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Sequence of numbers with three alternate common differences and common ratios

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Abstract This paper talks about two types of special sequences. The first is the arithmetic sequence of numbers with three alternate common differences; and the other, is the geometric sequence of numbers with three alternate common differences. The formulas for the general term a_n and the sum of the first n terms, denoted by S_n, are given respectively.

Key Words sequence, three alternate common ratios, alternate common differences

MSC 2010 20D06, 20D20

1 Arithmetic sequence of numbers with three alternate common differences

Definition 1.1. *A sequence of numbers $\{a_n\}$ is called a sequence of numbers with three alternating common differences if the following conditions are satisfied:*

(i) for all $k \in N$, $a_{3k-1} - a_{3k-2} = d_1$,

(ii) for all $k \in N$, $a_{3k} - a_{3k-1} = d_2$,

(iii) for all $k \in N$, $a_{3k+1} - a_{3k} = d_3$,

here d_1 (d_2, and d_3) is called the first (the second and the third) common differences of $\{a_n\}$.

Example 1.2. *The number sequence $1, 2, 4, 7, 8, 10, 13, 14, 16, \ldots$ is a sequence of numbers with three alternate common differences, where $d_1 = 1, d_2 = 2,$ and $d_3 = 3$.*

Obviously, $\{a_n\}$ has the following form

$$a_1,\ a_1 + d_1,\ a_1 + d_1 + d_2,\ a_1 + d_1 + d_2 + d_3, a_1 + 2d_1 + d_2 + d_3,\ a_1 + 2d_1 + 2d_2 + d_3,$$

$$a_1 + 2d_1 + 2d_2 + 2d_3,\ a_1 + 3d_1 + 2d_2 + 2d_3,\ a_1 + 3d_1 + 3d_2 + 2d_3, \ldots$$

Theorem 1.3. *The formula of the general term of a_n is*

$$a_n = a_1 + \left\lfloor \frac{n+1}{3} \right\rfloor d_1 + \left\lfloor \frac{n}{3} \right\rfloor d_2 + \left\lfloor \frac{n-1}{3} \right\rfloor d_3 \tag{1}$$

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Proof. We prove this theorem by induction on n.

Obviously, (1) holds for $n = 1$, 2, 3 and 4.

Suppose (1) holds when $n = k$, hence

$$a_k = a_1 + \left\lfloor \frac{k+1}{3} \right\rfloor d_1 + \left\lfloor \frac{k}{3} \right\rfloor d_2 + \left\lfloor \frac{k-1}{3} \right\rfloor d_3$$

We need to show that $\mathrm{P}(k+1)$ also holds for any $k \in N$.

(i.) If $k = 3m - 2$, where $m \in N$, then $a_{k+1} = a_k + d_1$

$$\begin{aligned} a_{k+1} &= a_1 + \left\lfloor \frac{k+1}{3} \right\rfloor d_1 + \left\lfloor \frac{k}{3} \right\rfloor d_2 + \left\lfloor \frac{k-1}{3} \right\rfloor d_3 + d_1 \\ &= a_1 + \left\lfloor \frac{3m-2+1}{3} \right\rfloor d_1 + \left\lfloor \frac{3m-2}{3} \right\rfloor d_2 + \left\lfloor \frac{3m-2-1}{3} \right\rfloor d_3 + d_1 \\ &= a_1 + (m-1)d_1 + (m-1)d_2 + (m-1)d_3 + d_1 \\ &= a_1 + \left\lfloor \frac{3m}{3} \right\rfloor d_1 + \left\lfloor m - 1 + \frac{2}{3} \right\rfloor d_2 + \left\lfloor m - 1 + \frac{1}{3} \right\rfloor d_3 \\ &= a_1 + \left\lfloor \frac{(k+1)+1}{3} \right\rfloor d_1 + \left\lfloor \frac{k+1}{3} \right\rfloor d_2 + \left\lfloor \frac{(k+1)-1}{3} \right\rfloor d_3 \end{aligned}$$

$\therefore$ $\mathrm{P}(k+1)$ holds for $k = 3m - 2$.

(ii.) If $k = 3m - 1$, where $m \in N$, then $a_{k+1} = a_k + d_2$

$$\begin{aligned} a_{k+1} &= a_1 + \left\lfloor \frac{k+1}{3} \right\rfloor d_1 + \left\lfloor \frac{k}{3} \right\rfloor d_2 + \left\lfloor \frac{k-1}{3} \right\rfloor d_3 + d_2 \\ &= a_1 + \left\lfloor \frac{3m-1+1}{3} \right\rfloor d_1 + \left\lfloor \frac{3m-1}{3} \right\rfloor d_2 + \left\lfloor \frac{3m-1-1}{3} \right\rfloor d_3 + d_2 \\ &= a_1 + md_1 + (m-1)d_2 + (m-1)d_3 + d_2 \\ &= a_1 + \left\lfloor m + \frac{1}{3} \right\rfloor d_1 + \left\lfloor \frac{3m}{3} \right\rfloor d_2 + \left\lfloor m - 1 + \frac{2}{3} \right\rfloor d_3 \\ &= a_1 + \left\lfloor \frac{(k+1)+1}{3} \right\rfloor d_1 + \left\lfloor \frac{k+1}{3} \right\rfloor d_2 + \left\lfloor \frac{(k+1)-1}{3} \right\rfloor d_3 \end{aligned}$$

$\therefore$ $\mathrm{P}(k+1)$ holds for $k = 3m - 1$.

(iii.) If $k = 3m$, where $m \in N$, then $a_{k+1} = a_k + d_3$

$$\begin{aligned} a_{k+1} &= a_1 + \left\lfloor \frac{k+1}{3} \right\rfloor d_1 + \left\lfloor \frac{k}{3} \right\rfloor d_2 + \left\lfloor \frac{k-1}{3} \right\rfloor d_3 + d_3 \\ &= a_1 + \left\lfloor \frac{3m+1}{3} \right\rfloor d_1 + \left\lfloor \frac{3m}{3} \right\rfloor d_2 + \left\lfloor \frac{3m-1}{3} \right\rfloor d_3 + d_3 \\ &= a_1 + md_1 + md_2 + (m-1)d_3 + d_3 \\ &= a_1 + \left\lfloor m + \frac{2}{3} \right\rfloor d_1 + \left\lfloor m + \frac{1}{3} \right\rfloor d_2 + \left\lfloor \frac{3m}{3} \right\rfloor d_3 \end{aligned}$$

$$= \quad a_1 + \left\lfloor \frac{(k+1)+1}{3} \right\rfloor d_1 + \left\lfloor \frac{k+1}{3} \right\rfloor d_2 + \left\lfloor \frac{(k+1)-1}{3} \right\rfloor d_3$$

$\therefore$ P$(k+1)$ holds for $k = 3m$.

Therefore, (1) holds when $n = k+1$. This proves the theorem. □

Theorem 1.4. *The formula of the general term of a_n can also be*

$$a_n = a_1 + \left\lfloor \frac{n-1}{3} \right\rfloor d + \left(\left\lfloor \frac{n+1}{3} \right\rfloor - \left\lfloor \frac{n-1}{3} \right\rfloor \right) d_1 + \left(\left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n-1}{3} \right\rfloor \right) d_2 \tag{2}$$

where $d = d_1 + d_2 + d_3$.

Formula (2) can be shown easily using induction on n. The proof for the theorem is ommited.

Now we proceed to the sum of the first n terms of the sequence.

Theorem 1.5. *The sum of the of the first n terms of the sequence, denoted by S_n, is given by*

$$S_n = na_1 + \frac{1}{2} d \sum_{i=0}^{2} \left\lfloor \frac{n+i}{3} \right\rfloor + 2 \left(\left\lfloor \frac{n+1}{3} \right\rfloor d_1 - \left\lfloor \frac{n}{3} \right\rfloor d_3 \right)$$

where $d = d_1 + d_2 + d_3$

Proof. Let $d = d_1 + d_2 + d_3$.

$$\begin{aligned}
S_n &= a_1 + (a_1 + d_1) + (a_1 + d_1 + d_2) + (a_1 + d_1 + d_2 + d_3) \\
&\quad + (a_1 + 2d_1 + d_2 + d_3) + (a_1 + 2d_1 + 2d_2 + d_3) + \dots \\
&\quad + \left(a_1 + \left\lfloor \frac{k+1}{3} \right\rfloor d_1 + \left\lfloor \frac{k}{3} \right\rfloor d_2 + \left\lfloor \frac{k-1}{3} \right\rfloor d_3 \right) \\
&= (a_1 + (1-1)d) + (a_1 + d_1 + (1-1)d) + (a_1 + d_1 + d_2 + (1-1)d) \\
&\quad + (a_1 + (2-1)d) + (a_1 + d_1 + (2-1)d) + (a_1 + d_1 + d_2 + (2-1)d) \\
&\quad + (a_1 + (3-1)d) + \dots + \left(a_1 + d_1 + d_2 + \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) d \right) \\
&\quad + \left(a_1 + d_1 + \left(\left\lfloor \frac{n+1}{3} \right\rfloor - 1 \right) d \right) + \left(a_1 + \left(\left\lfloor \frac{n+2}{3} \right\rfloor - 1 \right) d \right) \\
&= \left(\left\lfloor \frac{n+2}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor \right) a_1 + \frac{1}{2} \left\lfloor \frac{n+2}{3} \right\rfloor \left(\left\lfloor \frac{n+2}{3} \right\rfloor - 1 \right) d \\
&\quad + \left\lfloor \frac{n+1}{3} \right\rfloor d_1 + \frac{1}{2} \left\lfloor \frac{n+1}{3} \right\rfloor \left(\left\lfloor \frac{n+1}{3} \right\rfloor - 1 \right) d + \left\lfloor \frac{n}{3} \right\rfloor (d_1 + d_2) \\
&\quad + \frac{1}{2} \left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) d \\
&= na_1 + \frac{1}{2} \left(\left\lfloor \frac{n+2}{3} \right\rfloor \left(\left\lfloor \frac{n+2}{3} \right\rfloor - 1 \right) + \left\lfloor \frac{n+1}{3} \right\rfloor \left(\left\lfloor \frac{n+1}{3} \right\rfloor - 1 \right) \right) d_1 \\
&\quad + \frac{1}{2} \left(2 \left\lfloor \frac{n+1}{3} \right\rfloor + 2 \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor \left(\left\lfloor \frac{n}{3} \right\rfloor - 1 \right) \right) d_1 \\
&\quad + \frac{1}{2} \left(\left\lfloor \frac{n+2}{3} \right\rfloor \left(\left\lfloor \frac{n+2}{3} \right\rfloor - 1 \right) + \left\lfloor \frac{n+1}{3} \right\rfloor \left(\left\lfloor \frac{n+1}{3} \right\rfloor - 1 \right) \right) d_2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left(\left\lfloor\frac{n+2}{3}\right\rfloor\left(\left\lfloor\frac{n+2}{3}\right\rfloor-1\right)+\left\lfloor\frac{n+1}{3}\right\rfloor\left(\left\lfloor\frac{n+1}{3}\right\rfloor-1\right)\right)d_3\\
&+\frac{1}{2}\left(2\left\lfloor\frac{n}{3}\right\rfloor+\left\lfloor\frac{n}{3}\right\rfloor\left(\left\lfloor\frac{n}{3}\right\rfloor-1\right)\right)d_2+\frac{1}{2}\left(\left\lfloor\frac{n}{3}\right\rfloor\left(\left\lfloor\frac{n}{3}\right\rfloor-1\right)\right)d_3\\
= \;& na_1+\frac{1}{2}\left(\left\lfloor\frac{n+2}{3}\right\rfloor\left\lfloor\frac{n-1}{3}\right\rfloor+\left\lfloor\frac{n+1}{3}\right\rfloor\left\lfloor\frac{n+4}{3}\right\rfloor+\left\lfloor\frac{n}{3}\right\rfloor\left\lfloor\frac{n+3}{3}\right\rfloor\right)d_1\\
&+\frac{1}{2}\left(\left\lfloor\frac{n+2}{3}\right\rfloor\left\lfloor\frac{n-1}{3}\right\rfloor+\left\lfloor\frac{n+1}{3}\right\rfloor\left\lfloor\frac{n-2}{3}\right\rfloor+\left\lfloor\frac{n}{3}\right\rfloor\left\lfloor\frac{n+3}{3}\right\rfloor\right)d_2\\
&+\frac{1}{2}\left(\left\lfloor\frac{n+2}{3}\right\rfloor\left\lfloor\frac{n-1}{3}\right\rfloor+\left\lfloor\frac{n+1}{3}\right\rfloor\left\lfloor\frac{n-2}{3}\right\rfloor+\left\lfloor\frac{n}{3}\right\rfloor\left\lfloor\frac{n-3}{3}\right\rfloor\right)d_3\\
= \;& na_1+\frac{1}{2}\left(\left\lfloor\frac{n+2}{3}\right\rfloor\left(\left\lfloor\frac{n+2}{3}\right\rfloor-1\right)+\left\lfloor\frac{n+1}{3}\right\rfloor\left(\left\lfloor\frac{n+1}{3}\right\rfloor-1\right)\right)d\\
&+\frac{1}{2}\left\lfloor\frac{n}{3}\right\rfloor\left(\left\lfloor\frac{n}{3}\right\rfloor-1\right)d+2\left(\left\lfloor\frac{n+1}{3}\right\rfloor d_1-\left\lfloor\frac{n}{3}\right\rfloor d_3\right)
\end{aligned}
$$

□

Lemma 1.6. *For any positive integers p, q, and n,*

$$\left[\frac{p}{q}\right]+n=\left[\frac{p+nq}{q}\right]$$

Proof.

$$
\begin{aligned}
\left[\frac{p}{q}\right] \quad & \Rightarrow \quad k\leqslant\frac{p}{q}<k+1 \text{ where } k \text{ is an integer}\\
& \Rightarrow \quad m\leqslant\frac{p}{q}+n<m+1,\; m=n+k.\\
& \therefore \quad \left[\frac{p}{q}\right]+n=\left[\frac{p+nq}{q}\right].
\end{aligned}
$$

□

Theorem 1.7. *For any integer $m>0$*

$$\sum_{i=mq}^{n}\left\lfloor\frac{i}{m}\right\rfloor=\left\lfloor\frac{n}{m}\right\rfloor\left(n+1-m\left\lfloor\frac{n}{m}\right\rfloor\right)$$

where $q=\left\lfloor\frac{n}{m}\right\rfloor$.

Proof.

$$
\begin{aligned}
\sum_{i=mq}^{n}\left\lfloor\frac{i}{m}\right\rfloor &= \sum_{i=0}^{n-mq}\left\lfloor\frac{i+mq}{m}\right\rfloor\\
&= \sum_{i=0}^{n-mq}\left(q+\left\lfloor\frac{i}{m}\right\rfloor\right)\\
&= \sum_{i=0}^{n-mq}\left\lfloor\frac{i}{m}\right\rfloor+\sum_{i=0}^{n-mq}q
\end{aligned}
$$

$$\begin{aligned} &= \left\lfloor \frac{0}{m} \right\rfloor + \left\lfloor \frac{1}{m} \right\rfloor + \ldots + \left\lfloor \frac{n-mq}{m} \right\rfloor + q(n+1-mq) \\ &= \left\lfloor \frac{0}{m} \right\rfloor + \left\lfloor \frac{1}{m} \right\rfloor + \ldots + \left\lfloor \frac{n}{m} \right\rfloor - q + q(n+1-mq) \\ &= \left\lfloor \frac{n}{m} \right\rfloor \left(n+1-m\left\lfloor \frac{n}{m} \right\rfloor \right) \end{aligned}$$

□

Corollary 1.8. *For any integer $m > 0$,*

$$\sum_{i=0}^{n} \left\lfloor \frac{i}{m} \right\rfloor = \left\lfloor \frac{n}{m} \right\rfloor \left(n+1-\frac{m}{2}\left\lfloor \frac{n+m}{m} \right\rfloor \right)$$

Proof. Let $q = \left\lfloor \frac{n}{m} \right\rfloor$

$$\begin{aligned} \sum_{i=0}^{n} \left\lfloor \frac{i}{m} \right\rfloor &= \sum_{i=0}^{m-1} \left\lfloor \frac{i}{m} \right\rfloor + \sum_{i=m}^{2m-1} \left\lfloor \frac{i}{m} \right\rfloor + \ldots \\ &\quad + \sum_{i=m(q-1)}^{mq-1} \left\lfloor \frac{i}{m} \right\rfloor + \sum_{i=mq}^{n} \left\lfloor \frac{i}{m} \right\rfloor \\ &= \sum_{j=0}^{q-1} \left(\sum_{i=jm}^{(j+1)m-1} \left\lfloor \frac{i}{m} \right\rfloor \right) + \sum_{i=mq}^{n} \left\lfloor \frac{i}{m} \right\rfloor \\ &= \sum_{j=0}^{q-1} mj + \sum_{i=mq}^{n} \left\lfloor \frac{i}{m} \right\rfloor \\ &= \frac{mq}{2}(q-1) + q(n+1-mq) \\ &= q\left(\frac{mq}{2} - \frac{m}{2} + n + 1 - mq \right) \\ &= \left\lfloor \frac{n}{m} \right\rfloor \left(n+1-\frac{m}{2}\left\lfloor \frac{n+m}{m} \right\rfloor \right) \end{aligned}$$

□

Theorem 1.9. *The sum of the first n terms of the sequence can also be*

$$S_n = na_1 + \left\lfloor \frac{n+1}{3} \right\rfloor \left(n+2-\frac{3}{2}\left\lfloor \frac{n+4}{3} \right\rfloor \right) d_1 + \left\lfloor \frac{n}{3} \right\rfloor \left(n+1-\frac{3}{2}\left\lfloor \frac{n+3}{3} \right\rfloor \right) d_2$$

$$+ \left\lfloor \frac{n-1}{3} \right\rfloor \left(n-\frac{3}{2}\left\lfloor \frac{n+2}{3} \right\rfloor \right) d_3$$

Proof.

$$\begin{aligned} S_n &= \sum_{i=1}^{n} \left(a_1 + \left\lfloor \frac{i+1}{3} \right\rfloor d_1 + \left\lfloor \frac{i}{3} \right\rfloor d_2 + \left\lfloor \frac{i-1}{3} \right\rfloor d_3 \right) \\ &= na_1 + \sum_{i=1}^{n} \left\lfloor \frac{i+1}{3} \right\rfloor d_1 + \sum_{i=1}^{n} \left\lfloor \frac{i}{3} \right\rfloor d_2 + \sum_{i=1}^{n} \left\lfloor \frac{i-1}{3} \right\rfloor d_3 \end{aligned}$$

$$= na_1 + \left\lfloor \frac{n+1}{3} \right\rfloor \left(n + 2 - \frac{3}{2} \left\lfloor \frac{n+4}{3} \right\rfloor \right) d_1$$
$$+ \left\lfloor \frac{n}{3} \right\rfloor \left(n + 1 - \frac{3}{2} \left\lfloor \frac{n+3}{3} \right\rfloor \right) d_2 + \left\lfloor \frac{n-1}{3} \right\rfloor \left(n - \frac{3}{2} \left\lfloor \frac{n+2}{3} \right\rfloor \right) d_3$$

□

2 Geometric sequence of numbers with three alternate common ratios

Definition 2.1. *A sequence of numbers $\{a_n\}$ is called a sequence of numbers with three alternating common ratios if the following conditions are satisfied:*

(i) for all $k \in N$, $\frac{a_{3k-1}}{a_{3k-2}} = r_1$,

(ii) for all $k \in N$, $\frac{a_{3k}}{a_{3k-1}} = r_2$,

(iii) for all $k \in N$, $\frac{a_{3k+1}}{a_{3k}} = r_3$,

where r_1 , r_2, and r_3 are called the first, the second and the third common ratios of $\{a_n\}$ respectively.

Example 2.2. *The number sequence* $1, 1/2, 1/6, 1/24, 1/48, 1/144, 1/576, 1/1152, 1/3456, \ldots$ *is an example of the sequence where $r_1 = 1/2, r_2 = 1/3$, and $r_3 = 1/4$.*

Obviously, $\{a_n\}$ has the following form

$$a_1,\ a_1r_1,\ a_1r_1r_2,\ a_1r_1r_2r_3, a_1r_1^2r_2r_3,\ a_1r_1^2r_2^2r_3,\ a_1r_1^2r_2^2r_3^2,\ a_1r_1^3r_2^2r_3^2,\ \ldots$$

Theorem 2.3. *The formula of the general term of a_n is*

$$a_n = a_1 \cdot r_1^{e_{n+1}} \cdot r_2^{e_n} \cdot r_3^{e_{n-1}} \tag{3}$$

where $e_i = \left\lfloor \frac{i}{3} \right\rfloor$.

Proof. Let $e_i = \left\lfloor \frac{i}{3} \right\rfloor$ and use induction on n to prove theorem 2.3.

Obviously, (3) holds for $n = 1$, 2, 3 and 4.

Now suppose (3) holds when $n = k$, hence

$$a_k = a_1 \cdot r_1^{e_{k+1}} \cdot r_2^{e_k} \cdot r_3^{e_{k-1}} \tag{4}$$

We need to show that $\mathrm{P}(k+1)$ also holds for any $k \in N$.

(i.) If $k = 3m - 2$, where $m \in N$, then $a_{k+1} = a_k \cdot r_1$

$$\begin{aligned}
a_k &= a_1 \cdot r_1^{e_{k+1}} \cdot r_2^{e_k} \cdot r_3^{e_{k-1}} \cdot r_1 \\
&= a_1\ r_1^{e_{3m-2+1}}\ r_2^{e_{3m-2}}\ r_3^{e_{3m-2-1}} \cdot r_1 \\
&= a_1\ r_1^{m-1}\ r_2^{m-1}\ r_3^{m-1} \cdot r_1 \\
&= a_1\ r_1^{\lfloor \frac{3m}{3} \rfloor}\ r_2^{\lfloor m-1+\frac{2}{3} \rfloor}\ r_3^{\lfloor m-1+\frac{1}{3} \rfloor} \\
&= a_1\ r_1^{\lfloor \frac{(k+1)+1}{3} \rfloor}\ r_2^{\lfloor \frac{k+1}{3} \rfloor}\ r_3^{\lfloor \frac{(k+1)-1}{3} \rfloor}
\end{aligned}$$

$\therefore$ P$(k+1)$ holds for $k = 3m - 2$.

(ii.) If $k = 3m - 1$, where $m \in N$, then $a_{k+1} = a_k \cdot r_2$

$$\begin{aligned}
a_k &= a_1 \cdot r_1^{e_{k+1}} \cdot r_2^{e_k} \cdot r_3^{e_{k-1}} \cdot r_2 \\
&= a_1\ r_1^{e_{3m-1+1}}\ r_2^{e_{3m-1}}\ r_3^{e_{3m-1-1}} \cdot r_2 \\
&= a_1\ r_1^{m}\ r_2^{m-1}\ r_3^{m-1} \cdot r_2 \\
&= a_1\ r_1^{\lfloor m+\frac{1}{3} \rfloor}\ r_2^{\lfloor \frac{3m}{3} \rfloor}\ r_3^{\lfloor m-1+\frac{2}{3} \rfloor} \\
&= a_1\ r_1^{\lfloor \frac{(k+1)+1}{3} \rfloor}\ r_2^{\lfloor \frac{k+1}{3} \rfloor}\ r_3^{\lfloor \frac{(k+1)-1}{3} \rfloor}
\end{aligned}$$

$\therefore$ P$(k+1)$ holds for $k = 3m - 1$.

(iii.) If $k = 3m$, where $m \in N$, then $a_{k+1} = a_k \cdot r_3$

$$\begin{aligned}
a_k &= a_1 \cdot r_1^{e_{k+1}} \cdot r_2^{e_k} \cdot r_3^{e_{k-1}} \cdot r_3 \\
&= a_1\ r_1^{e_{3m+1}}\ r_2^{e_{3m}}\ r_3^{e_{3m-1}} \cdot r_3 \\
&= a_1\ r_1^{m}\ r_2^{m}\ r_3^{m-1} \cdot r_3 \\
&= a_1\ r_1^{\lfloor m+\frac{2}{3} \rfloor}\ r_2^{\lfloor m+\frac{1}{3} \rfloor}\ r_3^{\lfloor \frac{3m}{3} \rfloor} \\
&= a_1\ r_1^{\lfloor \frac{(k+1)+1}{3} \rfloor}\ r_2^{\lfloor \frac{k+1}{3} \rfloor}\ r_3^{\lfloor \frac{(k+1)-1}{3} \rfloor}
\end{aligned}$$

$\therefore$ P$(k+1)$ holds for $k = 3m$.

Therefore, (5) holds when $n = k + 1$ and this proves the theorem. □

Theorem 2.4. *The formula of the general term of a_n can also be*

$$a_n = a_1\ r^{e_{n-1}}\ r_1^{e_{n+1}-e_{n-1}}\ r_2^{e_n - e_{n-1}}$$

where $r = r_1 \cdot r_2 \cdot r_3$ and $e_i = \lfloor \frac{i}{m} \rfloor$.

The proof for theorem 2.4 is ommited but it can be easily verified using mathematical induction.

Theorem 2.5. *The formula for the sum of the first n terms of the sequence is given by*

$$S_n = a_1 \left(R \left(\frac{1-r^{e_{n-1}}}{1-r} \right) + 1 \right) + a_1 \, r^{e_{n-1}} \left(r_1 \left(\left\lfloor \frac{n+1}{3} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor \right) \right)$$

$$+ a_1 \, r^{e_{n-1}} \left((r_1 + r_1 r_2) \left(\left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n-1}{3} \right\rfloor \right) \right)$$

where $R = r_1 + r_1 \, r_2 + r_1 \, r_2 \, r_3$, $r = r_1 \, r_2 \, r_3$ *and* $e_{n-1} = \lfloor \frac{n-1}{3} \rfloor$

Proof. Let $p = e_{n-1} = \lfloor \frac{n-1}{3} \rfloor$, $R = r_1 + r_1 \, r_2 + r_1 \, r_2 \, r_3$ and $r = r_1 \, r_2 \, r_3$.

$$\begin{aligned}
S_n &= a_1 + a_1 r_1 + a_1 r_1 r_2 + a_1 r_1 r_2 r_3 + a_1 r_1^2 r_2 r_3 + a_1 r_1^2 r_2^2 r_3 + a_1 r_1^2 r_2^2 r_3^2 \\
&\quad + a_1 r_1^3 r_2^2 r_3^2 + a_1 r_1^3 r_2^3 r_3^2 + a_1 r_1^3 r_2^3 r_3^3 + \ldots + a_1 \, r_1^{e_{n-1}} \, r_2^{e_{n-2}} \, r_3^{e_{n-3}} \\
&\quad + a_1 \, r_1^{e_n} \, r_2^{e_{n-1}} \, r_3^{e_{n-2}} + a_1 \, r_1^{e_{n+1}} \, r_2^{e_n} \, r_3^{e_{n-1}} \\
&= a_1 + a_1 R + a_1 r R + a_1 r^2 R + \ldots + a_1 r^{p-1} R + a_1 \, r_1^{e_n} \, r_2^{e_{n-1}} \, r_3^{e_{n-2}} \\
&\quad + a_1 \, r_1^{e_{n+1}} \, r_2^{e_n} \, r_3^{e_{n-1}} \\
&= a_1 + a_1 R \left(1 + r + r^2 + \ldots + r^{p-1}\right) + a_1 r_1 r^p \left(\left\lfloor \frac{n+1}{3} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor \right) \\
&\quad + a_1 r^p (r_1 + r_1 r_2) \left(\left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n-1}{3} \right\rfloor \right) \\
&= a_1 + a_1 R \left(\frac{1-r^p}{1-r} \right) + a_1 r_1 r^p \left(\left\lfloor \frac{n+1}{3} \right\rfloor - \left\lfloor \frac{n}{3} \right\rfloor \right) \\
&\quad + a_1 r^p (r_1 + r_1 r_2) \left(\left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n-1}{3} \right\rfloor \right)
\end{aligned}$$

□

References

1 Zhang Xiong and Zhang Yilin, Sequence of numbers with alternate common differences, Scientia Magna, High American Press, 2007 Vol. 3, No. 1, 93-97.