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On some new fractional q-integral inequalities

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Abstract Several new fractional q-integral inequalities are presented by using the Reimann-Liouville fractional integral in three types and concerning the product of two and three functions. There is a relationship between our results and [1]&[5].

Key Words Fractional integral inequality, Q-integral inequality

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1 Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ .For $0 < q < 1$, the q-analog of the derivative, denoted by D_q is defined by (see [6])

$$D_q f(x) = \frac{f(x) - f(qx)}{x - qx}, \quad x \neq 0. \tag{1.1}$$

Whenever $f'(0)$ exists, $D_q F(0) = f'(0)$, and as $q \to 1^-$, the $q-$derivative reduces to the usual derivative.

The $q-$analog of integration from 0 to a is given by (see [7])

$$\int_0^a f(x) d_q x = a(1-q) \sum_{k=0}^{\infty} f(aq^k) q^k \quad , \tag{1.2}$$

and for $a = \infty$,

$$\int_0^\infty f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n) q^n \tag{1.3}$$

provided the sum converges absolutely. On a general interval $[a, b]$ the $q-$integral is defined by (see [3]-[4])

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{1.4}$$

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The $q-$Jackson integral and the $q-$derivative are related by the fundamental theorem of quantum calculus, which can be stated as follows (see [4], p.73):

If F is an anti $q-$derivative of the function f, namely $D_qF = f$, continuous at $x = a$, then

$$\int_a^b f(x)d_qx = F(b) - F(a). \tag{1.5}$$

For any bounded function f, we have

$$D_q \int_a^x f(t)d_qt = f(x), \tag{1.6}$$

and if $f$is continuous at 0, then

$$\int_a^x D_qf(t)d_qt = f(x) - f(0), \tag{1.7}$$

For $b > 0$ and $a = bq^n$, $n \in N$ we denote

$$[a, b]_q = \left\{bq^k : 0 \leqslant k \leqslant n\right\} \text{ and } (a, b] = [aq^{-1}, b]_q. \tag{1.8}$$

Let c be a complex number, the $q-$shifted factorial are defined by

$$(c; q)_0 = 1, (c; q)_n = \prod_{k=0}^{n-1} \left(1 - cq^k\right), \quad n = 1, 2, ... \tag{1.9}$$

$$(c; q)_\infty = \lim_{n\to\infty} (c; q)_n = \prod_{k=0}^{\infty} \left(1 - cq^k\right). \tag{1.10}$$

For x complex we denote

$$[x]_q = \frac{1 - q^x}{1 - q}. \tag{1.11}$$

The $q-$analogue of the Gamma function is defined by Jackson [3] as follows

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \qquad x \neq 0, -1, -2, ... \quad , \tag{1.12}$$

and it is satisfying the following

$$\Gamma_q(x + 1) = [x]_q\Gamma_q(x), \quad \Gamma_q(1) = 1, \tag{1.13}$$

and tends to $\Gamma(x)$ as $q \to 1^-$. The $q-$integral representation of the Gamma function is (see [2]) as follows

$$\Gamma_q(x) = K_q(x) \int_0^\infty t^{x-1}e_q(-t)d_qt, \tag{1.14}$$

where

$$e_q(t) = \frac{1}{((1 - q)\,t; q)_\infty}, \tag{1.15}$$

and

$$K_q(t) = \frac{(1-q)^{-x}}{1+(1-q)^{-1}} \times \frac{(-(1-q);q)_\infty \left(-(1-q)^{-1};q\right)_\infty}{(-(1-q)q^t;q)_\infty \left(-(1-q)^{-1}q^{1-t};q\right)_\infty}. \tag{1.16}$$

The $q-$fractional function is defined by the following : If n is a positive integer, then

$$(t-s)^{(n)} = (t-s)(t-qs)...(t-q^{n-1}s). \tag{1.17}$$

If n is not a positive integer, then

$$(t-s)^{(n)} = x^n \prod_{k=0}^{\infty} \frac{1-(s/t)q^k}{1-(s/t)q^{n+k}}. \tag{1.18}$$

The $q-$derivative of the $q-$factorial function with respect to t is

$$D_q\,(t-s)^{(n)} = \frac{1-q^n}{1-q}(t-s)^{(n-1)}, \tag{1.19}$$

and the $q-$derivative of the $q-$factorial function with respect to t is

$$D_q\,(t-s)^{(n)} = \frac{1-q^n}{1-q}(t-qs)^{(n-1)}. \tag{1.20}$$

We define the fractional $q-$integral by the following

$$I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qx)^{(\alpha-1)} f(x) d_q x \quad , \tag{1.21}$$

$$J_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-x)^{(\alpha-1)} f(x) d_q x \quad , \tag{1.22}$$

and

$$K_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-x)^{(\alpha-1)} f(x) d_q x \quad . \tag{1.23}$$

In particular

$$I_q^1 f(t) = J_q^1 f(t) = K_q^1 f(t) \int_0^t f(x) d_q x \quad .$$

Öğünmez and Özkan [5] proved the following result

Theorem 1.1. *Let f and g be two synchronous functions on $[0,\infty)$. Then for all $t>0$, $v>0$, we have*

$$I_q^v\,(fg)\,(t) \geqslant \frac{\Gamma_q(v+1)}{t^{(v)}} I_q^v f(t) I_q^v g(t). \tag{1.24}$$

2 Lemmas

The following lemmas are needed for our aim

Lemma 2.1.

$$I_q^{\alpha}(1) = \frac{1}{\Gamma_q(\alpha+1)} t^{(\alpha)}, \tag{2.1}$$

$$J_q^{\alpha}(1) = \frac{1}{\Gamma_q(\alpha+1)} t^{\alpha}. \tag{2.2}$$

Proof.

$$\begin{aligned} I_q^{\alpha}(1) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qx)^{(\alpha-1)} d_q x \\ &= -\frac{1}{\Gamma_q(\alpha)} \frac{1-q}{1-q^{\alpha}} \int_0^t D_q (t-x)^{(\alpha-1)} d_q x \\ &= -\frac{1}{\Gamma_q(\alpha)} \frac{1-q}{1-q^{\alpha}} \left(0 - t^{(\alpha)}\right) \\ &= \frac{1}{[\alpha]_q \Gamma_q(\alpha)} t^{(\alpha)} = \frac{1}{\Gamma_q(\alpha+1)} t^{(\alpha)}. \end{aligned}$$

$$\begin{aligned} J_q^{\alpha}(1) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-x)^{\alpha-1} d_q x \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^t u^{\alpha-1} d_q u, \qquad (t-x=u) \\ &= \frac{1}{\Gamma_q(\alpha)} t(1-q) \sum_{k=0}^{\infty} \left(tq^k\right)^{\alpha-1} q^k \\ &= \frac{1}{\Gamma_q(\alpha)} t^{\alpha} (1-q) \sum_{k=0}^{\infty} (q^{\alpha})^k = \frac{1}{\Gamma_q(\alpha)} \frac{1-q}{1-q^{\alpha}} t^{\alpha} \\ &= \frac{t^{\alpha}}{[\alpha]_q \Gamma_q(\alpha)} = \frac{t^{\alpha}}{\Gamma_q(\alpha+1)}. \end{aligned}$$

□

Lemma 2.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *and define*

$$\overline{f}(x) = \int_0^x f(u) d_q u, \tag{2.3}$$

then, for $\alpha > -1$,

$$J_q^{\alpha} \overline{f}(x) = J_q^{\alpha+1} f(t), \tag{2.4}$$

$$I_q^{\alpha} \overline{f}(x) = \frac{1}{\Gamma_q(\alpha+1)} K_q^{\alpha+1} f(x). \tag{2.5}$$

Proof.

$$\begin{aligned}
J_q^{\alpha}\bar{f}(t) &= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-x)^{\alpha-1}\int_0^x f(u)d_qud_qx \\
&= \frac{1}{\Gamma_q(\alpha)}\int_0^t f(u)\int_u^t (t-x)^{\alpha-1}d_qxd_qu \\
&= \frac{1}{\Gamma_q(\alpha)}\int_0^t f(u)\int_0^{t-u} v^{\alpha-1}d_qvd_qu \\
&= \frac{1}{\Gamma_q(\alpha)}(1-q)\sum_{k=0}^{\infty}\left(q^k\right)^{\alpha-1}q^k\int_0^t f(u)(t-u)^{\alpha}d_qu \\
&= \frac{1}{\Gamma_q(\alpha)}(1-q)\sum_{k=0}^{\infty}(q^{\alpha})^k\int_0^t f(u)(t-u)^{\alpha}d_qu \\
&= \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)}\frac{1-q}{1-q^{\alpha}}J_q^{\alpha+1}f(t) = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)\,[\alpha]_q}J_q^{\alpha+1}f(t) \\
&= J_q^{\alpha+1}f(t).
\end{aligned}$$

$$\begin{aligned}
I_q^{\alpha}\bar{f}(t) &= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qx)^{(\alpha-1)}\int_0^x f(u)d_qud_qx \\
&= \frac{1}{\Gamma_q(\alpha)}\int_0^t f(u)\int_u^t (t-qx)^{(\alpha-1)}d_qxd_qu \\
&= \frac{1}{\Gamma_q(\alpha)}\int_0^t f(u)\int_u^t D_q(t-x)^{(\alpha)}d_qxd_qu \\
&= \frac{1}{[\alpha]_q\,\Gamma_q(\alpha)}\int_0^t (t-u)^{(\alpha)}f(u)d_qu \\
&= \frac{1}{\Gamma_q(\alpha+1)}K_q^{\alpha}f(t).
\end{aligned}$$

□

3 Main Results

Theorem 3.1. *Let f and g be two synchronous on $[0,\infty]$, $h \geqslant 0$, then for all $t>0$, $\alpha,\beta>0$, we have*

$$\begin{aligned}
&\frac{t^{\beta}}{\Gamma_q(\beta+1)}J_q^{\alpha}(fgh)(t) + \frac{t^{\alpha}}{\Gamma_q(\alpha+1)}J_q^{\beta}(fgh)(t) \\
\geqslant\; & J_q^{\alpha}f(t)J_q^{\beta}(gh)(t) + J_q^{\beta}f(t)J_q^{\alpha}(gh)(t) + J_q^{\alpha}g(t)J_q^{\alpha}(fh)(t) \\
&+J_q^{\beta}g(t)J_q^{\alpha}(fh)(t) - J_q^{\alpha}h(t)J_q^{\beta}(fg)(t) - J_q^{\beta}h(t)J_q^{\alpha}(fg)(t). \qquad (3.1)
\end{aligned}$$

In particular,

$$\frac{t^{\alpha}}{\Gamma_q(\alpha+1)} J_q^{\alpha}(fgh)(t) \geqslant J_q^{\alpha} f(t) J_q^{\alpha}(gh)(t) + J_q^{\alpha} g(t) J_q^{\alpha}(fh)(t) - J_q^{\alpha} h(t) J_q^{\alpha}(fg)(t).$$

$$\frac{t^{\beta}}{\Gamma_q(\beta+1)} J_q^{\alpha}(fg)(t) + \frac{t^{\alpha}}{\Gamma_q(\alpha+1)} J_q^{\beta}(fg)(t) \geqslant J_q^{\alpha} f(t) J_q^{\beta} g(t) + J_q^{\beta} f(t) J_q^{\alpha} g(t).$$

Proof. As

$$(t-x)^{\alpha-1}(t-y)^{\beta-1}(f(x)-f(y))(g(x)-g(y))(h(x)+h(y)) \geqslant 0,$$

then, we have

$$\int_0^t \int_0^t (t-x)^{\alpha-1}(t-y)^{\beta-1}(f(x)-f(y))(g(x)-g(y))(h(x)+h(y))\, d_qx d_qy \geqslant 0.$$

By opening the above, we obtain

$$\begin{aligned}
& \int_0^t (t-x)^{\alpha-1} f(x)g(x)h(x) d_qx \int_0^t (t-y)^{\beta-1} d_qy + \int_0^t (t-y)^{\beta-1} f(y)g(y)h(y) d_qy \int_0^t (t-x)^{\alpha-1} d_qx \\
\geqslant\ & \int_0^t (t-x)^{\alpha-1} f(x)h(x) d_qx \int_0^t (t-y)^{\beta-1} g(y) d_qy + \int_0^t (t-x)^{\alpha-1} g(x)h(x) d_qx \int_0^t (t-y)^{\beta-1} f(y) d_qy \\
& - \int_0^t (t-x)^{\alpha-1} h(x) d_qx \int_0^t (t-y)^{\beta-1} f(y)g(y) d_qy - \int_0^t (t-x)^{\alpha-1} f(x)g(x) d_qx \int_0^t (t-y)^{\beta-1} h(y) d_qy \\
& + \int_0^t (t-x)^{\alpha-1} f(x) d_qx \int_0^t (t-y)^{\beta-1} g(y)h(y) d_qy + \int_0^t (t-x)^{\alpha-1} g(x) d_qx \int_0^t (t-y)^{\beta-1} f(y)h(y) d_qy.
\end{aligned}$$

Dividing the above inequality by $\Gamma_q(\alpha)\Gamma_q(\beta)$, noticing that

$$J_q^{\alpha}(1) = \frac{1}{\Gamma_q(\alpha+1)} t^{\alpha},$$

we obtain

$$\begin{aligned}
& \frac{t^{\beta}}{\Gamma_q(\beta+1)} J_q^{\alpha}(fgh)(t) + \frac{t^{\alpha}}{\Gamma_q(\alpha+1)} J_q^{\beta}(fgh)(t) \\
\geqslant\ & J_q^{\alpha} f(t) J_q^{\beta}(gh)(t) + J_q^{\beta} f(t) J_q^{\alpha}(gh)(t) + J_q^{\alpha} g(t) J_q^{\alpha}(fh)(t) \\
& + J_q^{\beta} g(t) J_q^{\alpha}(fh)(t) - J_q^{\alpha} h(t) J_q^{\beta}(fg)(t) - J_q^{\beta} h(t) J_q^{\alpha}(fg)(t).
\end{aligned}$$

□

The following is a similar result but dealing with I_q^{α}, I_q^{β}.

Theorem 3.2. *Let f and g be two synchronous on $[0,\infty]$, $h \geqslant 0$, then for all $t > 0$, $\alpha, \beta > 0$,*

$$\frac{1}{\Gamma_q(\beta+1)} t^{(\beta)} I_q^{\alpha}(fgh)(t) + \frac{1}{\Gamma_q(\alpha+1)} t^{(\alpha)} I_q^{\beta}(fgh)(t)$$

$$\geqslant I_q^\alpha f(t) I_q^\beta (gh)(t) + I_q^\beta f(t) I_q^\alpha (gh)(t) + I_q^\alpha g(t) I_q^\alpha (fh)(t) + I_q^\beta g(t) I_q^\alpha (fh)(t) - I_q^\alpha h(t) I_q^\beta (fg)(t) - I_q^\beta h(t) I_q^\alpha (fg)(t). \tag{3.2}$$

In particular,

$$\frac{1}{\Gamma_q(\alpha+1)} t^{(\alpha)} I_q^\alpha (fgh)(t) \geqslant I_q^\alpha f(t) I_q^\alpha (gh)(t) + I_q^\alpha g(t) I_q^\alpha (fh)(t) - I_q^\alpha h(t) I_q^\alpha (fg)(t). \tag{3.3}$$

$$\frac{1}{\Gamma_q(\beta+1)} t^{(\beta)} I_q^\alpha (fg)(t) + \frac{1}{\Gamma_q(\alpha+1)} t^{(\alpha)} I_q^\beta (fg)(t) \geqslant I_q^\alpha f(t) I_q^\beta g(t) + I_q^\beta f(t) I_q^\alpha g(t). \tag{3.4}$$

Remark. It may be mentioned that (1.24) follows from (3.4) by putting $\alpha = \beta$.

Theorem 3.3. *Let f and g be two functions on $[0, \infty]$, then for $t > 0$, $\alpha, \beta > 0$, we have*

$$\frac{t^{(\beta)}}{\Gamma_q(\beta+1)} J_q^\alpha \left(f^2\right)(t) + \frac{t^{(\alpha)}}{\Gamma_q(\alpha+1)} J_q^\beta \left(g^2\right)(t) \geqslant 2 J_q^\alpha f(t) J_q^\beta g(t). \tag{3.5}$$

In particular

$$\frac{t^{(\alpha)}}{\Gamma_q(\alpha+1)} J_q^\alpha \left(f^2\right)(t) \geqslant \left(J_q^\alpha f(t)\right)^2. \tag{3.6}$$

Proof. Since

$$(t-x)^{\alpha-1} (t-y)^{\beta-1} (f(x) - g(y))^2 \geqslant 0,$$

then, we have

$$\int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} (f(x) - g(y))^2 \, d_q x d_q y \geqslant 0.$$

The above implies

$$\int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} f^2(x) \, d_q x d_q y + \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} g^2(y) \, d_q x d_q y \geqslant 2 \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} f(x) g(y) d_q x d_q y.$$

Dividing the above inequality by $\Gamma_q(\alpha) \Gamma_q(\beta)$, noticing that

$$J_q^\alpha(1) = \frac{t^\alpha}{\Gamma_q(\alpha+1)},$$

We obtain

$$\frac{t^{(\beta)}}{\Gamma_q(\beta+1)} J_q^\alpha \left(f^2\right)(t) + \frac{t^{(\alpha)}}{\Gamma_q(\alpha+1)} J_q^\beta \left(g^2\right)(t) \geqslant 2 J_q^\alpha f(t) J_q^\beta g(t).$$

□

Theorem 3.4. *Let f and g be two functions on $[0,\infty]$, then for $t>0$, $\alpha,\beta>0$, we have*

$$J_q^{\alpha}\left(f^2\right)(t)J_q^{\beta}\left(g^2\right)(t)+J_q^{\beta}\left(f^2\right)(t)J_q^{\alpha}\left(g^2\right)(t)\geqslant 2J_q^{\alpha}\left(fg\right)(t)J_q^{\beta}\left(fg\right)(t) \tag{3.7}$$

In particular

$$J_q^{\alpha}\left(f^2\right)(t)I_q^{\beta}\left(g^2\right)(t)\geqslant\left(J_q^{\alpha}\left(fg\right)(t)\right)^2. \tag{3.8}$$

Proof. Since

$$(t-x)^{\alpha-1}(t-y)^{\beta-1}\left(f(x)g(y)-f(y)g(x)\right)^2\geqslant 0,$$

then, we have

$$\int_0^t\int_0^t (t-x)^{\alpha-1}(t-y)^{\beta-1}\left(f(x)g(y)-f(y)g(x)\right)^2 d_qx d_qy\geqslant 0.$$

□

The result follows by opening the last inequality and dividing by $\Gamma_q(\alpha)\Gamma_q(\beta)$ as in the previous result.

Other inequalities dealing with $\bar{f}$, $\bar{g}$ are presented in the following two results

Theorem 3.5. *Let f and g be two functions on $[0,\infty]$, then for $t>0$, $\alpha,\beta>-1$, we have*

$$\begin{aligned}&\frac{t^{(\beta)}}{\Gamma_q(\beta+1)}J_q^{\alpha}\left(\overline{fg}\right)(t)\frac{t^{(\alpha)}}{\Gamma_q(\alpha+1)}J_q^{\beta}\left(\overline{fg}\right)(t)\\ \geqslant\ & J_q^{\alpha}\left(\overline{f}\right)(t)J_q^{\beta}\left(\overline{g}\right)(t)+J_q^{\beta}\left(\overline{f}\right)(t)J_q^{\alpha}\left(\overline{g}\right)(t)\end{aligned} \tag{3.9}$$

Proof. The proof follows from Theorem 3.1, (3.2), and via Lemma 2.2. □

Theorem 3.6. *Let f and g be two functions on $[0,\infty]$, then for $t>0$, $\alpha,\beta>0$ we have*

$$\begin{aligned}&t^{(\beta)}K_q^{\alpha+1}\left(\overline{fg}\right)(t)+t^{(\alpha)}K_q^{\beta+1}\left(\overline{fg}\right)(t)\\ \geqslant\ & K_q^{\alpha}\left(\overline{f}\right)(t)K_q^{\beta}\left(\overline{g}\right)(t)+K_q^{\beta}\left(\overline{f}\right)(t)K_q^{\alpha}\left(\overline{g}\right)(t)\end{aligned} \tag{3.10}$$

Proof. Replacing f, g by $\bar{f}$, $\bar{g}$ in (3.4), we obtain

$$\frac{1}{\Gamma_q(\beta+1)}t^{(\beta)}I_q^{1}\left(\overline{fg}\right)(t)+\frac{1}{\Gamma_q(\alpha+1)}t^{(\alpha)}I_q^{\beta}\left(\overline{fg}\right)(t)\geqslant I_q^{\alpha}\left(\bar{f}\right)(t)I_q^{\beta}\left(\overline{g}\right)(t)+I_q^{\beta}\left(\overline{f}\right)(t)I_q^{\alpha}\left(\overline{g}\right)(t).$$

□

The result follows via Lemma 2.2. Our last result concerning some bounds

Theorem 3.7. *Let f and g be two differentiable functions on $[0,\infty]$. Define*

$$\|f'\|_{\infty}=\sup_{x\in[0,\infty]}|f'(x)|<\infty,$$

then for $t>0$, $\alpha,\beta>1$ we have

$$\left|\frac{t^{\beta}}{\Gamma(\beta+1)}J_q^{\alpha}\left(fg\right)(t)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}J_q^{\beta}\left(fg\right)(t)-J_q^{\alpha}\left(f\right)(t)J_q^{\beta}\left(g\right)(t)-J_q^{\beta}f(t)J_q^{\alpha}g(t)\right|$$

$$\leqslant \quad t^{\alpha+\beta+2} C_{\alpha,\beta} \left\| f' \right\|_\infty \left\| g' \right\|_\infty , \tag{3.11}$$

where

$$C_{\alpha,\beta} = \max\left\{ \frac{1}{\Gamma_q(\alpha+1)\Gamma_q(\beta+3)}, \frac{1}{\Gamma_q(\beta+1)\Gamma_q(\alpha+3)} \right\}.$$

In particular

$$\left| \frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha (fg)(t) - I^\alpha f(t) I^\alpha g(t) \right| \leqslant \frac{\left\| f' \right\|_\infty \left\| g' \right\|_\infty}{\Gamma(\alpha+1)\Gamma_q(\alpha+3)} t^{2\alpha+2}. \tag{3.12}$$

Proof.

$$\begin{aligned}
J &= \left| \frac{t^\beta}{\Gamma(\beta+1)} I^\alpha (fg)(t) + \frac{t^\alpha}{\Gamma(\alpha+1)} I^\beta (fg)(t) - I^\alpha (f)(t) I^\beta (g)(t) - I^\beta f(t) I^\alpha g(t) \right| \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left| \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} (f(x)-f(y))(g(x)-g(y))\, dxdy \right| \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left| \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} \int_y^x f'(u)du \int_y^x g'(v)dvdxdy \right| \\
&\leqslant \frac{\left\| f' \right\|_\infty \left\| g' \right\|_\infty}{\Gamma(\alpha)\Gamma(\beta)} \left| \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} \int_y^x du \int_y^x dvdxdy \right| \\
&= \frac{\left\| f' \right\|_\infty \left\| g' \right\|_\infty}{\Gamma(\alpha)\Gamma(\beta)} \left| \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} |x-y|^2\, dxdy \right|.
\end{aligned}$$

If $x \geqslant y$, $|x-y| = x-y \leqslant t-y$, hence

$$\begin{aligned}
J &\leqslant \frac{\left\| f' \right\|_\infty \left\| g' \right\|_\infty}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta+1}\, dxdy \\
&= \left\| f' \right\|_\infty \left\| g' \right\|_\infty J_q^\alpha(1) J_q^{\beta+2}(1) \\
&= C_{\alpha,\beta} t^{\alpha+\beta+2} \left\| f' \right\|_\infty \left\| g' \right\|_\infty \\
&= \frac{t^{\alpha+\beta+2}}{\alpha(\beta+2)} \frac{\left\| f' \right\|_\infty \left\| g' \right\|_\infty}{\Gamma(\alpha)\Gamma(\beta)} \leqslant \frac{t^{\alpha+\beta+2}}{\Gamma(\alpha)\Gamma(\beta)} C_{\alpha,\beta} \left\| f'_\infty \right\| \left\| g' \right\|_\infty .
\end{aligned}$$

Similarly, if $x \leqslant y$, we have

$$J \leqslant \frac{\left\| f' \right\|_\infty \left\| g' \right\|_\infty}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-x)^{\alpha+1} (t-y)^{\beta-1}\, dxdy \leqslant C_{\alpha,\beta} t^{\alpha+\beta+2} \left\| f' \right\|_\infty \left\| g' \right\|_\infty .$$

□

4 Conclusion

Three definitions of fractional q-integral are given in order to introduce generalizations for the results of [1] and [5] as well as other results are presented including some bounds.

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