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The Drazin inverses of combinations of two idempotents on a Hilbert space

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Abstract The paper studied the criteria and representation of the Drazin inverse of combinations of two idempotent operators on a Hilbert space. By using the methods of splitting operator's matrix into blocks and space decompositions, the existence and calculation formulas of Drazin inverse of the combinations $aP + bQ + cPQ$ of two idempotent operators P and Q are obtained under the conditions $PQP = 0, PQP = P$ and $PQP = PQ$ respectively. These generalized the related results of Deng Chunyuan's work, which characterized the Drazin inverse of the sum and difference of two idempotents.

Key Words idempotent operator, Drazin inverse, combination

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space, the set of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathbf{B}(\mathcal{H})$. For an operator $T \in \mathbf{B}(\mathcal{H})$, $\mathcal{N}(T)$ and $\mathcal{R}(T)$ denote the null space and the range of T, respectively. An operator $P \in \mathbf{B}(\mathcal{H})$ is said to be idempotent if $P^2 = P$. If P satisfies $P^2 = P = P^*$ then P is called orthogonal projector, where P^* is the conjugate operator of $P \in \mathbf{B}(\mathcal{H})$. Let $T \in \mathbf{B}(\mathcal{H})$, if there exists an operator $T^D \in \mathbf{B}(\mathcal{H})$ and nonnegative integer k such that

$$TT^D = T^D T, \ \ T^D T T^D = T^D, \ \ T^{k+1} T^D = T^k,$$

then T^D is called a Drazin inverse of T. The least integer k such that the above identities are hold is called the index of T, denoted by $\mathrm{ind}(T) = k$. Specifically, if $k = 0$, then T is invertible and $T^D = T^{-1}$. For Drazin invertible operator $T \in \mathbf{B}(\mathcal{H})$, the Drazin inverse T^D of T is unique [1].

The set of all idempotents in $\mathbf{B}(\mathcal{H})$ is invariant under similarity, that is, if P is an idempotent operator and $S \in \mathbf{B}(\mathcal{H})$ is an invertible operator, then $S^{-1}PS$ is also an idempotent operator. Moreover the Drazin invertibility is also invariant under similarity, that is, if T is Drazin invertible and S is invertible, then $S^{-1}TS$ is Drazin invertible and $(S^{-1}TS)^D = S^{-1}T^D T$. Two facts are well known on a Hilbert space, one is that the orthogonal operator P is Drazin invertible and $P^D = P$, another is that

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for any idempotent operator P, there exists an invertible operator S such that $S^{-1}PS$ is an orthogonal projector [2]. In the following discussion, given two idempotent operators P and Q on $\mathcal{H}$, without loss of generality, we may assume that P is orthogonal.

The concept of a Drazin inverse was shown to be very useful in various applied mathematical settings which can be found in references [6, 3, 4, 5].

The problem of finding the Drazin inverse $(P \pm Q)^D$ of the sum and difference of two idempotents P and Q was first considered by Drazin in 1958 in his celebrated paper [7]. Herein, it was proved that

$$(P+Q)^D = P^D + Q^D \ \text{ provided } \ PQ = QP = 0.$$

The general question of how to express $(P+Q)^D$ as a function of P, Q, P^D, Q^D, without side condition, is very difficult and remains open [8].

In 2009, Deng Chunyuan extended Drazin's result to the three different cases

$$(i) PQP = 0; \ (ii) PQP = P; \ (iii) PQP = PQ,$$

see [9]. These cases are useful in several applications, such as in the splitting of operators and iteration theory. Zhang Shifang and Wu Junde discussed the Drazin inverse of the linear combinations of two idempotents in a Banach algebras and represent the Drazin inverse as a function of P, Q, PQ, QP, PQP, QPQ [10].

In 2010, Zuo considered a special combination $aP + bQ - cPQ$ of two idempotent matrices over complex numbers, and obtained that

$$r(aP + bQ - cPQ) = \begin{cases} r(P-Q), & \text{when} c = a+b \\ r(P+Q), & \text{when} c \neq a+b, \end{cases}$$

where $r(A)$ represents the rank of the matrix A [11]. Later, Xie and Zuo found that the Fredholmness, nullity and index of $aP+bQ+cPQ$ is independent of choices of scalars $a, b, c \in \mathbb{C}$ with $ab \neq 0, a+b+c \neq 0$ [12, 13]. After that, Liu, Wu and Yu discussed the group invertibility of combinations of two idempotents and represent the group inverse as a function of P, Q, PQ, QP, PQP, QPQ [14].

Under the above works, we consider the Drazin invertibility of combinations $aP + bQ + cPQ$ of two idempotent operators P and Q on $\mathcal{H}$. Under the conditions $PQP = 0, PQP = P$ and $PQP = PQ$, the representations for the Drazin inverse of $aP + bQ + cPQ$ as a functions of P, Q, PQ, QP, PQP, QPQ are obtained by using the technique of splitting matrices into blocks and space decompositions.

The following two Lemmas which were proved for a bounded linear operator [15] and for arbitrary elements in a Banach algebra [16].

Lemma 1.1 *Let $A \in \mathbf{B}(X), B \in \mathbf{B}(Y)$ and $C \in \mathbf{B}(Y, X)$. If A and B are Drazin invertible, then*

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \quad N = \begin{pmatrix} B & 0 \\ C & A \end{pmatrix}$$

are Drazin invertible and

$$M^D = \begin{pmatrix} A^D & X \\ 0 & B^D \end{pmatrix}, \quad N^D = \begin{pmatrix} B^D & 0 \\ X & A^D \end{pmatrix},$$

where $X = (A^D)^2[\sum_{i=0}^{\infty}(A^D)^i C B^i](I - BB^D) + (I - AA^D)[\sum_{i=0}^{\infty} A^i C (B^D)^i](B^D)^2 - A^D C B^D$.

Lemma 1.2 *Let* $A \in \mathbf{B}(X), B \in \mathbf{B}(Y)$ *and* $C \in \mathbf{B}(Y, X)$. *If* A *is invertible and* $B^k = 0$, *then*

$$M = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$$

are Drazin invertible and

$$M^D = \begin{pmatrix} A^{-1} & 0 \\ X & 0 \end{pmatrix},$$

where $X = \sum_{i=0}^{k-1} B^{k-1-i} C A^{i-k-1}$.

Lemma 1.3 **[1]** *Let* $A, B \in \mathbf{B}(\mathcal{H})$. *Then the following conditions are equivalent.*

(i) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$;

(ii) There exists $D \in \mathbf{B}(\mathcal{H})$ *such that* $B = AD$.

2 Main results

Let P and Q be two idempotent operators on the Hilbert space $\mathcal{H}$. For any given $a, b, c \in \mathbb{C}, ab \neq 0$, we discuss the Drazin invertibility of combinations of $aP + bQ + cPQ$ under some conditions and give the formula of its Drazin inverse.

Firstly, we consider the problem of Drazin invertibility of $aP + bQ + cPQ$ under the hypothesis of $PQP = 0$.

Theorem 2.1 *Let* P *and* Q *be two idempotents in* $B(\mathcal{H})$, *and* $a, b, c \in \mathbb{C}, ab \neq 0$. *If* $PQP = 0$, *then* $aP + bQ + cPQ$ *is Drazin invertible and* $(aP + bQ + cPQ)^D =$

$$\frac{1}{a}P + \frac{1}{b}Q - (\frac{1}{a} + \frac{1}{b} + \frac{c}{ab})PQ - (\frac{1}{a} + \frac{1}{b})QP + (\frac{1}{a} + \frac{2}{b} + \frac{c}{ab})QPQ.$$

Proof. Let P and Q be two idempotent operators in $B(\mathcal{H})$. With out loss of generality, we assume that P is an orthogonal projector. By Lemma 1.3, the condition $PQP = 0$ implies that $\mathcal{R}(QP) \subseteq \mathcal{N}(P)$ and $\mathcal{R}(QP) \subseteq \mathcal{R}(Q)$. Observing that $Q(\overline{\mathcal{R}(QP)} \oplus \mathcal{R}(P)) \subseteq \overline{\mathcal{R}(QP)}$, the space $\mathcal{H}$ can be decomposed as

$$\mathcal{H} = \overline{\mathcal{R}(QP)} \oplus \mathcal{R}(P) \oplus (\mathcal{R}(QP)^{\perp} \ominus \mathcal{R}(P)).$$

Then P and Q can be represented as

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & Q_{12} & Q_{13} \\ 0 & 0 & Q_{23} \\ 0 & 0 & Q_{33} \end{pmatrix},$$

where $\overline{\mathcal{R}(QP)}$ denotes the closure of $\mathcal{R}(QP)$. On the other hand, $Q^2 = Q$ gives that $Q_{33}^2 = Q_{33}$ and

$\mathcal{R}(QP)^{\perp} \ominus \mathcal{R}(P) = \mathcal{R}(Q_{33}) \oplus \mathcal{R}(Q_{33})^{\perp}$. It follows that P and Q can be written as

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & Q_{12} & Q'_{13} & Q''_{13} \\ 0 & 0 & Q'_{23} & Q''_{23} \\ 0 & 0 & I & Q''_{33} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

under the space decomposition $\mathcal{H} = \overline{\mathcal{R}(QP)} \oplus \mathcal{R}(P) \oplus \mathcal{R}(Q_{33})\mathcal{R}(Q_{33})^{\perp}$. The idempotency of Q implies that

$$Q'_{23}Q''_{33} = Q''_{23}, \;\; Q_{12}Q'_{23} + Q'_{13} = 0, \;\; Q_{12}Q''_{23} + Q'_{13}Q''_{33} = 0.$$

Direct calculations show that

$$aP + bQ + cPQ = \begin{pmatrix} bI & bQ_{12} & bQ'_{13} & bQ''_{13} \\ 0 & aI & (b+c)Q'_{23} & (b+c)Q''_{23} \\ 0 & 0 & bI & bQ''_{33} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It is clear that the condition $a, b \neq 0$ implies the invertibility of

$$\begin{pmatrix} bI & bQ_{12} & bQ'_{13} \\ 0 & aI & (b+c)Q'_{23} \\ 0 & 0 & bI \end{pmatrix}$$

on $\overline{\mathcal{R}(QP)} \oplus \mathcal{R}(P) \oplus \mathcal{R}(Q_{33})$ and its inverse is

$$\begin{pmatrix} \frac{1}{b}I & -\frac{1}{a}Q_{12} & -\frac{a+b+c}{ab}Q'_{13} \\ 0 & \frac{1}{a}I & -\frac{b+c}{ab}Q'_{23} \\ 0 & 0 & \frac{1}{b}I \end{pmatrix}.$$

Moreover,

$$\begin{pmatrix} \frac{1}{b}I & -\frac{1}{a}Q_{12} & -\frac{a+b+c}{ab}Q'_{13} \\ 0 & \frac{1}{a}I & -\frac{b+c}{ab}Q'_{23} \\ 0 & 0 & \frac{1}{b}I \end{pmatrix}^2 \begin{pmatrix} bQ''_{13} \\ (b+c)Q''_{23} \\ bQ''_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{b}Q''_{13} - (\frac{1}{a} + \frac{2}{b} + \frac{c}{ab})Q'_{13}Q''_{33} \\ -(\frac{1}{a} + \frac{c}{ab})Q''_{23} \\ \frac{1}{b}Q''_{33} \end{pmatrix}.$$

Applying $B = 0$ to the formula of representing Drazin inverse of upper triangle block matrix in Lemma 1.1, we have

$$(aP + bQ + cPQ)^D = \begin{pmatrix} \frac{1}{b}I & -\frac{1}{a}Q_{12} & -\frac{a+b+c}{ab}Q'_{13} & \frac{1}{b}Q''_{13} - (\frac{1}{a} + \frac{2}{b} + \frac{c}{ab})Q'_{13}Q''_{33} \\ 0 & \frac{1}{a}I & -\frac{b+c}{ab}Q'_{23} & -(\frac{1}{a} + \frac{c}{ab})Q''_{23} \\ 0 & 0 & \frac{1}{b}I & \frac{1}{b}Q''_{33} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, through direct calculations, we have

$$PQ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & Q'_{23} & Q''_{23} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad QP = \begin{pmatrix} 0 & Q_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$QPQ = \begin{pmatrix} 0 & 0 & Q_{12}Q'_{23} & Q_{12}Q''_{23} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $(aP + bQ + cPQ)^D = \frac{1}{a}P + \frac{1}{b}Q - (\frac{1}{a} + \frac{1}{b} + \frac{c}{ab})PQ - (\frac{1}{a} + \frac{1}{b})QP + (\frac{1}{a} + \frac{2}{b} + \frac{c}{ab})QPQ$. $\square$

From Theorem 2.1, we immediately have the representations for the Drazin inverse of $P + Q$ and $P - Q$ under the assumption that $PQP = 0$, which is the results of Theorem 2.1 of [9].

Corollary 2.1 *Let P and Q be two idempotents in $B(\mathcal{H})$. Assume that $PQP = 0$, then the following statements hold.*

(i)$(P + Q)^D = P + Q - 2(PQ + QP) + 3QPQ$.

(ii)$(P - Q)^D = P - Q - QPQ$.

If either of the stronger condition $QP = 0$ or $PQ = 0$ is satisfied, then by Theorem 2.1, we obtain the following results.

Corollary 2.2 *Let P and Q be two idempotents in $B(\mathcal{H})$. Then the following statements hold.*

(i) If $QP = 0$, then for any $a, b \in \mathbb{C}, ab \neq 0, a + b \neq 0$, $(aP + bQ)^D = \frac{1}{a}P + \frac{1}{b}Q - (\frac{1}{a} + \frac{1}{b})PQ$.

(ii) If $PQ = 0$, then for any $a, b \in \mathbb{C}, ab \neq 0, a + b \neq 0$, $(aP + bQ)^D = \frac{1}{a}P + \frac{1}{b}Q - (\frac{1}{a} + \frac{1}{b})QP$.

Next we give the representations for the Drazin inverse of $aP + bQ + cPQ$ under the assumption that $PQP = P$.

Theorem 2.2 *Let P and Q be two idempotents in $B(\mathcal{H})$, then for any $a, b, c \in \mathbb{C}, ab \neq 0$, the combinations $aP+bQ+cPQ$ are Drazin invertible under the condition $PQP = P$. The Drazin inverses of $aP+bQ+cPQ$ can be represented as following:*

(i) If $a + b + c \neq 0$, then $(aP + bQ + cPQ)^D =$

$$\frac{a^2 + ac}{(a + b + c)^3}P + \frac{1}{b}Q + \frac{(b + c)(a + c)}{(a + b + c)^3}PQ + \frac{ab}{(a + b + c)^3}QP + [\frac{b^2 + bc}{(a + b + c)^3} - \frac{1}{b}]QPQ.$$

(ii) If $a + b + c = 0$, then $(aP + bQ - cPQ)^D = \frac{1}{b}(Q - QPQ)$.

Proof. If $PQP = P$, then P and Q can be written as

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & Q_1 \\ Q_2 & Q_3 \end{pmatrix}$$

under the space decomposition of $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(P)^{\perp}$. The idempotency of Q yields that $Q_1Q_2 = 0, Q_1Q_3 = 0, Q_3Q_2 = 0$ and $Q_2Q_1 + Q_3^2 = Q_3$. It follows that $\mathcal{R}(Q_2) \subseteq \mathcal{N}(Q_1)$, $\mathcal{R}(Q_2) \subseteq \mathcal{N}(Q_3)$, $\mathcal{R}(Q_3) \subseteq \mathcal{N}(Q_1)$. With respect to the space decomposition $\mathcal{H} = \overline{\mathcal{R}(Q_1)} \oplus \mathcal{R}(Q_1)^{\perp} \oplus \overline{\mathcal{R}(Q_2)} \oplus \mathcal{R}(Q_2)^{\perp}$, P and Q can be represented as

$$P = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & 0 & 0 & Q_{11}' \\ 0 & I & 0 & 0 \\ Q_{21} & Q_{22} & 0 & Q_{31} \\ 0 & 0 & 0 & Q_{32} \end{pmatrix},$$

where $Q_{11}Q_{32} = 0, Q_{32}^2 = Q_{32}$ and $Q_{21}Q_{11} + Q_{31}Q_{32} = Q_{31}$. So, under the space decomposition of $\mathcal{H} = \overline{\mathcal{R}(Q_1)} \oplus \mathcal{Q}_{\infty}{}^{\perp} \oplus \overline{\mathcal{R}(Q_2)} \oplus \mathcal{R}(Q_{32}) \oplus \mathcal{R}(Q_{32})^{\perp}$, the operators P and Q can then be further written as

$$P = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & 0 & 0 & 0 & Q_{11}'' \\ 0 & I & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & Q_{31}' & Q_{31}'' \\ 0 & 0 & 0 & I & Q_{32}'' \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $Q_{21}Q_{11}'' + Q_{31}'Q_{32}'' = Q_{31}''$.

(i) If $a + b + c \neq 0$, then

$$aP + bQ + cPQ = \begin{pmatrix} (a+b+c)I & 0 & 0 & 0 & (b+c)Q_{11}'' \\ 0 & (a+b+c)I & 0 & 0 & 0 \\ bQ_{21} & bQ_{22} & 0 & bQ_{31}' & bQ_{31}'' \\ 0 & 0 & 0 & bI & bQ_{32}'' \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $b \neq 0$, let $a' = \frac{a}{b}, c' = \frac{c}{b}$, then we consider the following combination

$$a'P + Q + c'PQ = \begin{pmatrix} (a'+1+c')I & 0 & 0 & 0 & (1+c')Q_{11}'' \\ 0 & (a'+1+c')I & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & Q_{31}' & Q_{31}'' \\ 0 & 0 & 0 & I & Q_{32}'' \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$S=\begin{pmatrix} I & 0 & 0 & 0 & \frac{1+c'}{(a'+1+c')}Q''_{11} \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & Q''_{32} \\ 0 & 0 & I & 0 & Q''_{32} \\ 0 & 0 & 0 & 0 & I \end{pmatrix},$$

then

$$S^{-1}=\begin{pmatrix} I & 0 & 0 & 0 & -\frac{1+c'}{(a'+1+c')}Q''_{11} \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & -Q''_{32} \\ 0 & 0 & I & 0 & -Q''_{32} \\ 0 & 0 & 0 & 0 & I \end{pmatrix}.$$

Direct calculation shows that

$$S(a'P+Q+c'PQ)S^{-1}=\begin{pmatrix} (a'+1+c')I & 0 & 0 & 0 & 0 \\ 0 & (a'+1+c')I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ Q_{21} & Q_{22} & Q'_{31} & 0 & \frac{a'}{(a'+1+c')}Q_{21}Q''_{11} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows from Lemma 1.2 that

$$(a'P+Q+c'PQ)^D=S^{-1}(S(a'P+Q+c'PQ)S^{-1})^DS=$$

$$\begin{pmatrix} \frac{1}{(a'+1+c')}I & 0 & 0 & 0 & \frac{1+c'}{(a'+1+c')}Q''_{11} \\ 0 & \frac{1}{(a'+1+c')}I & 0 & 0 & 0 \\ \frac{1}{(a'+1+c')^2}Q_{21} & \frac{1}{(a'+1+c')^2}Q_{22} & 0 & Q'_{31} & \frac{1+c'}{(a'+1+c')^3}Q_{21}Q''_{11}+Q'_{31}Q''_{32} \\ 0 & 0 & 0 & I & Q''_{32} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$=\frac{a'^2+a'c'}{(a'+1+c')^3}P+Q+\frac{(1+c')(a'+c')}{(a'+1+c')^3}PQ+\frac{a'}{(a'+1+c')^3}QP+[\frac{1+c'}{(a'+1+c')^3}-1]QPQ.$$

Moreover, since $(cT)^D=\frac{1}{c}T^D$ holds for any $c\neq 0$ and any Drazin invertible operator $T\in\mathbf{B}(\mathcal{H})$. Hence

$$(aP+bQ+cPQ)^D=[b(a'P+Q+c'PQ)]^D=\frac{1}{b}(a'P+Q+c'PQ)^D=$$

$$\frac{a^2+ac}{(a+b+c)^3}P+\frac{1}{b}Q+\frac{(b+c)(a+c)}{(a+b+c)^3}PQ+\frac{ab}{(a+b+c)^3}QP+(\frac{b^2+bc}{(a+b+c)^3}-\frac{1}{b})QPQ.$$

(ii) If $a+b+c=0$, then

$$(aP+bQ+cPQ)^D = \begin{pmatrix} 0 & 0 & 0 & 0 & -aQ''_{11} \\ 0 & 0 & 0 & 0 & 0 \\ bQ_{21} & bQ_{22} & 0 & bQ'_{31} & bQ''_{31} \\ 0 & 0 & 0 & bI & bQ''_{32} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^D$$

$$= \frac{1}{b}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q''_{31} & Q'_{31}Q''_{32} \\ 0 & 0 & 0 & I & Q''_{32} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{b}(Q-QPQ).$$

□

Now we can derive some special cases from Theorem 2.2. These results are those of Theorem 2.3 in [9].

Corollary 2.3 *Let P and Q be two idempotents in $B(\mathcal{H})$. Assume that $PQP=P$, then the following statements hold.*

(i) $(P+Q)^D = \frac{1}{4}P+Q+\frac{1}{8}(PQ+QP)-\frac{7}{8}QPQ$.

(ii) $(P-Q)^D = Q(P-I)Q$.

We can also derive some special cases from Theorem 2.2 by the stronger condition $PQ=P$ or $QP=P$.

Corollary 2.4 *Let P and Q be two idempotents in $B(\mathcal{H})$, then the following statements hold.*

(i) If $PQ=P$, then $(aP+bQ+cPQ)^D = \frac{a+c}{(a+b+c)^2}P+(\frac{b}{(a+b+c)^2}-\frac{1}{b})QP+\frac{1}{b}Q$.

(ii) If $QP=P$, then $(aP+bQ+cPQ)^D = \frac{a}{(a+b+c)^2}P+(\frac{b+c}{(a+b+c)^2}-\frac{1}{b})PQ+\frac{1}{b}Q$.

Next we give the representations for the Drazin inverse of $aP+bQ+cPQ$ under the assumption that $PQP=PQ$.

Theorem 2.3 *Let P and Q be two idempotents in $B(\mathcal{H})$, then for any $a,b,c\in\mathbb{C}$, $ab\neq 0$, the combinations $aP+bQ+cPQ$ are Drazin invertible under the condition $PQP=PQ$. The Drazin inverses of $aP+bQ+cPQ$ can be represented as following:*

(i) If $a+b+c\neq 0$, then $(aP+bQ+cPQ)^D =$

$$\frac{1}{a}P+\frac{1}{b}Q+[\frac{a}{(a+b+c)^2}+\frac{c}{(a+b+c)^2}-\frac{1}{a}]PQ-(\frac{1}{a}+\frac{1}{b})QP+$$

$$[\frac{1}{a}+\frac{b}{a^2}-\frac{b(b+c)(2a+b+c)}{a^2(a+b+c)^2}]QPQ.$$

(ii) If $a+b+c=0$, then $(aP+bQ+cPQ)^D = \frac{1}{a}P+\frac{1}{b}Q-\frac{1}{a}PQ-(\frac{1}{a}+\frac{1}{b})QP-\frac{1}{a}QPQ$.

Proof. If $PQP = PQ$, then P and Q can be written as

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{pmatrix}$$

under the space decomposition of $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(P)^{\perp}$. The idempotency of Q yields that $Q_1^2 = Q_1, Q_3^2 = Q_3, Q_3Q_2 = 0$ and $Q_2Q_1 + Q_3^2 = Q_2$. With respect to the space decomposition $\mathcal{H} = \mathcal{R}(Q_1)^{\perp} \oplus \mathcal{R}(Q_1) \oplus \mathcal{R}(Q_3^*) \oplus \mathcal{R}(Q_3^*)$, P and Q can be further represented as

$$P = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ Q_{11} & I & 0 & 0 \\ Q_{21} & 0 & I & 0 \\ Q_{23} & Q_{24} & Q_{31} & 0 \end{pmatrix},$$

where $Q_{24}Q_{11} + Q_{31}Q_{21} = Q_{23}$.

(i) If $a + b + c \neq 0$, then

$$aP + bQ + cPQ = \begin{pmatrix} aI & 0 & 0 & 0 \\ (b+c)Q_{11} & (a+b+c)I & 0 & 0 \\ bQ_{21} & 0 & bI & 0 \\ bQ_{23} & bQ_{24} & bQ_{31} & 0 \end{pmatrix}.$$

Since $ab \neq 0$ and $a + b + c \neq 0$ then the submatrix

$$\begin{pmatrix} aI & 0 & 0 \\ (b+c)Q_{11} & (a+b+c)I & 0 \\ bQ_{21} & 0 & bI \end{pmatrix}$$

of $aP + bQ + cPQ$ is invertible and it's inverse is

$$\begin{pmatrix} \frac{1}{a}I & 0 & 0 \\ -\frac{b+c}{a(a+b+c)}Q_{11} & \frac{1}{a+b+c}I & 0 \\ -\frac{1}{a}Q_{21} & 0 & \frac{1}{b}I \end{pmatrix}$$

By using the results of Lemma 1.2 we have

$$(aP + bQ + cPQ)^D = \begin{pmatrix} \frac{1}{a}I & 0 & 0 & 0 \\ -\frac{b+c}{a(a+b+c)}Q_{11} & \frac{1}{a+b+c}I & 0 & 0 \\ -\frac{1}{a}Q_{21} & 0 & \frac{1}{b}I & 0 \\ X & \frac{b}{(a+b+c)^2}Q_{24} & \frac{1}{b}Q_{31} & 0 \end{pmatrix},$$

where $X = -\frac{1}{a}Q_{23} + [\frac{1}{a} + \frac{b}{a^2} - \frac{b(b+c)(2a+b+c)}{a^2(a+b+c)^2}]Q_{24}Q_{11}$. The coefficients of P, Q, PQ, QP, QPQ in the expression of $(aP + bQ + cPQ)^D$ can be obtained by solving some linear equations. Then we have

$$(aP + bQ + cPQ)^D = \frac{1}{a}P + \frac{1}{b}Q + [\frac{a}{(a+b+c)^2} + \frac{c}{(a+b+c)^2} - \frac{1}{a}]PQ - (\frac{1}{a} + \frac{1}{b})QP+$$

$$[\frac{1}{a}+\frac{b}{a^2}-\frac{b(b+c)(2a+b+c)}{a^2(a+b+c)^2}]QPQ.$$

(ii) If $a+b+c=0$, then

$$aP+bQ+cPQ=\begin{pmatrix} aI & 0 & 0 & 0 \\ -aQ_{11} & 0 & 0 & 0 \\ bQ_{21} & 0 & bI & 0 \\ bQ_{23} & bQ_{24} & bQ_{31} & 0 \end{pmatrix}.$$

By using the method in Theorem 2.1, we have

$$(aP+bQ+cPQ)^D=\begin{pmatrix} \frac{1}{a}I & 0 & 0 & 0 \\ -\frac{1}{a}Q_{11} & 0 & 0 & 0 \\ -\frac{1}{a}Q_{21} & 0 & \frac{1}{b}I & 0 \\ -\frac{1}{a}Q_{31}Q_{21} & 0 & \frac{1}{b}Q_{31} & 0 \end{pmatrix}$$

$$=\frac{1}{a}P+\frac{1}{b}Q-\frac{1}{a}PQ-(\frac{1}{a}+\frac{1}{b})QP-\frac{1}{a}QPQ.$$

□ Now we can derive some special cases from Theorem 2.3. These results are those of Theorem 2.6 in [9].

Corollary 2.5 *Let P and Q be two idempotents in $B(\mathcal{H})$. Assume that $PQP=PQ$, then the following statements hold.*

(i) $(P+Q)^D=P+Q-2QP-\frac{3}{4}PQ+\frac{5}{4}QPQ.$

(ii) $(P-Q)^D=P-Q-PQ+QPQ.$

We can also derive the formulaes of Drazin inverses of linear combinations of P and Q under the condition $PQP=PQ$.

Corollary 2.6 *Let P and Q be two idempotents in $B(\mathcal{H})$. Assume that $PQP=PQ$, then the following statements hold.*

$$(aP+bQ)^D=\begin{cases} \frac{1}{a}P+\frac{1}{b}Q+[\frac{1}{a+b}-\frac{b}{(a+b)^2}-\frac{1}{a}]PQ & \\ -\frac{a+b}{ab}QP+[\frac{b}{(a+b)^2}+\frac{1}{a}]QPQ, & \text{when } a+b\neq 0 \\ \frac{1}{a}(P-Q-PQ+QPQ), & \text{when } a+b=0. \end{cases}$$

References

1 R.G. Douglas. On majorization factorization and range inclusion of operators in Hilbert space[J]. Proc. Amer. Math. Soc., 17(1966): 413-416.

2 Guorong Wang, Yiming Wei , Sanzheng Qiao. Generalized inverse: theory and computations[M]. Graduate Series in Mathematics, Beijing: Science Press, 2004.

3 R.E. Hartwig , J. Levine. Applications of the Drazin inverse to the Hill cryptographic system[J]. Crytologia, 5(1981):67-77.

4 C.D. Meyer. The condition number of a finite Markov chains and perturbation bounds for the limiting probabilities[J]. SIMA J. Algebraic Discrete Methods, 1(1980):273-283.

5 B. Simeon, C. Fuhrer, P. Rentrop. The Drazin inverse in multibody system dynamics[J]. Numer. Math., 64(1993):521-536.

6 S.L. Campbell, C.D. Meyer. Generalized inverse of linear transformations[M]. London: Pitman Press, 1979.

7 M.P. Drazin. Pseudoinverse in associative rings and semigroups[J]. Amer. Math. Monthly, 65(1958):506-514.

8 R.E. Hartwig, Guorong Wang, Yiming Wei. Some additive results on Drazin inverse[J]. Linear Algebra Appl., 322(2001):207-217.

9 Chunyuan Deng. The Drazin inverses of sum and difference of idempotents[J]. Linear Algebra and its Applications, 430(2009): 1282-1291.

10 Shifang Zhang, Junde Wu. The Drazin inverse of the linear combinations of two idempotents in the Banach algebras[J]. http://arxiv,org/abs/0906,14.6vl. 2009.

11 Kezheng Zuo. Nonsingularity of the difference and the sum of two idempotents matrices[J]. Linear Algebra and its Applications, 433(2010):476-482.

12 Tao Xie, Kezheng Zuo. Fredholmness of combinations of two idempotents[J]. European J. Pure and Appl. Math., 3(4)(2010):678-685.

13 Tao Xie, Kezheng Zuo, Shengli Yu, Huina Liu. Nullity and fredholmness of combinations of the differnce and the sum of two idempotents[J]. South Asian Journal of Math., 2(3)(2012):285-291.

14 Xiaoji Liu, Lingling Wu, Yaoming Yu. The group inverse of the combinations of two idempotent matrices[J]. Linear and Multilinear Algebra, 59(1)(2011):101-115.

15 D.S. Djordjrvic, P.S. Stanimirovic. On the generalized Drazin inverse and generalized resolvent[J]. Czechoslovak Math. J., 126(2001): 671-634.

16 G.N. Castro, J.J. Koliha. New additive results for the g-Drazin inverse[J]. Preceedings of the Royal Society of Edinburgh, 134(1)(2004):1085-1097.