by its neighbors is complete. Throughout the following G denotes a connected graph with at least two vertices.

The following theorems are used in the sequel.

Theorem 1.1. [2, 5] Each extreme vertex of a connected graph G belongs to every hull set(geodetic set) of G.

Theorem 1.2. [2] For a connected graph G, $h(G) = p$ if and only if $G = K_p$.

Theorem 1.3. [5] The geodetic number of a tree T is the number of end-vertices in T

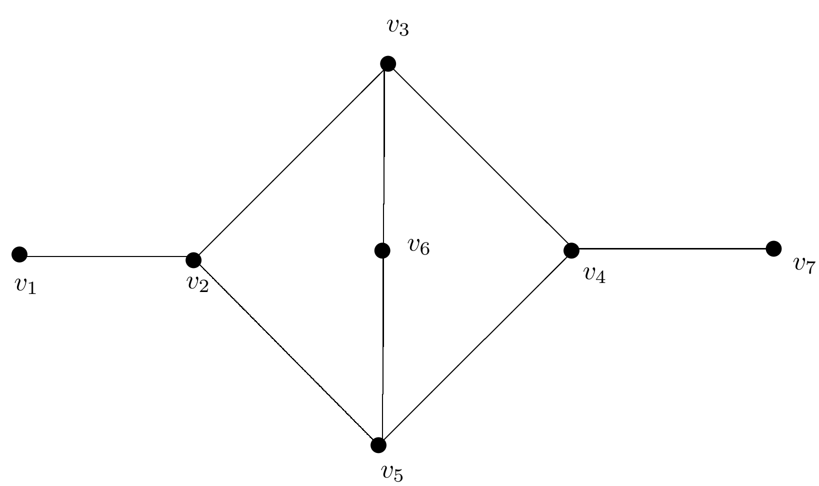

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Figure 1.1

2 The Connected Hull Number of a Graph

Definition 2.1. A set $S \subseteq G$ is called a connected hull set of a graph G if S is a hull set such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected hull set of G is the connected hull number of G and is denoted by $h_c(G)$. A connected hull set of cardinality $h_c(G)$ is called a h_c-set of G or a minimum connected hull set of G.

Example 2.2. For the graph G given in Figure 2.1. $S = \{v_1, v_4, v_7\}$is the minimum hull set of G and so $h(G) = 3$. Here the induced subgraph $G[S]$ is not connected,so that S is not a connected hull set of G. Now, it is clear that $S_1 = \{v_1, v_2, v_3, v_4, v_7\}$ is a minimum connected hull set of G and so $h_c(G) = 5$.

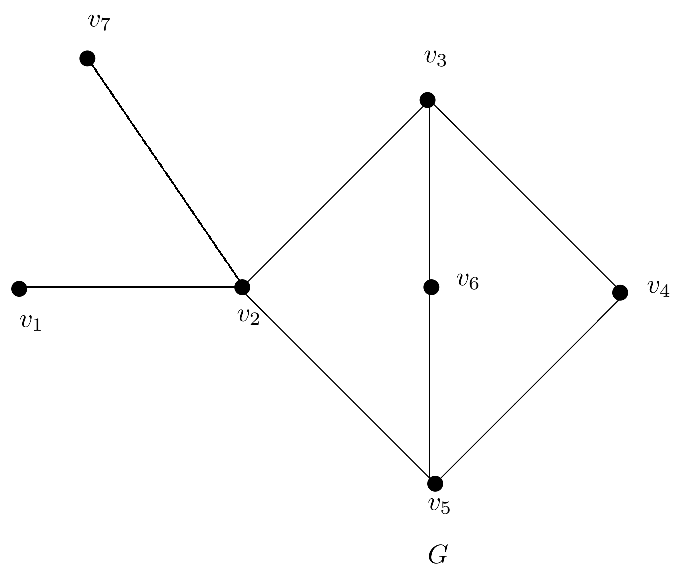

Figure 2.1

Theorem 2.3. *Every extreme vertex of a connected graph G belongs to every connected hull set of G. In particular, every end-vertex of G belongs to every connected hull set of G.*

Proof. Since every connected hull set is also a hull set, the result follows from Theorem 1.1. □

Corollary 2.4. For the complete graph $K_p (p \geqslant 2), h_c(K_p) = p$.

Theorem 2.5. *Let G be a connected graph with cut-vertices and let S be a connected hull set of G. If v is a cut-vertex of G, then every component of $G - v$ contains an element of S.*

Proof. Suppose that there is a component B of G at a cut-vertex v such that B contains no vertex of S. Let $u \in V(B)$. Since S is a connected hull set, there exists a pair of vertices x and y in S such that u lies on $I^k[x, y]$ in $G, (k \geqslant 1)$.Let us assume that u lies on $z - w$ geodesic in $I^k[x, y]$.Now $P : x = u_0, u_1, ..., u_l = z, ...u, ..., u_m = w, ..., u_n = y$ is a path in G. Since v is a cut-vertex of G, the $x - u$ subpath of P and the $u - y$ subpath of P both contain v, it follows that P is not a path, contrary to assumption. □

Corollary 2.6. *Let G be a connected graph with cut-vertices and let S be a connected hull set of G. Then every branch of G contains an element of S.*

Theorem 2.7. *Every cut-vertex of a connected graph G belongs to every connected hull set of G.*

Proof. Let v be any cut-vertex of G and let $G_1, G_2, ..., G_r (r \geqslant 2)$ be the components of $G - \{v\}$. Let S be any connected hull set of G. Then by Theorem 2.5,S contains at least one element from each $G_i (1 \leqslant i \leqslant r)$. Since $G[S]$ is connected, it follows that $v \in S$. □

Corollary 2.8. *For a connected graph G with k extreme vertices and l cut-vertices,$h_c(G) \geqslant max \{2, k + l\}$.*

Proof. This follows from Theorems 2.3 and 2.7. □

Corollary 2.9. *For any non-trivial tree T of order $p, h_c(T) = p$.*

Proof. This follows from Corollary 2.8. □

Theorem 2.10. *For a connected graph G of order $p, 2 \leqslant h(G) \leqslant h_c(G) \leqslant p$.*

Proof. Any hull set needs at least two vertices and so $h(G) \geqslant 2$. Since every connected hull set is also a hull set, it follows that $h(G) \leqslant h_c(G)$. Also, since $V[G]$ induces a connected hull set of G, it is clear that $h_c(G) \leqslant p$. □

Remark 2.11. The bounds in Theorem 2.10 are sharp. For any non-trivial path $P, h(P) = 2$. For the complete graph $K_p, h(K_p) = h_c(K_p)$. By Corollary 2.1, For any non-trivial tree $T, h_c(T) = p$. Also, all the inequalities in the theorem are strict. For the graph G given in Figure 2.1, $h(G) = 3, h_c(G) = 5$ and $p = 7$ so that $2 < h(G) < h_c(G) < p$.

Corollary 2.12. *Let G be any connected graph. If $h_c(G) = 2$, then $h(G) = 2$.*

The following Theorems 2.13 and 2.14 characterize graphs for which $h_c(G) = 2$ and $h_c(G) = p$ respectively.

Theorem 2.13. *Let G be a connected graph of order $p \geqslant 2$. Then $G = K_2$ if and only if $h_c(G) = 2$.*

Proof. If $G = K_2$, then $h_c(G) = 2$. Conversely, let $h_c(G) = 2$. Let $S = \{u, v\}$ be a minimum connected hull set of G. Then uv is an edge. If $G \neq K_2$, then there exists a vertex w different from u and v. Thus w cannot lie on any $I^k[u, v](k \geqslant 1)$ so that S is not a h_c-set, which is a contradiction. Thus $G = K_2$. □

Theorem 2.14. *Let G be a connected graph. Then every vertex of G is either a cut-vertex or an extreme vertex if and only if $h_c(G) = p$.*

Proof. Let G be a connected graph with every vertex of G either a cut-vertex or an extreme vertex. Then the result follows from Theorem 2.3 and Theorem 2.7.Conversely, suppose $h_c(G) = p$. Suppose that there is a vertex x in G which is neither a cut-vertex nor an extreme vertex. Since x is not an extreme vertex, $N(x)$ does not induce a complete subgraph.Let $S = V - \{x\}$.Then $I^k [S] = V$ and so S is a hull set of G. Clearly, x lies on $I^k [S]$. Also, since x is not a cut-vertex of $G, < G - x >$ is connected. Thus $V - \{x\}$ is a connected hull set of G and so $h_c(G) \leqslant |V - \{x\}| = p - 1$, which is a contradiction. □

We leave the following problem as an open question.

Problem 2.15. *Characterize graphs G for which $h_c(G) = h(G)$.*

We denote the vertex connectivity of a connected graph G by $\kappa(G)$ or κ.

Theorem 2.16. *If G is a non-complete connected graph such that it has a minimum cutset, then $h_c(G) \leqslant p - \kappa(G) + 1$.*

Proof. Since G is non-complete, it is clear that $1 \leqslant \kappa(G) \leqslant p - 2$. Let $U = \{u_1, u_2, \ldots, u_\kappa\}$ be a minimum cutset of G. Let $G_1, G_2, \cdots, G_r (r \geqslant 2)$be the components of $G - U$ and let $S = V(G) - U$. Then every vertex $u_i (1 \leqslant i \leqslant \kappa)$ is adjacent to at least one vertex of G_j for every $j (1 \leqslant j \leqslant r)$. It is clear that S is a hull set of G and $G[S]$ is not connected. Also, it is clear that $G[S \cup \{x\}]$ is a connected hull set for any vertex x in U so that $h_c(G) \leqslant p - \kappa(G) + 1$. □

Remark 2.17. *The bound in Theorem 2.16 is sharp. For the cycle $G = C_4, h_c(G) = 3$. Also, $\kappa(G) = 2, p - \kappa(G) + 1 = 3$. Thus $h_c(G) = p - \kappa(G) + 1$.*

In view of Theorem 2.10, we have the following realization result.

Theorem 2.18. *For any three positive integers a, b, c with $2 \leqslant a < b < c$ there exists a connected graph G such that $h(G) = a, h_c(G) = b$ and $g(G) = c$.*

Proof. Case 1. Suppose $a = 2; b < c$. Let $P_b : v_1, v_2, ..., v_b$ be a path of length $b - 1$. Add $2c - 4$ new vertices $w_1, w_2, ..., w_{c-2}, u_1, u_2, ..., u_{c-2}$ to P_b and join each $w_i (1 \leqslant i \leqslant c - 2)$ with v_1 and v_3 and join each $u_i (1 \leqslant i \leqslant c - 2)$ with v_2 and each $w_i (1 \leqslant i \leqslant c - 2)$ there by producing the graph G of Figure 2.2. First we prove that $h(G) = a$. Let $S = \{v_1, v_b\}$. Then S is a hull set of G so that $h(G) = 2$. Let $S_1 = \{v_3, v_4, ..., v_b\}$ be the set of all cut vertices and end vertices of G. By Theorems 2.3 and 2.7, each connected hull set contains S_1. It is clear that S_1 is not a connected hull set of G. It is easily verified that $S_1 \cup \{x\}$ where $x \notin S_1$ is not a connected hull set of G so that $h_c(G) \geqslant b$. However $S_1 \cup \{v_1, v_2\}$ is a connected hull set of G so that $h_c(G) = b$. Next we show that $g(G) = c$. By Theorem 1.1 every geodetic set of G contains v_b. It is easily observed that every geodetic set contain each $u_i (1 \leqslant i \leqslant c - 2)$. Let $S_2 = \{v_b, u_1, u_2, ..., u_{c-2}\}$. It is clear that S_2 is not a geodetic set of G and so $g(G) \geqslant c$. However $S_2 \cup \{v_1\}$ is a geodetic set of G so that $g(G) = c$.

Case 2. Let $2 \leqslant a < b < c$. Let $P_{b-a+2} : v_1, v_2, ..., v_{b-a+2}$ be a path of length $b - a + 1$. Add $2c - a - 1$ new vertices $w_1, w_2, ..., w_{c-a}, u_1, u_2, ..., u_{c-a}, z_1, z_2, ..., z_{a-1}$ to P_{b-a+1} and join each $z_i (1 \leqslant i \leqslant a - 1)$ with v_{b-a+2} and join each $w_i (1 \leqslant i \leqslant c - a)$ with v_1 and v_3 and join each $u_i (1 \leqslant i \leqslant c - a)$ with v_2 and each $w_i (1 \leqslant i \leqslant c - a)$ there by producing the graph G of Figure 2.3. First we prove that $h(G) = a$. Let $S = \{z_1, z_2, ..., z_{a-1}\}$ be the set of all extreme vertices of G. By Theorem 1.1, every hull set of G contains S. It is clear that S is not a hull set of G and so $h(G) \geqslant a$ .But $S \cup \{v_1\}$ is a hull set of G so that $h(G) = a$.Next we show that $h_c(G) = b$. Let $S_1 = S \cup \{v_3, v_4, ..., v_{b-a+1}\}$ be the set of all extreme vertices and cut vertices of G. By Theorems 2.3 and 2.7, each connected hull set contains S_1. It is clear that S_1 is not a connected hull set of G. It is easily verified that $S_1 \cup \{x\}$ where $x \notin S_1$ is not a connected hull set of G so that $h_c(G) \geqslant b$.However $S_1 \cup \{v_1, v_2\}$ is a connected hull set of G sothat $h_c(G) = b$. Next we show that $g(G) = c$. By Theorem 1.1 every geodetic set of G contains S. It is easily observed that every geodetic set contain each $u_i (1 \leqslant i \leqslant c - a)$. Let $S_2 = S \cup \{u_1, u_2, ..., u_{c-a}\}$. It is clear that S_2 is not a geodetic set of G and so $g(G) \geqslant c$. However $S_2 \cup \{v_1\}$ is a geodetic set of G so that $g(G) = c$. □

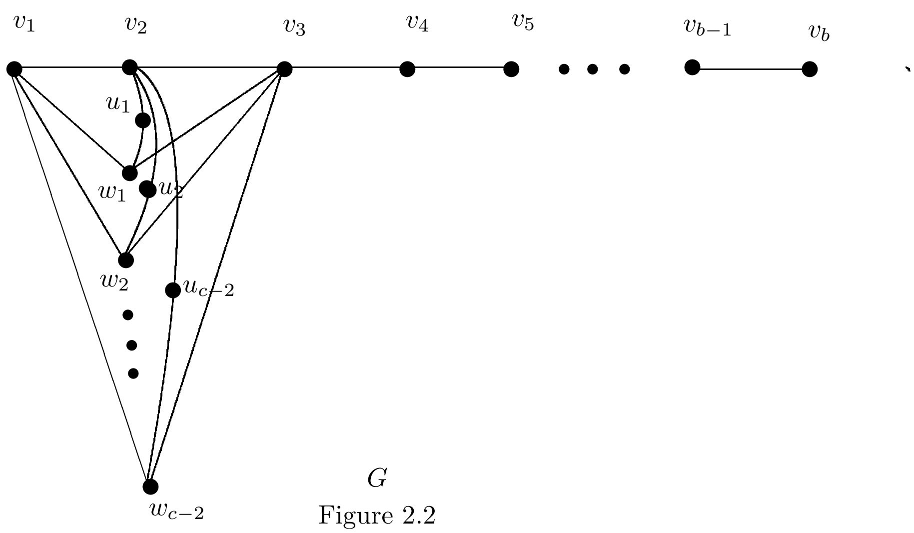

Figure 2.2

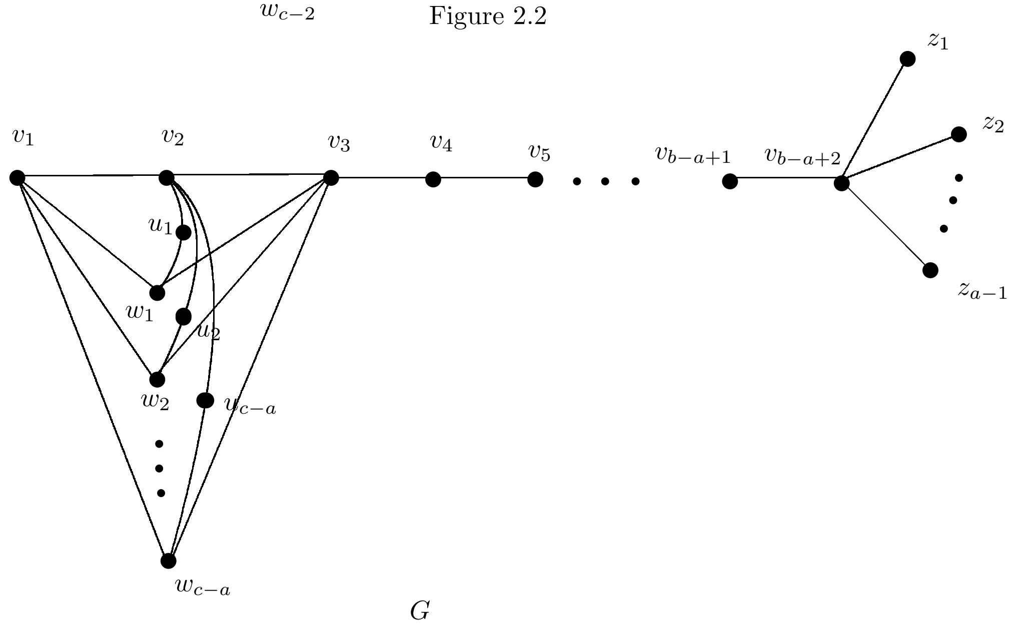

Figure 2.3

3 The Forcing Connected Hull Number of a Graph

Definition 3.1. *Let G be a connected graph and S a h_c-set of G. A subset $T \subseteq S$ is called a forcing subset for S if S is the unique h_c-set containing T. A forcing subset for S of minimum cardinality* *is a minimum forcing subset of S. The forcing connected hull number of S, denoted by $f_{hc}(S)$, is the cardinality of a minimum forcing subset of S. The forcing connected hull number of G, denoted by $f_{hc}(G)$, is $f_{hc}(G) = min\{f_{hc}(S)\}$, where the minimum is taken over all h_c-sets S in G.*

Example 3.2. *For the complete bipartrite graph $G = K_{2,3}$ with bipartite sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$,the sets $S_1 = \{x_1, x_2, y_1\}, S_2 = \{x_1, x_2, y_2\}, S_3 = \{x_1, x_2, y_3\}, S_4 = \{x_1, y_2, y_1\}, S_5 = \{x_2, y_2, y_1\}, S_6 = \{x_1, y_2, y_3\}, S_7 = \{x_2, y_2, y_3\}, S_8 = \{x_1, y_1, y_3\}$ and $S_9 = \{x_2, y_1, y_3\}$ are the h_c-sets of G such that $f_c(S_1) = f_c(S_2) = f_c(S_3) = f_c(S_4) = f_c(S_5) = f_c(S_6) = f_c(S_7) = f_c(S_8) = f_c(S_9) = 3$. Thus $f_c(G) = 3$.*

The next theorem follows immediately from the definition of the connected hull number and the forcing connected hull number of a connected graph G.

Theorem 3.3. *For any connected graph $G, 0 \leqslant f_{hc}(G) \leqslant h_c(G) \leqslant p$.*

Remark 3.4. *The bounds in Theorem 3.3 are sharp.For any non-trivial tree T, by Corollary 2.9, the set of all vertices is the unique h_c-set of G. It follows that $f_{hc}(T) = 0$ and $h_c(T) = p$. For the complete bipartrite graph $G = K_{2,3}$ with bipartite sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$, the sets $S_1 = \{x_1, x_2, y_1\}, S_2 = \{x_1, x_2, y_2\}, S_3 = \{x_1, x_2, y_3\}, S_4 = \{x_1, y_2, y_1\}, S_5 = \{x_2, y_2, y_1\}, S_6 = \{x_1, y_2, y_3\}, S_7 = \{x_2, y_2, y_3\}, S_8 = \{x_1, y_1, y_3\}$ and $S_9 = \{x_2, y_1, y_3\}$ are the h_c-sets of G so that $h_c(G) = 3$.Also,it is easily seen that $f_{hc}(G) = 3$. Thus $f_{hc}(G) = h_c(G)$. Also, the inequality in the theorem can be strict. For the graph G given in Figure 2.1,the sets $S_1 = \{v_1, v_2, v_3, v_4, v_7\}, S_2 = \{v_1, v_2, v_4, v_5, v_7\}, S_3 = \{v_1, v_2, v_5, v_6, v_7\}, S_4 = \{v_1, v_2, v_3, v_6, v_7\}$ are the h_c-sets of G so that $h_c(G) = 5$.Also $f_{hc}(S_1) = f_{hc}(S_2) = f_{hc}(S_3) = f_{hc}(S_4) = 2$ so that $f_{hc}(G) = 2$. Thus $0 < f_{hc}(G) < h_c(G) < p$.*

Definition 3.5. *A vertex v of a connected graph G is said to be a connected hull vertex of G if v belongs to every h_c-set of G.*

Example 3.6. *For the graph G given in Figure 2.1, $S_1 = \{v_1, v_2, v_3, v_4, v_7\}, S_2 = \{v_1, v_2, v_4, v_5, v_7\}, S_3 = \{v_1, v_2, v_5, v_6, v_7\}$ and $S_4 = \{v_1, v_2, v_3, v_6, v_7\}$ are the only four h_c-sets of G. It is clear that v_1, v_2 and v_7 are the connected hull vertices of G.*

The proof of the following theorems and corollary are straight forward, therefore we omit it.

Theorem 3.7. *Let G be a connected graph. Then*
a) $f_{hc}(G) = 0$ if and only if G has a unique h_c-set.
b) $f_{hc}(G) = 1$ if and only if G has at least two h_c-sets, one of which is a unique h_c-set containing one of its elements.
c) $f_{hc}(G) = h_c(G)$ if and only if no connected h_c-set of G is the unique h_c-set containing any of its proper subsets.

Theorem 3.8. *Let G be a connected graph and let $\Im$ be the set of relative complements of the minimum forcing subsets in their respective connected h_c-sets in G. Then $\bigcap_{F \in \Im}$ is the set of all connected hull vertices of G.*

Corollary 3.9. *Let G be a connected graph and S a h_c-set of G. Then no connected hull vertex of G belongs to any minimum forcing subset of S.*

Theorem 3.10. *Let G be a connected graph and W be the set of all connected hull vertices of G. Then $f_{hc}(G) \leqslant h_c(G) - |W|$.*

Proof. Let S be any h_c-set of G.Then $h_c(G) = |S|$, $W \subseteq S$ and W is the unique h_c-set containing $S - W$. Thus $f_{hc}(G) \leqslant |S - W| = |S| - |W| = h_c(G) - |W|$. □

Corollary 3.11. *If G is a connected graph with k extreme vertices and l cut-vertices, then $f_{hc}(G) \leqslant h_c(G) - (k + l)$.*

Proof. This follows from Theorems 1.1 and 2.7. □

Remark 3.12. *The bound in Theorem 3.10 is sharp. For the graph G given in Figure 2.1 $S_1 = \{v_1, v_2, v_3, v_4, v_7\}$, $S_2 = \{v_1, v_2, v_4, v_5, v_7\}$, $S_3 = \{v_1, v_2, v_5, v_6, v_7\}$ and $S_4 = \{v_1, v_2, v_3, v_6, v_7\}$ are the only four h_c-sets so that $h_c(G) = 5$.Also, it is easily seen that $f_{hc}(G) = 2$ and $W = \{v_1, v_2, v_7\}$ is the set of connected hull vertices of G. Thus $f_{hc}(G) = h_c(G) - |W|$.*

Theorem 3.13. *If G is a connected graph with $h_c(G) = 2$, then $f_{hc}(G) = 0$.*

Proof. If $h_c(G) = 2$, then by Theorem 2.13, $G = K_2$. By Theorem 3.7(a), $f_{hc}(G) = 0$. □

Theorem 3.14. *For every pair a, b of integers with $a \geqslant 0, b > 2a + 2$, there exists a connected graph G such that $f_{hc}(G) = a$ and $h_c(G) = b$.*

Proof. We consider two cases
Case 1. Suppose $a = 0$. Let $G = K_b$. Then by Corollary 2.4, $h_c(G) = b$ and by Theorem 3.7(a), $f_{hc}(G) = 0$.
Case 2. Suppose $a \geqslant 1$. Then $b > 4$. Let $P_i : e_i, f_i, l_i, c_i, e_i (1 \leqslant i \leqslant a)$ be a copy of cycle C_4.Let Q_i be the graph obtained from P_i by adding a new vertex p_i and the edge $p_i l_i, p_i f_i$ and $p_i c_i (1 \leqslant i \leqslant a)$. The graph W_a is obtained from Q_i's by identifying e_i of Q_i and l_{i-1} of $Q_{i-1} (2 \leqslant i \leqslant a + 1)$.Let G be the graph given in Figure 3. 1 is obtained from W_a by adding new vertices $x, z_1, z_2, ..., z_{b-2a-2}$ and joining the edges $l_a z_1, l_a z_2, ..., l_a z_{b-2a-2}$ and $e_1 x$. Now we prove that $h_c(G) = b$. Let $Z = \{x, z_1, z_2, ..., z_{b-2a-2}, e_1, l_1, l_2, ..., l_a\}$ be the set of end vertices and cut vertices of G. Then by Theorems 2.3, every connected hull set of G contains Z. Now Z is a hull set. However $G[Z]$ is not connected.Therefore Z is not a connected hull set of G . Let $H_i = \{f_i, c_i\} (1 \leqslant i \leqslant a)$. It is easily observed that every connected hull set contains atleast one vertex from each $H_i (1 \leqslant i \leqslant a)$ so that $h_c(G) \geqslant b$. Now $S = Z \cup \{f_1, f_2, ..., f_a\}$ is a connected hull set of G so that $h_c(G) = b$. Next, we show that $f_{hc}(G) = a$. Since every h_c-set contains Z it follows from Theorem 3.10, that $f_{hc}(G) \leqslant h_c(G) - |Z| = a$. Now since $h_c(G) = b$ and every minimum connected hull set of G contains Z, it is easily seen that every minium connected hullset S is of the form $Z \cup \{v_1, v_2, ..., v_a\}$,where $v_i \in H_i (1 \leqslant i \leqslant a)$. Let T be any proper subset of S with $|T| < a$.Then there is a vertex $u_j (1 \leqslant i \leqslant a)$ such that $u_j \notin T$. Let w_j be a vertex of H_j distinct from u_j, then $S_1 = S \cup [S - \{u_j\}] \cup \{w_j\}$ is a h_c-set properly containing T. Thus S is not the

unique h_c-set containing T. Thus T is not a forcing subset of S. This is true for all minimum connected hull sets of G and so it follows that $f_{hc}(G) = a$. □

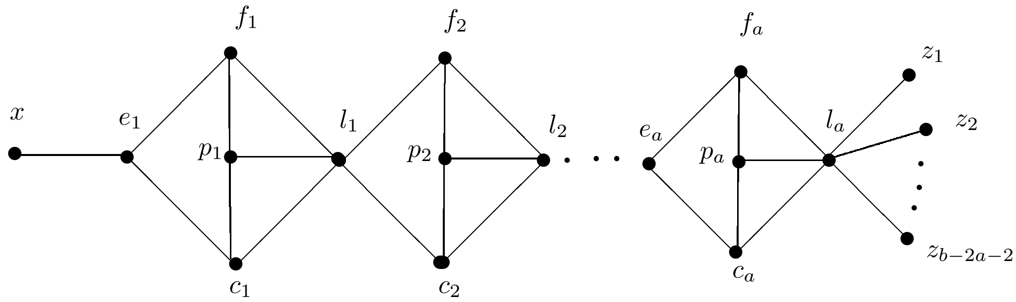

Figure 3.1

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