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NSE characterization of the group PSL(2,49)

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Abstract Let G be a finite group and $\pi_e(G)$ be the set of element orders of G. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. Set nse(G):=$\{m_k|k \in \pi_e(G)\}$. In this paper, we prove that if G is a group such that $nse(G) = nse(PSL(2, 49))$, then $G \cong PSL(2, 49)$.

Key Words element order, set of the numbers of elements of the same order, Sylow subgroup

MSC 2010 20D06, 20D20

1 Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p. Also the set of element orders of G is denoted by $\pi_e(G)$. A finite group G is called a simple K_n-group, if G is a simple group with $|\pi(G)| = n$. Set m_i=$m_i(G)$=$|\{g \in G|$ the order of g is $i\}|$. In fact, m_i is the number of elements of order i in G, and nse(G) := $\{m_i|i \in \pi_e(G)\}$, the set of sizes of elements with the same order. Throughout this paper, we denote by ϕ the Euler totient function. If G is a finite group, then we denote by P_q a Sylow $q-$subgroup of G and $n_q(G)$ is the number of Sylow $q-$subgroup of G, that is $n_q(G)$=$|Syl_q(G)|$. All further unexplained notations are standard and refer to [1], for example. In [8], it is proved that all simple K_4-groups can be uniquely determined by nse(G) and $|G|$. But, in [9], it is proved that the groups A_4, A_5 and A_6, and in [6], the groups PSL(2, q) for $q \in \{7, 8, 11, 13\}$ are uniquely determined by only nse(G). In [6], the authors gave the following problem:

Problem: *Let G be a group such that $nse(G) = nse(PSL(2, q))$, where q is a prime power. Is G isomorphic to $PSL(2, q)$?*

In [5] we gave a positive answer to this problem and show that the group PSL(2, q) is characterizable by only nse(G) for $q = 25$. In this paper, we give a positive answer to this problem for $q = 49$. In fact the main theorem of our paper is as follow:

Main Theorem: *Let G be a group such that $nse(G) = nse(PSL(2, 49))$, then $G \cong PSL(2, 49)$.*

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2 Preliminary Results

In this section we bring some preliminary lemmas to be used in the proof of main theorem.

Lemma 2.1. [3] *Let G be a finite solvable group and $|G| = m \cdot n$, where $m = p_1^{\alpha_1}...p_r^{\alpha_r}$, $(m,n) = 1$. Let $\pi = \{p_1, ..., p_r\}$ and h_m be the number of $\pi-$Hall subgroups of G. Then $h_m = q_1^{\beta_1}...q_s^{\beta_s}$, satisfies the following conditions for all $i \in \{1, 2, ..., s\}$:*

1. *$q_i^{\beta_i} \equiv 1$ (mod p_j), for some p_j.*
2. *The order of some chief factor of G is divisible by $q_i^{\beta_i}$.*

Lemma 2.2. [4] *If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, PSL(2, 7), PSL(2, 8), PSL(2, 17), PSL(3, 3), PSU(3, 3) or PSU(4, 2).*

Lemma 2.3. [10] *Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*
(1) A_7, A_8, A_9, A_{10}.
(2) M_{11}, M_{11}, J_2.
(3) *(a) $L_2(r)$, where r is a prime and satisfies $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a \geqslant 1$, $b \geqslant 1$, $c \geqslant 1$, $v > 3$, is a prime*;
(b) *$L_2(2^m)$, where satisfies $2^m - 1 = u$, $2^m + 1 = 3t^b$, with $m \geqslant 2$, u, t are primes, $t > 3$, $b \geqslant 1$; (c) $L_2(3^m)$, where m satisfies $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, with $m \geqslant 2$, u, t are odd primes, $b \geqslant 1$, $c \geqslant 1$*;
(d) $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, $2F_4(2)'$.

Lemma 2.4. [8] *Let G be a finite group, $P \in Syl_p(G)$, where $p \in \pi(G)$. Let G have a normal series $K \trianglelefteq L \trianglelefteq G$. If $P \leqslant L$ and $p \nmid |K|$, then the following hold:*
(1) $N_{G/K}(PK/K) = N_G(P)K/K$;
(2) $|G : N_G(P)| = |L : N_L(P)|$, that is $n_p(G) = n_p(L)$;
(3) $|L/K : N_{L/K}(PK/K)|t = |G : N_G(P)| = |L : N_L(P)|$, that is $n_p(L/K)t = n_p(G) = n_p(L)$ for some positive integer t, and $|N_K(P)|t = |K|$.

Lemma 2.5. [2] *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.6. [9] *Let G be a group containing more than two elements. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. If $s = sup\{m_k | k \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leqslant s(s^2 - 1)$.*

Lemma 2.7. [7] *Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow $p-$subgroup of G and $n = p^s m$, where $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n*

is always a multiple of p^s*.*

Let G be a group such that $nse(G) = nse(PSL(2, 49))$. By Lemma 2.6, we can assume that G is finite. Let m_n be the number of elements of order n. We note that $m_n = k\phi(n)$, where k is the number of cyclic subgroups of order n in G. Also we note that if $n > 2$, then $\phi(n)$ is even. If $n \in \pi_e(G)$, then by Lemma 2.5 and the above notation we have:

$$\begin{cases} \phi(n) \mid m_n \\ \\ n \mid \sum_{d|n} m_d \end{cases} \qquad (*)$$

In the proof of the main theorem, we almost apply $(*)$ and the above comments.

3 Proof of the Main Theorem

Let G be a group, such that nse(G)=nse(PSL(2, 49))={1, 225, 2400, 2450, 4704, 4900, 9800, 23520}. At first we prove that $\pi(G) \subseteq \{2, 3, 5, 7\}$. Since $1225 \in$ nse(G), it follows that by $(*)$, $2 \in \pi(G)$ and $m_2 = 1255$. Let $2 \neq p \in \pi(G)$, by $(*)$, $p \mid (1 + m_p)$ and $(p - 1) \mid m_p$, which implies that $p \in \{3, 5, 7, 11, 29, 43\}$. If $11 \in \pi_e(G)$, then by $(*)$, $m_{11} = 9800$. On the other hand, by $(*)$, we conclude that if $22 \in \pi_e(G)$, then $m_{22} = 9800$, 23520, 4900, 2450 or 2400 and $22 \mid (1 + m_2 + m_{11} + m_{22})$, which is a contradiction. That is $22 \notin \pi_e(G)$. Thus the group P_{11} acts fixed point freely on the set of elements of order 2, and $|P_{11}| \mid m_2$, which is a contradiction. Hence $11 \notin \pi(G)$ and similarly we can prove that 29 and $43 \notin \pi(G)$. Therefore $\pi(G) \subseteq \{2, 3, 5, 7\}$. If 3, 5 and $7 \in \pi(G)$, then $m_3 = 2450$, $m_5 = 4704$ and $m_7 = 2400$, by $(*)$. Now let $3 \in \pi(G)$, we can see easily that G does not contain any elements of order 27. Hence $\exp(P_3) = 3$ or 9. Let $\exp(P_3) = 3$, by Lemma 2.5, with consider $m = |P_3|$, we have $|P_3| \mid (1 + m_3) = 2451$. Hence $|P_3| = 3$, then $n_3 = m_3/\phi(3) = 1225 \mid |G|$. Now let $\exp(P_3) = 9$, by $(*)$ we have $m_9 = 2400$ or 4704. By Lemma 2.5, $|P_3| \mid (1 + m_3 + m_9)$, then $|P_3| = 9$ or 27. If $|P_3| = 9$, then $n_3 = m_9/\phi(9) = 400$ or 784 and if $|P_3| = 27$, then by Lemma 2.7, $9 \mid m_9$, which is a contradiction. Therefore if $3 \in \pi(G)$, then $5 \in \pi(G)$. If $5 \in \pi(G)$, then we can see easily that G does not contain any elements of order 125. Hence $\exp(P_5) = 5$ or 25. Let $\exp(P_5) = 5$, by Lemma 2.5, $|P_5| \mid (1+m_5) = 4705$. Hence $|P_5| = 5$, then $n_5 = m_5/\phi(5) = 1176 \mid |G|$. Now let $\exp(P_5) = 25$, by $(*)$ we have $m_{25} = 23520$. By Lemma 2.5, $|P_5| \mid (1+m_5+m_{25}) = 28225$, then $|P_{25}| = 25$. Thus $n_5 = m_{25}/\phi(25) = 1176$. Therefore if $5 \in \pi(G)$, then 3 and $7 \in \pi(G)$. By the above discussion in follow, we show that $\pi(G)$ could not be the sets $\{2\}$ and $\{2, 7\}$, and so $\pi(G)$ must be equal to $\{2, 3, 5, 7\}$.

Case a. Let $\pi(G) = \{2\}$, then $\pi_e(G) \subseteq \{1, 2, 2^2, \ldots, 2^6\}$. Since nse($G$) has eight elements and $|\pi_e(G)| \leqslant 7$, which is a contradiction. Therefore this case impossible.

Case b. Let $\pi(G) = \{2, 7\}$. By $(*)$, $7^4 \notin \pi_e(G)$, then we have $exp(P_7) = 7$, 49 or 343. If $exp(P_7) = 7$, then $|P_7| \mid (1 + m_7) = 2401$. Hence $|P_7| \mid 7^4$. If $|P_7| = 7$, then $n_7 = m_7/\phi(7) = 400 \mid |G|$, since $5 \notin \pi(G)$ we get a contradiction. If $|P_7| = 49$, then $|G| = 2^m \times 49 = 49000 + 2400k_1 + 2450k_2 + 4704k_3 + 4900k_4 + 9800k_5 + 23520k_6$, where m, k_1, k_2, k_4, k_3, k_5 and k_6 are non-negative integers. Since $\pi_e(G) \subseteq \{1, 2, 2^2, 2^3, 2^4, 2^5, 2^6\} \bigcup \{7, 7 \times 2^2, \ldots, 7 \times 2^5\}$, then $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 + k_6 \leqslant 5$. Therefore $49000 \leqslant |G| \leqslant 23520 \times 5 + 49000$, then $m = 10$ or 11. If $m = 10$, then $1176 = 2400k_1 + 2450k_2 + 4704k_3 + 4900k_4 + 9800k_5 + 23520k_6$. It is easy to check that the equation has no solution. If $m = 11$, then $51352 = 2400k_1 + 2450k_2 + 4704k_3 + 4900k_4 + 9800k_5 + 23520k_6$. It is easy to check that the equation has no solution. If $|P_7| = 343$, then $|G| = 2^m \times 343 = 49000 + 2400k_1 + 2450k_2 + 4704k_3 + 4900k_4 + 9800k_5 + 23520k_6$, then $m = 8$. Therefore $38808 = 2400k_1 + 2450k_2 + 4704k_3 + 4900k_4 + 9800k_5 + 23520k_6$, it is easy to check that the equation has a solution $(k_1, k_2, k_3, k_4, k_5, k_6) = (0, 0, 2, 0, 3, 0)$. By $(*)$, we have if 4, $8 \in \pi_e(G)$, then $m_4 = 2450$ and $m_8 = 4900$. We know that $2^7 \notin \pi_e(G)$, so $exp(P_2) = 2$, 4, 8, 16, 32 or 64. It known that if $exp(P_2) = 2^i$ for $1 \leqslant i \leqslant 6$, then $|P_2| \mid (1 + m_2 + \ldots + m_{2^i})$, by Lemma 2.5. Also by $(*)$, $m_{16} \in \{2400, 4704, 23520\}$, $m_{32} \in \{2400, 4704, 23520\}$ and $m_{64} \in \{2400, 4704, 23520\}$. By an easy computer calculation, $|P_2| \mid 2^7$. Since $|P_2| = 2^8$, this is impossible. Similarly if $|P_7| = 2401$ we can get a contradiction. Now suppose that $exp(P_7) = 49$. By $(*)$, we have $m_{49} = 4704$ or 23520. Since $|P_7| \mid (1 + m_7 + m_{49})$ we can conclude that $|P_7| = 49$, then $n_7 = m_{49}/\phi(49)$. If $m_{49} = 4704$, then by Sylow theorem we get a contradiction. If $m_{49} = 23520$, then by $5 \notin \pi(G)$ we get a contradiction. Let $exp(P_7) = 343$. We know $|P_7| = 7^n$, where $n \geqslant 3$. If $|P_7| = 343$, then by $m_{343} = 4704$ or 23520, we get a contradiction. If $|P_7| = 7^n$ where $n \geqslant 4$, by Lemma 2.7 $343 \mid m_{343}$, which is a contradiction.

Therefore $\pi(G) = \{2, 3, 5, 7\}$. It known that $\exp(P_3) = 3$ or 9, we prove that $\exp(P_3) \neq 9$. If $\exp(P_3) = 9$, then we know that $|P_3| = 9$ and $n_3 = 400$ or 784. Since every Sylow 3-subgroups of order has two elements of order 3, then $m_3 \leqslant 400 \times 2$ or 784×2, but $m_3 = 2450$, a contradiction. Hence $\exp(P_3) = 3$, then $|P_3| = 3$. Now we show that G does not contain any element of order 21. Suppose that $21 \in \pi_e(G)$ we know that if P and Q are Sylow $3-$subgroups of G, then P and Q are conjugate, which implies that $C_G(P)$ and $C_G(Q)$ are conjugate in G. Therefore $m_{21} = \phi(21) \cdot n_3 \cdot k$, where k is the number of cyclic subgroups of order 7 in $C_G(P_3)$. Since $n_3 = 1225$, we have $2450 \mid m_{21}$. On the other hand we have, $21 \mid (1 + m_3 + m_7 + m_{21})$, which is a contradiction. Hence $21 \notin \pi_e(G)$. Since $21 \notin \pi_e(G)$, then the group P_3 acts fixed point freely on the set of elements of order 7, and so $|P_3| \mid m_7 = 2400$, which implies that $|P_3| = 3$. Also the group P_7 acts fixed point freely on the set of elements of order 3, and so $|P_7| \mid m_3 = 2450$, which implies that $|P_7| \mid 49$. Also we can prove that $15 \notin \pi_e(G)$, then the group P_5 acts fixed point freely on the set of elements of order 3, and so $|P_5| \mid m_3 = 2450$, which implies that $|P_5| \mid 25$. We know that, $|P_2| \mid 2^7$, now we prove that if $exp(P_2) = 8$, then $|P_2| \mid 2^7$ and if $exp(P_2) \neq 8$, then $|P_2| \mid 2^6$. Let $exp(P_2) = 8$, then $|P_2| \mid (1 + m_2 + m_4 + m_8) = 8576$, hence $|P_2| \mid 2^7$. Now suppose that $exp(P_2) \neq 8$, by $(*)$ we can show that $2^6 \notin \pi_e(G)$, then $exp(P_2) = 2$, 4, 16 or 32. Hence $|P_2| \mid (1 + m_2 + \ldots + m_{2^i})$, where $1 \leqslant i \leqslant 5$ and $i \neq 3$. Also it known that $m_{16} \in \{2400, 4704, 23520\}$ and $m_{32} \in \{2400, 4704, 23520\}$. By an easy computer calculation $|P_2| \mid 2^6$. Therefore $|G| = 2^n \times 3 \times 5^m \times 7^k$, where $n \leqslant 7$, $m \leqslant 2$ and $k \leqslant 2$. We claim that G is unsolvable group. Suppose that G is a solvable group as $n_5 = 1176$, then by Lemma 2.1, $49 \equiv 1 \pmod 5$, which is a contradiction. Hence G is an unsolvable group. Since G is an unsolvable group such that $3 \mid |G|$ but $9 \nmid |G|$, so G has a normal series $1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$, where N is a maximal solvable normal subgroup of G and H/N is an unsolvable minimal normal subgroup of G/N. Then H/N is a non-abelian simple K_3-group or K_4-group. Let H/N be a non-abelian simple K_3-group, then by Lemma 2.2, $H/N \cong A_5$ or PSL(2, 7). Let $H/N \cong A_5$, if $P_3 \in Syl_3(G)$, then $P_3N/N \in Syl_3(H/N)$ and $n_3(H/N)t = n_3(G)$ for some positive integer t and $3 \nmid t$, by Lemma 2.4. Since $n_3(H/N) = n_3(A_5) = 10$, then $1225 = 10t$, which is a contradiction. Now let $H/N \cong$ PSL(2, 7), if $P_3 \in Syl_3(G)$, then $P_3N/N \in Syl_3(H/N)$ and $n_3(H/N)t = n_3(G)$ for some positive integer t and $3 \nmid t$, by Lemma 2.4. Since $n_3(H/N) = n_3(PSL(2,7)) = 28$, then $1225 = 28t$, which is a contradiction. Hence H/N is a non-abelian simple K_4-group. By Lemma 2.3, $H/N \equiv$PSL(2,49). Now set $\overline{H} := H/N \cong$PSL(2,49) and $\overline{G} := G/N$. On the other hand:

$$\mathrm{PSL}(2,49) \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leqslant \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leqslant Aut(\overline{H}).$$

Let $K = \{x \in G \mid xN \in C_{\overline{G}}(\overline{H})\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence PSL(2, 49)$\leqslant G/K \leqslant Aut$(PSL(2,49)), then $G/K \cong$PSL(2,49), PSL(2,49).2_1, PSL(2,49).2_2, PSL(2,49).2_3 or PSL(2,49).2^2. Therefore $|G| = 2^n \times 3 \times 25 \times 49$, where $n \leqslant 7$. It known that $N \leqslant K$, as $|K| \mid 8$, $n \leqslant 7$ and N is a maximal solvable normal subgroup of G, then $N = K$. Hence G/N is isomorphic to one of the groups: PSL(2,49), $PSL(2,49).2_1$, $PSL(2,49).2_2$, $PSL(2,49).2_3$ or PSL(2,49).2^2. Let $|G| = 2^7 \times 3 \times 25 \times 49$. We know that $exp(P_2) = 8$, then $\pi_e(G) \subseteq \{1, 2, 4, 8\} \bigcup \{3, 6, 12, 24\} \bigcup \{5, 25, 10, 20, 40, 50, 100, 200\} \bigcup \{7, 49, 14, 28, 56, 98, 196, 392\} \bigcup \{35, 175, 245, 1225\}$. Thus $|\pi_e(G)| \leqslant 28$. Therefore $421400 = 2400k_1 + 2450k_2 + 4704k_3 + 4900k_4 + 9800k_5 + 23520k_6$, where $0 \leqslant k_1+k_2+k_3+k_4+k_5+k_6 \leqslant 20$, it is easy to check that this equation has no solution. Hence $|G| = 2^n \times 3 \times 25 \times 49$ where $n \leqslant 6$, as $49000 \leqslant |G|$, then $4 \leqslant n \leqslant 6$. If G/N is isomorphic to PSL(2,49).2^2, then $N = 1$. Since nse(G) $\neq$nse(PSL(2,49).2^2), this is impossible. If G/N is isomorphic to one of the groups: PSL(2,49).2_1, PSL(2,49).2_2 or PSL(2, 49).2_3, it is clear that $|G| = 2^5 \times 3 \times 25 \times 49$ or $|G| = 2^6 \times 3 \times 25 \times 49$. If $|G| = 2^5 \times 3 \times 25 \times 49$, then $N = 1$. Since nse(G) $\neq$nse(PSL(2,49).2_1), nse(G) $\neq$nse(PSL(2,49).2_2) and nse(G) $\neq$nse(PSL(2,49).2_3), this is impossible. Now let $|G| = 2^6 \times 3 \times 25 \times 49$, then $|N| = 2$. So G has a normal subgroup G of order 2, generated by a central involution z. Let $G/N \cong$PSL(2,49).2_1 =PGL(2, 49) and f be the natural homomorphism from G to G/N. Then f takes the identity and the element z to the identity of G/N, and takes the other 1224 elements of order 2 to $(1225-1)/2 = 612$ elements of order 2 in $G/N \cong$ PSL(2, 49).2_1. On the other hand, suppose that Nx and Ny are elements of order 2 in the same conjugacy class in $G/N \cong$ PSL(2,49).2_1. Then $Ny = Nx^{Ng} = N(x^g)$ for some g in G, so $y = x^g$ or zx^g, and then since $zx^g = (z^g)(x^g)$, we have $y = x^g$ or $(zx)^g$. Hence if x has order 2, then so does y. It follows from this that the images in G/N of elements of order 2 in G form a union of conjugacy classes in G/N. But that's impossible. There are two conjugacy classes of elements of order 2 in $G/N \cong$PSL(2,49).2_1: one of size 1225 containing elements of order 2 in PSL(2,49), and the other of size 1176. As $1 + 612 = 613$ images in G/N of the elements of order 2 from G make up a union of conjugacy classes. We get a contradiction. Similarly if $G/N \cong$ PSL(2,49).2_2 or PSL(2,49).2_3, we can get a contradiction. Hence no such group G exists. If $G/N \cong$PSL(2,49), then $|G| = 2^4 \times 3 \times 25 \times 49$, $|G| = 2^5 \times 3 \times 25 \times 49$ or $|G| = 2^6 \times 3 \times 25 \times 49$. If $|G| = 2^4 \times 3 \times 25 \times 49$, then $N = 1$ and $G \cong$PSL(2,49). If $|G| = 2^5 \times 3 \times 25 \times 49$, then $|N| = 2$. Similar to the above discussion we get a contradiction. Let $|G| = 2^6 \times 3 \times 25 \times 49$, then $|N| = 4$. Consider the action of G on N by conjugation. The kernel is $C_G(N)$, and $G/C_G(N)$ is isomorphic to a subgroup of $Aut(N)$. Here N has order 4, so $Aut(N)$ is isomorphic to either $Aut(C_4) = C_2$ or $Aut(V_4) = S_3$. In particular, $|G/C_G(N)|$ is at most 6. But also $C_G(N)$ contains N, since N is abelian, and then since G/N is PSL(2,49), which is simple and has order greater than 6, it follows that $|G/C_G(N)| = 1$. Thus $G = C_G(N)$, so N is central. Now we know that N is central in G, if x has order 2 in G, and lies outside N, then Nx is one of the 1225 elements of order 2 in PSL(2,49)$= G/N$. Choose any such x, then if Ny is any element of order 2 in G/N, then Ny is conjugate to Nx, so $Ny = (Ng)^{-1}Nx(Ng) = N(g^{-1}xg)$ for some g in G, so Ny= $\{a(g^{-1}xg) : a \in N\}$, and in particular, we find that Ny contains at least two elements of order 2. Hence G has at least $2 \times 1225 = 2450$ elements of order 2, a contradiction. Thus the proof is completed. □

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