South Asian Journal of Mathematics 2012 , Vol. 2 (5) : 492 ∼ 496
www.sajm-online.com ISSN 2251-1512 RESEARCH ARTICLE

Compatible mappings on fixed point theorem in Menger spaces

M. Rangamma ①, A. Padma ①*

① Department of Mathematics O.U Hyderabad,Andhra Pradesh, India
E-mail: padmavanch@yahoo.com

Received: 8-5-2012; Accepted: 9-28-2012 *Corresponding author

Abstract In this paper, the concept of semi-compatibility and weak compatibility in Menger space has been applied to prove a common fixed point theorem for five self maps.

Key Words Probabilistic metric space, Menger space, compatible maps, semi-compatible maps

MSC 2010 47H10, 54H25

1 Introduction

Menger [4] was introduce a number of generalizations of metric space. One such generalization is Menger space. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function F_{xy}. a generalization of Banach Contraction Principle on a complete Menger space which is fixed-point theory in Menger space. [Sehgal and Bharucha-Reid 9] Jungck and Rhoades [3] termed a pair of self maps to be coincidentally commuting or equivalently weakly compatible if they commute at their coincidence points. Sessa [10] initiated the tradition of improving commutativity in fixed-point theorems by introducing the notion of weak commuting maps in metric spaces. Jungck [2] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [5]. Cho, Sharma and Sahu [1] introduced the concept of semi-compatibility in a d-complete topological space. Popa [7] proved interesting fixed point results using implicit real functions and semi-compatibility in d-complete topological space. In the sequel, Pathak and Verma [6] proved a common fixed point theorem in Menger space using compatibility and weak compatibility. In this paper a fixed point theorem for five self maps has been proved using the concept of semi-compatible maps and weak compatible maps.

2 Preliminaries

Definition 2.1. *A triangular norm $*$ (shortly t-norm) is a binary operation on the unit interval* $[0, 1]$ *such that for all* $a, b, c, d \in [0, 1]$ *the following conditions are satisfied:*

Citation: M. Rangamma, A. Padma, Compatible mappings on fixed point theorem in Menger spaces, South Asian J Math, 2012, 2(5), 492-496.

(1) $a * 1 = a$;

(2) $a * b = b * a$;

(3) $a * b \leqslant c * d$ *whenever* $a \leqslant c$ *and* $b \leqslant d$;

(4) $a * (b * c) = (a * b) * c$.

Examples of t-norms are $a * b = max\{a + b - 1, 0\}$ and $a * b = min\{a, b\}$.

Definition 2.2.([8]) *A probabilistic metric space (PM-space) is an ordered pair* (X, F) *consisting of a non empty set* X *and a function* $F : X \times X \to L$*, where* L *is the collection of all distribution functions and the value of* F *at* $(u, v) \in X \times X$ *is represented by* $F_{u,v}$*. The function* F_{uv} *assumed to satisfy the following conditions:*

$(PM\text{-}1)$ $F_{uv}(x) = 1$*, for all* $x > 0$*, if and only if* $u = v$*;*

$(PM\text{-}2)$ $F_{u,v}(0) = 0$; $(PM\text{-}3)$ $F_{u,v} = F_{v,u}$;

$(PM\text{-}4)$ *If* $F_{u,v}(x) = 1$ *and* $F_{v,w}(y) = 1$ *then* $F_{u,w}(x + y) = 1, \forall u, v, w \in X$ *and* $x, y > 0$.

Definition 2.3.([8]) *A Menger space is a triplet* (X, F, t) *where* (X, F) *is a PM-space and* $*$ *is a t-norm such that the inequality.*

(PM-5) $F_{u,w}(x + y) > F_{u,v}(x) * F_{v,w}(y)$*, for all* $u, v, w \in X, x, y \geqslant 0$.

Definition 2.4. *A mapping* $F : R \to R^+$ *is called a distribution if it is non-decreasing left continuous with* $inf\{F(t)|t \in R\} = 0$ *and* $sup\{F(t)|t \in R\} = 1$*. We shall denote by L the set of all distribution functions while* H *will always denote the specific distribution function defined by*

$$H(x) = \begin{cases} 0 & if\ t < 0, \\ 1 & if\ t \geqslant 0. \end{cases}$$

Proposition 2.1.([9]) *Let* (X, d) *be a metric space. Then the metric* d *induces a distribution function* F *defined by* $F_{xy}(t) = H(t - d(x, y))$ *for all* $x, y \in X$ *and* $t > 0$*. If t-norm* $*$ *is* $a * b = min\{a, b\}$ *for all* $a, b \in [0, 1]$ *then* $(X, F, *)$ *is a Menger space. Further,* $(X, F, *)$ *is a complete Menger space if* (X, d) *is complete.*

Definition 2.5.([5]) (a). *Let* $(X, F, *)$ *be a Menger space and* $*$ *be a continuous t-norm. (a) A sequence* $\{x_n\}$ *in* X *is said to be*

(i) Converge to a point x *in* S *(written* $x_n \to x$*) iff for every* $\varepsilon > 0$ *and* $\lambda \in (0, 1)$*, there exists an integer* $n_0 = n_0(\varepsilon, \lambda)$ *such that* $F_{x_n,x}(\varepsilon) > 1 - \lambda$ *for all* $n \geqslant n_0$

(ii) Cauchy if for every $\varepsilon > 0$ *and* $\lambda \in (0, 1)$*, there exists an integer* $n_0 = n_0(\varepsilon, \lambda)$ *such that* $F_{x_n,x_{n+p}}(\varepsilon) > 1 - \lambda$ *for all* $n \geqslant n_0$ *and* $p > 0$.

(iii) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.6. *Self mappings* A *and* S *of a Menger space* (X, F, t) *are said to be (i). Weak compatible if they commute at their coincidence points i.e.* $Ax = Sx$ *for* $x \in X$ *implies* $ASx = SAx$*. (ii). Compatible if* $F_{ASxn,SAxn}(x) \to 1$ *for all* $x > 0$*, whenever* x_n *is a sequence in* X *such that* $Ax_n, Sx_n \to u$ *for some* u *in* X*, as* $n \to \infty$*. (iii). semi-compatible if* $F_{ASx,Su}(x) \to 1$ *for all* $x > 0$*, whenever* x_n *is a sequence in*

X such that $Ax_n, Sx_n \to u$, for some u in X, as $n \to \infty$. Now, we give an example of pair of self maps (S,T) which is semi-compatible but not compatible. Further we observe here that the pair (T,S) is not semi-compatible though (S,T) is semi-compatible.

Lemma 2.1.([11]) *Let $\{x_n\}$ be a sequence in a Menger space $(X,F,*)$ with continuous t-norm $*$ and $t*t>t$. If there exists a constant $k \in (0,1)$ such that $Fx_n, x_{n+1}(kt) > Fx_{n_1}, x_n(t)$ for all $t>0$ and $n=1,2,3,\cdots$, then $\{x_n\}$ is a Cauchy sequence in X.*

3 Main Result

Theorem 3.1. *Let A,B,S,T,P and Q be self maps of a complete Menger space $(X,F,*)$ with $t*t>t$ satisfying:*

(a) *$P(x)\forall ST(x), Q(x),\ \forall AB(X)$;*

(b) *$AB=BA, ST=TS, PB=BL, QT=TQ$;*

(c) *Either Por AB is continuous;*

(d) *(P,AB) is semi-compatible and (Q,ST) is weak compatible;*

(e) *there exists a constant $q \in (0,1)$ such that $M_{(Px,Qy)}(qt) \geqslant M_{(ABX,STy,)}(t) * M_{(Px,ABx,t)}(t) * M_{(Qy,STy,)}(t) * M_{(Px,STy)}(t)$ A,B,S,T,P and Q have a unique common fixed point in X.*

Proof. Let $x_0 \in X$. From (a) there exist $x_1, x_2 \in X$ such that $Px_0 = STx_1$ and $Qx_1 = ABx_2$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Px2_{n-2} = STx_{2n-1} = y_{2n-1}$ and $Qx_{2n-1} = ABx_{2n} = y_{2n}$ for $n=1,2,3,\cdots$.

Step 1. Put $x=x_{2n}$ and $y=x_{2n+1}$ in (e), we get

$$\begin{aligned} M_{(Px_{2n}Qx_{2n+1})}(qt) &\geqslant M_{(ABx_{2n}S,Tx_{2n+1}))}(t)*M_{(Px_{2n},ABx_{2n}))}(t)*M_{(Qx_{2n+1},STx_{2n+1})}(t)*M_{(Px_{2n},S,Tx_{2n+1})}(t) \\ &\geqslant M_{(y_{2n}y_{2n+1})}(t)*M_{y_{2n+1}y_{2n})}(t)M_{(y_{2n+2}y_{2n+1})}(t)*M_{y_{2n+1}y_{2n+2})}(t) \\ &\geqslant M_{(y_{2n}y_{2n+1}))}(t)*M_{y_{2n+1}y_{2n+2})}(t). \end{aligned}$$

We have $M_{y_{2n+1}y_{2n+2})}(qt) \geqslant M_{(y_{2n}y_{2n+1})}(t)$. Similarly , $M_{y_{2n+2}y_{2n+3})}(qt) \geqslant M_{(y_{2n+1}y_{2n+2})}(t)$. Thus, we have

$$M_{y_{2n+2}y_{2n+3})}(qt) \geqslant M_{(y_ny_{n+1})}(t)$$

for $n=1,2,3\cdots$. $M_{y_ny_{n+3})}(t) \geqslant M_{(y_ny_{n+1})}(t/q) > M(y_{n-2},y_{n-1},t/q^2) > M(y_{n-2},y_{n-1},t/q^n)$ and hence $M(y_n,y_{n+1},t) \to 1$ as $n\to\infty$ for any $t>0$. For each $s>0$ and $t>1-\varepsilon$ for all $n>n_0$. we can choose $n_0 \in N$ such that $M(y_n,y_{n+1}(t) > 1-\varepsilon$ for all $n>n_0$. For $m,n \in N$, we suppose $m \geqslant n$. Then we have $M(y_n,y_m)(t) \geqslant M(y_n,y_{n+1})(t/m-n) * M(y_{n+1},y_{n+2})(t/m-n) * \cdots * M(y_{m-1},y_m)(t/m-n)$ $\geqslant (1-\varepsilon)*(1-\varepsilon)*\cdots*(1-\varepsilon)$ $(m-n$ times$)$ $\geqslant (1-\varepsilon)$ and hence $\{y_n\}$ is a Cauchy sequence in X. Since $(X,M,*)$ is complete, $\{y_n\}$ converges to some point $z\in X$. Also its subsequences converges to the same point i.e. $z \in X$, i.e., $\{Qx_{2n+1}\} \to z$ and $STx_{2n+1} \to z$ $\{Px_{2n}\} \to z$ and $\{ABx_{2n}\} \to z$.

Case I. Suppose AB is continuous. Since AB is continuous, we have $(AB)^2x_{2n} \to ABz$ and $ABPx_{2n} \to ABz$. As (P,AB) is compatible pair of type $(|\beta)$, we have $M(PPx_{2n},(AB)(AB)x_{2n},t)=1$, for all $t>0$ or, $M(PPx_{2n},ABz,t)=1$. Therefore, $PPx_{2n} \to ABz$.

Step 2. Put $x=ABx_{2n}$ and $y=x_{2n+1}$ in (e), we get $M(PABx_{2n},Qx_{2n+1},qt) \geqslant M(ABABx_{2n},STx_{2n+1},$

$t)*M(PABx_{2n}, ABABx_{2n}, t)*M(Qx_{2n+1}, STx_{2n+1}, t)*M(PABx_{2n}, STx_{2n+1}, t)$. Taking $n \to \infty$, we get $M(ABz, z, qt) \geqslant M(ABz, z, t)*M(ABz, A)Bz, t*M(z, z, t)*M(ABz, z, t) \geqslant M(ABz, z, t) * M(ABz, z, t)$ i.e. $M(ABz, z, qt) \geqslant M(ABz, z, t)$. Therefore, we get

$$ABz = z. \tag{3}$$

Step 3. Put $x = z$ and $y = x_{2n+1}$ in (e), we have $M_{p_z Qx_{2n+1}}(qt) \geqslant MAB_z STx_{2n+1)}(t) * M_{Pz,AB_z}(t)$ $*M_{Qx_{2n+1},STx_{2n+1}}(t) * M_{Pz,ST_{2n+1}}(t)$. Taking $n \to \infty$ and using equation (1), we get $M(Pz, z,)(qt) \geqslant M(z, z,)(t) * M(Pz, z,)(t) * M(z, z,)(t) * M(Pz, z,)(t) \geqslant M(Pz, z)(t)M(Pz, z,)(t)$ i.e. $M(Pz, z,)(qt) \geqslant M(Pz, z).(t)$ we get, $Pz = z$. Therefore, $ABz = Pz = z$.

Step 4. Putting $x = Bzandy = x_{2n+1}$, we get $M(PBz, Qx_{2n+1})(qt) \geqslant M(ABBz, STx_{2n+1})(qt)$ $*M(PBz, ABBz,)(qt)) *M(Qx_{2n+1}, STx_{2n+1},)(qt) *M(PBz, STx_{2n+1},)(qt)$ $As, BP = PB, AB = BA$, so we have $P(Bz) = B(Pz) = Bz$ and $(AB)(Bz) = (BA)(Bz) = B(ABz) = Bz$ Taking $n \to \infty$ and using (1), we get $M(Bz, z,)(qt) \geqslant M(Bz, z,)(t) * M(Bz, Bz,)(t) * M(z, z,)(t) * M(Bz, Bz,)(t) * M(Bz, z,)(t)$ $\geqslant M(Bz, z,)(t)*M(Bz, z,)(t)$ i.e $M(Bz, z,)(qt) \geqslant M(Bz, z,)(t)$ we get $Bz = z$ and also we have $ABz = z$ $,\forall Az = z$ Therefore, $Az = Bz = Pz = z$.

Step 5. As $P(X)\forall ST(X)$, there exists $u \in X$ such that $z = Pz = STu$. Putting $x = x_{2n}$ and $y = u$ in (e), we get $M(Px_{2n}, Qu,)(qt) > M(ABx_{2n}, STu)(t)*M(Px_{2n}, ABx_{2n})(t)M(Qu, STu, (t)(Px_{2n}, STu, t)(t)$ Taking $n- \to M\infty$ and using (1) and (2), we get $M(z, Qu)(qt) \geqslant M(z, z,)(t)*M(z, z, t)(t)*M(Qu, z,)(t)*$ $M(z, z,)(t) \geqslant M(Qu, z,)(t)i.e.M(z, Qu,)(qt) \geqslant M(z, Qu,)(t)$ Therefore, we get $Qu = z$. Hence $STu = z = Qu$. Since (Q, ST) is weak compatible, therefore, we have $QSTu = STQu$. Thus $Qz = STz$.

Step 6. Putting $x = x_{2n}$ and $y = z$ in (e), we get $M(Px_{2n}, Qz)(qt) \geqslant M(ABx_{2n}, STz)(t) *M(Px_{2n}, AB$ $x_{2n})(t) *M(Qz, STz,)(t)*M(Px_{2n}, STz)(t)$. Taking $n \to \infty$ and using (2) and step 5, we get $M(z, Qz)(qt)$ $\geqslant M(z, Qz)(t) *M(z, z)(t) *M(Qz, Qz)(t) *M(z, Qz)(t) \geqslant M(z, Qz)(t)*M(z, Qz)(t)$ i.e., $M(z, Qz)(qt) \geqslant M(z, Qz)(t)$. Therefore, we get $Qz = z$.

Step 7. Putting $x = x_{2n}$ and $y = Tz$ in (e), we get $M(Px_{2n}, Q)Tz(qt) > M(ABx_{2n}, STTz)t) *$ $M(Px_{2n}, ABx_{2n})(t)$ $M(QTz, STTz)(t)*M(Px_{2n}, STTz)(t)$. As $QT = TQ$ and $ST = TS$, we have $QTz = TQz = Tz$ $ST(Tz) = T(STz) = TQz = Tz$. Taking $n \to \infty$ we get $M(z, Tz,)(qt) > M(z, Tz,)(t)*M(z, z), (t)*M(Tz, Tz,)(t)*M(z, Tz,)(t) \geqslant M(z, Tz,)(t)*M(z, Tz)(t)$, i.e. $M(z, Tz)(qt) \geqslant M(z, Tz)(t)$. Therefore, we get $Tz = z$. Now $STz = Tz = z$ implies $Sz = z$.

Hence $Sz = Tz = Qz = z$. we get $Az = Bz = Pz = Qz = Tz = Sz = z$. Hence, z is the common fixed point of A, B, S, T, P and Q.

Case II. Suppose P is continuous. As P is continuous, $P2x_{2n} \twoheadrightarrow Pz$ and $P(AB)x_{2n} \twoheadrightarrow Pz$. As (P, AB) is compatible $M(PPx_{2n}, (AB)(AB)x_{2n},)(t) = 1$, for all $t > 0$ or, $M(Pz, (AB)(AB)x_{2n},)(t) = 1$. Therefore, $(AB)2x_{2n} \to Pz$.

Step 8. Putting $x = Px_{2n}$ and $y = x_{2n+1}$ in condition (e), we have $M(PPx_{2n}, Qx_{2n+1})(qt) > M(ABPx_{2n}, STx_{2n+1}))(t)*M(PPx_{2n}, ABPx_{2n})(t) *M(Qx_{2n+1}, STx_{2n+1})(t)*M(PPx_{2n}, STx_{2n+1})(t)$. Taking $n \to \infty$, we get $M(Pz, z, qt) > M(Pz, z, t) *M(Pz, Pz, t) *M(z, z, t) *M(Pz, z, t) > M(Pz, z, t)*$ $M(Pz, z, t)i.e.M(Pz, z, qt) > M(Pz, z, t)$. Therefore by using lemma 2.2, we have $Pz = z$.

Step 9. Put $x = ABx_{2n}$ and $y = x_{2n+1}$ in (e), we get $M(PABx_{2n}, Qx_{2n+1})(qt) \geqslant M(ABABx_{2n}, STx_{2n+1})$ $(t) *M(PABx_{2n}, ABABx_{2n})(t) *M(Qx_{2n+1}, STx_{2n+1})(t) * M(PABx_{2n}, STx_{2n+1})(t)$. Taking $n \to \infty$, we get $M(ABz, z)(qt) \geqslant M(ABz, z)(t) * M(ABz, ABz)(t) * M(z, z, t) * M(ABz, z)(t) \geqslant M(ABz, z)(t) * M(ABz, z)(t)$

Uniqueness: Let u be another common fixed point of A, B, S, T, P and Q. Then $Au = Bu = Pu = Qu = Su = Tu = u$. Put $x = z$ and $y = u$ in (e), we get $M(Pz, Qu,)(qt) > M(ABz, STu,)(t)$*$M(Pz, ABz)(, t)$ $*M(Qu, Stu)(, t) * M(Pz, STu,)(t)$. Taking $n \to \infty$, we get $M(z, u,)(qt) > M(z, u)(, t)$ $*M(z, z,)(t) * M(u, u)(t) * M(z, u,)(t) > M(z, u,)(t) * M(z, u,)(t)$ i.e. $M(z, u,)(qt) > M(z, u,)(t)$. we get $z = u$. □

Corollary 3.1. *let A, S, P and Q be self maps of a complete Menger Space $(X, F, *)$ with $t * t \geqslant t$ conditions are satisfied:*

(*a*) $P(X) \leqslant S(X), Q(X) \leqslant A(X)$;

(*b*) *either A or P is continuous;*

(*c*) *(P, A) is semi- compatible and (Q, S) is weak-compatible;*

(*d*) *there exists $q \in (0, 1)$ such that $M_{(Px,Qy)}(qt) \geqslant M_{(ABX,STy)}(t) * M_{(Px,ABx,t)}(t) * M_{(Qy,STy)}(t) * M_{(Px,STy)}(t)$. Then A, S, P and Q have a unique common fixed point in X.*

References

1 Y.J. Cho, B.K. Sharma and R.D. Sahu, Semi-compatibility and fixed points, Math. Japon. 42 (1), (1995), 91-98.

2 G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. and Math. Sci. 9(4), (1986), 771-779.

3 G. Jungck and B.E. Rhoades, Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. 29(1998), 227-238.

4 K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. USA. 28(1942), 535 -537.

5 S.N. Mishra, Common fixed points of compatible mappings in PM-spaces, Math. Japon. 36(2), (1991), 283-289.

6 H.K. Pathak and R.K. Verma, Common fixed point theorems for weakly compatible mappings in Menger space and application, Int. Journal of Math. Analysis, Vol. 3, 2009, No. 24, 1199-1206.

7 V. Popa, Fixed points for non-surjective expansion ma ppings satisfying an implicit relation, Bul. Stiint. Univ. Baia Mare Ser. B Fasc. Mat.-Inform. 18 (1), (2002), 105-108.

8 B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 313-334.

9 V.M. Sehgal and A.T. Bharucha-Reid, Fixed points of contraction maps on probabilistic metric spaces, Math. System Theory 6(1972), 97- 102.

10 S. Sessa, On a weak commutativity condition of mappings in fixed point consideration, Publ. Inst. Math. Beograd 32(46), (1982), 146-153.

11 B. Singh and S. Jain, A fixed point theorem in Menger space through weak compatibility, J. Math. Anal. Appl., 301 (2005), 439-448.

12 A. Jain, M. Sharma and B. Singh, Fixed point theorem using compatibility of type (β) in Fuzzy metric space, Chh. J. Sci. & Tech., Vol. 3 & 4, (2006-2007), 53-62.