On some new fractional $q$-integral inequalities

W.T. SULAIMAN ⋆*

⋆ Department of Computer Engineering, College of Engineering, University of Mosul, Iraq
E-mail: waadsulaiman@hotmail.com

Received: 7-1-2012; Accepted: 9-8-2012 *Corresponding author

Abstract  Several new fractional $q$-integral inequalities are presented by using the Reimann-Liouville fractional integral in three types and concerning the product of two and three functions. There is a relationship between our results and [1]&[5].

Key Words  Fractional integral inequality, $Q$-integral inequality

MSC 2010  26D15, 39A12

1 Introduction

Let $f : \mathbb{R} \to \mathbb{R}$. For $0 < q < 1$, the $q$-analog of the derivative, denoted by $D_q$, is defined by (see [6])

$$D_q f(x) = \frac{f(x) - f(qx)}{x - qx}, \quad x \neq 0. \quad (1.1)$$

Whenever $f'(0)$ exists, $D_qF(0) = f'(0)$, and as $q \to 1^-$, the $q$-derivative reduces to the usual derivative.

The $q$–analog of integration from 0 to a is given by (see [7])

$$\int_{0}^{a} f(x) d_q x = a(1-q) \sum_{k=0}^{\infty} f(aq^k)q^k, \quad (1.2)$$

and for $a = \infty$,

$$\int_{0}^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f(q^n)q^n \quad (1.3)$$

provided the sum converges absolutely. On a general interval $[a, b]$ the $q$–integral is defined by (see [3]-[4])

$$\int_{a}^{b} f(x) d_q x = \int_{0}^{b} f(x) d_q x - \int_{0}^{a} f(x) d_q x. \quad (1.4)$$
The $q$-Jackson integral and the $q$-derivative are related by the fundamental theorem of quantum calculus, which can be stated as follows (see [4], p.73):

If $F$ is an anti $q$-derivative of the function $f$, namely $D_q F = f$, continuous at $x = a$, then

$$\int_a^b f(x) d_q x = F(b) - F(a).$$  (1.5)

For any bounded function $f$, we have

$$D_q \int_a^x f(t) d_q t = f(x),$$  (1.6)

and if $f$ is continuous at 0, then

$$\int_a^x D_q f(t) d_q t = f(x) - f(0).$$  (1.7)

For $b > 0$ and $a = bq^n$, $n \in \mathbb{N}$ we denote

$$[a, b]_q = \{bq^k : 0 \leq k \leq n\} \text{ and } (a, b] = [aq^{-1}, b]_q.$$  (1.8)

Let $c$ be a complex number, the $q$-shifted factorial are defined by

$$(c; q)_0 = 1, (c; q)_n = \prod_{k=0}^{n-1} (1 - cq^k), \quad n = 1, 2, \ldots .$$  (1.9)

$$(c; q)_{\infty} = \lim_{n \to \infty} (c; q)_n = \prod_{k=0}^{\infty} (1 - cq^k).$$  (1.10)

For $x$ complex we denote

$$[x]_q = \frac{1 - q^x}{1 - q}.$$  (1.11)

The $q$-analogue of the Gamma function is defined by Jackson [3] as follows

$$\Gamma_q(x) = \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}} (1 - q)^{1-x}, \quad x \neq 0, -1, -2, \ldots .$$  (1.12)

and it is satisfying the following

$$\Gamma_q(x + 1) = [x]_q \Gamma_q(x), \quad \Gamma_q(1) = 1,$$  (1.13)

and tends to $\Gamma(x)$ as $q \to 1^-$. The $q$-integral representation of the Gamma function is (see [2]) as follows

$$\Gamma_q(x) = K_q(x) \int_0^\infty t^{x-1} e_q(-t) d_q t,$$  (1.14)

where

$$e_q(t) = \frac{1}{((1 - q) t; q)_{\infty}}.$$  (1.15)
and

\[ K_q(t) = \frac{(1 - q)^{-x}}{1 + (1 - q)^{-1}} \times \frac{(- (1 - q); q)_\infty}{(- (1 - q) q^k; q)_\infty} \left( - (1 - q)^{-1} q^{1-k}; q \right)_\infty. \]  

(1.16)

The \(q\)-fractional function is defined by the following: If \(n\) is a positive integer, then

\[ (t - s)^{(n)} = (t - s)(t - qs)...(t - q^{n-1}s). \]  

(1.17)

If \(n\) is not a positive integer, then

\[ (t - s)^{(n)} = x^n \prod_{k=0}^{\infty} \frac{1 - (s/t)q^k}{1 - (s/t)q^{n+k}}. \]  

(1.18)

The \(q\)-derivative of the \(q\)-factorial function with respect to \(t\) is

\[ D_q (t - s)^{(n)} = \frac{1 - q^n}{1 - q} (t - s)^{(n-1)}, \]  

(1.19)

and the \(q\)-derivative of the \(q\)-factorial function with respect to \(t\) is

\[ D_q (t - s)^{(n)} = \frac{1 - q^n}{1 - q} (t - qs)^{(n-1)}. \]  

(1.20)

We define the fractional \(q\)-integral by the following

\[ I_q^a f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qx)^{(\alpha-1)} f(x) dq_x x, \]  

(1.21)

\[ J_q^a f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - x)^{(\alpha-1)} f(x) dq_x x, \]  

(1.22)

and

\[ K_q^a f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - x)^{(\alpha-1)} f(x) dq_x x. \]  

(1.23)

In particular

\[ I_q^1 f(t) = J_q^1 f(t) = K_q^1 f(t) \int_0^t f(x) dq_x x. \]

Öğünmez and Özkan [5] proved the following result

**Theorem 1.1.** Let \(f\) and \(g\) be two synchronous functions on \([0, \infty)\). Then for all \(t > 0, v > 0\), we have

\[ I_q^v (fg)(t) \geq \frac{\Gamma_q(v+1)}{\Gamma(v)} I_q^v f(t) I_q^v g(t). \]  

(1.24)

452
2 Lemmas

The following lemmas are needed for our aim

Lemma 2.1.

\[ I_q^\alpha(1) = \frac{1}{\Gamma_q(\alpha + 1)} I^{(\alpha)}, \quad (2.1) \]
\[ J_q^\alpha(1) = \frac{1}{\Gamma_q(\alpha + 1)} t^\alpha. \quad (2.2) \]

Proof.

\[ I_q^\alpha(1) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qx)^{(\alpha - 1)} dq \]
\[ = -\frac{1}{\Gamma_q(\alpha)} \frac{1 - q}{1 - q^\alpha} \int_0^t D_q(t - x)^{(\alpha - 1)} dq \]
\[ = -\frac{1}{\Gamma_q(\alpha)} \frac{1 - q}{1 - q^\alpha} \left( 0 - I^{(\alpha)} \right) \]
\[ = \frac{1}{[\alpha]_q \Gamma_q(\alpha)} I^{(\alpha)} = \frac{1}{\Gamma_q(\alpha + 1)} I^{(\alpha)}. \]

\[ J_q^\alpha(1) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - x)^{(\alpha - 1)} dq \]
\[ = \frac{1}{\Gamma_q(\alpha)} \int_0^t u^{\alpha - 1} dq u, \quad (t - x = u) \]
\[ = \frac{1}{\Gamma_q(\alpha)} t \left( 1 - q \right) \sum_{k=0}^{\infty} (tq^k)^{(\alpha - 1)} q^k \]
\[ = \frac{1}{\Gamma_q(\alpha)} t^\alpha \sum_{k=0}^{\infty} (q^\alpha)^k = \frac{1}{\Gamma_q(\alpha)} \frac{1 - q}{1 - q^\alpha} t^\alpha \]
\[ = \frac{t^\alpha}{[\alpha]_q \Gamma_q(\alpha)} = \frac{t^\alpha}{\Gamma_q(\alpha + 1)}. \]

\[ \square \]

Lemma 2.2. Let \( f : \mathbb{R} \to \mathbb{R} \) and define

\[ \mathcal{F}(x) = \int_0^x f(u) dq u, \quad (2.3) \]

then, for \( \alpha > -1 \),

\[ J_q^\alpha \mathcal{F}(x) = J_q^{\alpha + 1} f(t), \quad (2.4) \]
\[ I_q^\alpha \mathcal{F}(x) = \frac{1}{\Gamma_q(\alpha + 1)} K_q^{\alpha + 1} f(x). \quad (2.5) \]
Proof.

\[
J_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-x)^{\alpha-1} \int_0^x f(u) d_q u d_q x
\]

\[
= \frac{1}{\Gamma_q(\alpha)} \int_0^t f(u) \int_u^t (t-x)^{\alpha-1} d_q x d_q u
\]

\[
= \frac{1}{\Gamma_q(\alpha)} \int_0^t f(u) \int_0^{t-u} v^{\alpha-1} d_q v d_q u
\]

\[
= \frac{1}{\Gamma_q(\alpha)} (1-q) \sum_{k=0}^{\infty} (q^k)^{\alpha-1} q^k \int_0^t f(u)(t-u)^{\alpha} d_q u
\]

\[
= \frac{1}{\Gamma_q(\alpha)} (1-q) \sum_{k=0}^{\infty} (q^\alpha)^k \int_0^t f(u)(t-u)^{\alpha} d_q u
\]

\[
= \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} \frac{1-q}{1-q^\alpha} J_q^{\alpha+1} f(t) = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\alpha)} [\alpha_q] J_q^{\alpha+1} f(t)
\]

\[
= J_q^{\alpha+1} f(t).
\]

\[
I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qx)^{\alpha-1} \int_0^x f(u) d_q u d_q x
\]

\[
= \frac{1}{\Gamma_q(\alpha)} \int_0^t f(u) \int_u^t (t-qx)^{\alpha-1} d_q x d_q u
\]

\[
= \frac{1}{\Gamma_q(\alpha)} \int_0^t f(u) \int_0^t D_q(t-x)^{\alpha} d_q x d_q u
\]

\[
= \frac{1}{[\alpha_q] \Gamma_q(\alpha)} \int_0^t (t-u)^{\alpha} f(u) d_q u
\]

\[
= \frac{1}{\Gamma_q(\alpha+1)} K_q^\alpha f(t).
\]

\[
\square
\]

3 Main Results

Theorem 3.1. Let \(f\) and \(g\) be two synchronous on \([0, \infty]\), \(h \geq 0\), then for all \(t > 0\), \(\alpha, \beta > 0\), we have

\[
\frac{t^\beta}{\Gamma_q(\beta+1)} J_q^\alpha (fgh)(t) + \frac{t^\alpha}{\Gamma_q(\alpha+1)} J_q^\beta (fgh)(t)
\]

\[
\geq J_q^\alpha f(t) J_q^\beta (gh)(t) + J_q^\beta f(\alpha) J_q^\alpha (gh)(t) + J_q^\alpha g(t) J_q^\alpha (fh)(t)
\]

\[
+ J_q^\beta g(t) J_q^\alpha (fh)(t) - J_q^\alpha h(t) J_q^\beta (fg)(t) - J_q^\beta h(t) J_q^\alpha (fg)(t).
\]

(3.1)
In particular,
\[
\frac{t^\alpha}{\Gamma_q(\alpha + 1)} J^\alpha_q (fgh) (t) \geq J^\alpha_q f(t) J^\alpha_q (gh) (t) + J^\alpha_q g(t) J^\alpha_q (fh) (t) - J^\beta_q h(t) J^\alpha_q (fg) (t),
\]
\[
\frac{t^\beta}{\Gamma_q(\beta + 1)} J^\alpha_q (fg) (t) + \frac{t^\alpha}{\Gamma_q(\alpha + 1)} J^\beta_q (fgh) (t) \geq J^\alpha_q f(t) J^\beta_q g(t) + J^\beta_q f(t) J^\alpha_q g(t).
\]

**Proof.** As
\[
(t - x)^{\alpha - 1} (t - y)^{\beta - 1} (f(x) - f(y)) (g(x) - g(y)) (h(x) + h(y)) \geq 0,
\]
then, we have
\[
\int_0^t \int_0^t (t - x)^{\alpha - 1} (t - y)^{\beta - 1} (f(x) - f(y)) (g(x) - g(y)) (h(x) + h(y)) \, dx \, dy \geq 0.
\]

By opening the above, we obtain
\[
\int_0^t (t - x)^{\alpha - 1} f(x) g(x) h(x) \, dx \int_0^t (t - y)^{\beta - 1} \, dy + \int_0^t (t - y)^{\beta - 1} f(y) g(y) h(y) \, dy \int_0^t (t - x)^{\alpha - 1} \, dx
\]
\[
\geq \int_0^t (t - x)^{\alpha - 1} f(x) h(x) \, dx \int_0^t (t - y)^{\beta - 1} g(y) \, dy + \int_0^t (t - x)^{\alpha - 1} g(x) h(x) \, dx \int_0^t (t - y)^{\beta - 1} f(y) \, dy
\]
\[
- \int_0^t (t - x)^{\alpha - 1} h(x) \, dx \int_0^t (t - y)^{\beta - 1} f(y) \, dy - \int_0^t (t - x)^{\alpha - 1} f(x) g(x) \, dx \int_0^t (t - y)^{\beta - 1} h(y) \, dy
\]
\[
+ \int_0^t (t - x)^{\alpha - 1} f(x) \, dx \int_0^t (t - y)^{\beta - 1} g(y) h(y) \, dy + \int_0^t (t - x)^{\alpha - 1} g(x) \, dx \int_0^t (t - y)^{\beta - 1} f(y) h(y) \, dy.
\]

Dividing the above inequality by \( \Gamma_q(\alpha) \Gamma_q(\beta) \), noticing that
\[
J^\alpha_q (1) = \frac{1}{\Gamma_q(\alpha + 1)} t^\alpha,
\]
we obtain
\[
\frac{t^\beta}{\Gamma_q(\beta + 1)} J^\alpha_q (fgh) (t) + \frac{t^\alpha}{\Gamma_q(\alpha + 1)} J^\beta_q (fgh) (t)
\]
\[
\geq J^\alpha_q f(t) J^\beta_q (gh) (t) + J^\beta_q f(t) J^\alpha_q (gh) (t) + J^\alpha_q g(t) J^\alpha_q (fh) (t)
\]
\[
+ J^\beta_q g(t) J^\beta_q (fh) (t) - J^\beta_q h(t) J^\beta_q (fg) (t) - J^\alpha_q h(t) J^\alpha_q (fg) (t).
\]

The following is a similar result but dealing with \( I^\alpha_q, I^\beta_q \).

**Theorem 3.2.** Let \( f \) and \( g \) be two synchronous on \([0, \infty], h \geq 0\), then for all \( t > 0, \alpha, \beta > 0 \),
\[
\frac{1}{\Gamma_q(\beta + 1)} I^{(\beta)} I^\alpha_q (fgh) (t) + \frac{1}{\Gamma_q(\alpha + 1)} I^{(\alpha)} I^\beta_q (fgh) (t)
\]

455
\[ I_q^\alpha f(t) f^\beta (gh) (t) + I_q^\alpha f(t) I_q^\alpha (gh) (t) + I_q^\alpha g(t) I_q^\alpha (fh) (t) + I_q^\alpha g(t) I_q^\alpha (fh) (t) - I_q^\alpha h(t) I_q^\alpha (fg) (t) - I_q^\alpha h(t) I_q^\alpha (fg) (t). \] (3.2)

In particular,
\[ \frac{1}{\Gamma_q(\alpha + 1)} t^{(\alpha)} I_q^\alpha (fg) (t) \geq I_q^\alpha f(t) I_q^\alpha (gh) (t) + I_q^\alpha g(t) I_q^\alpha (fh) (t) - I_q^\alpha h(t) I_q^\alpha (fg) (t). \] (3.3)

\[ \frac{1}{\Gamma_q(\beta + 1)} t^{(\beta)} I_q^\beta (fg) (t) + \frac{1}{\Gamma_q(\alpha + 1)} t^{(\alpha)} I_q^\alpha (fg) (t) \geq I_q^\alpha f(t) I_q^\beta g(t) + I_q^\beta f(t) I_q^\alpha g(t). \] (3.4)

Remark. It may be mentioned that (1.24) follows from (3.4) by putting \( \alpha = \beta. \)

Theorem 3.3. Let \( f \) and \( g \) be two functions on \([0, \infty], \) then for \( t > 0, \alpha, \beta > 0, \) we have
\[ \frac{t^{(\beta)}}{\Gamma_q(\beta + 1)} J_q^\alpha (f^2) (t) + \frac{t^{(\alpha)}}{\Gamma_q(\alpha + 1)} J_q^\beta (g^2) (t) \geq 2 J_q^\alpha f(t) J_q^\beta g(t). \] (3.5)

In particular,
\[ \frac{t^{(\alpha)}}{\Gamma_q(\alpha + 1)} J_q^\alpha (f^2) (t) \geq (J_q^\alpha f(t))^2. \] (3.6)

Proof. Since
\[ (t - x)^{\alpha - 1} (t - y)^{\beta - 1} (f(x) - g(y))^2 \geq 0, \]
then, we have
\[ \int\limits_0^t (t - x)^{\alpha - 1} (t - y)^{\beta - 1} (f(x) - g(y))^2 \, dx \, dy \geq 0. \]

The above implies
\[ \int\limits_0^t \int\limits_0^t (t - x)^{\alpha - 1} (t - y)^{\beta - 1} f^2 (x) \, dx \, dy + \int\limits_0^t \int\limits_0^t (t - x)^{\alpha - 1} (t - y)^{\beta - 1} g^2 (y) \, dy \, dx \geq 2 \int\limits_0^t \int\limits_0^t (t - x)^{\alpha - 1} (t - y)^{\beta - 1} f(x) g(y) \, dx \, dy. \]

Dividing the above inequality by \( \Gamma_q(\alpha) \Gamma_q(\beta), \) noticing that
\[ J_q^\alpha (1) = \frac{t^\alpha}{\Gamma_q(\alpha + 1)}, \]
We obtain
\[ \frac{t^{(\beta)}}{\Gamma_q(\beta + 1)} J_q^\alpha (f^2) (t) + \frac{t^{(\alpha)}}{\Gamma_q(\alpha + 1)} J_q^\beta (g^2) (t) \geq 2 J_q^\alpha f(t) J_q^\beta g(t). \]
Theorem 3.5. Let \( f \) and \( g \) be two functions on \([0, \infty]\), then for \( t > 0, \alpha, \beta > 0 \), we have

\[
J_q^\alpha (f^2) (t) J_q^\beta (g^2) (t) + J_q^\alpha (f^2) (t) J_q^\beta (g^2) (t) \geq 2 J_q^\alpha (fg) (t) J_q^\beta (fg) (t) \tag{3.7}
\]

In particular

\[
J_q^\alpha (f^2) (t) I_q^\beta (g^2) (t) \geq (J_q^\alpha (fg) (t))^2. \tag{3.8}
\]

Proof. Since

\[
(t - x)^{\alpha - 1} (t - y)^{\beta - 1} (f(x)g(y) - f(y)g(x))^2 \geq 0,
\]

then, we have

\[
\int_0^t \int_0^t (t - x)^{\alpha - 1} (t - y)^{\beta - 1} (f(x)g(y) - f(y)g(x))^2 \, dq \, dx y \geq 0.
\]

The result follows by opening the last inequality and dividing by \( \Gamma_q (\alpha) \Gamma_q (\beta) \) as in the previous result.

Other inequalities dealing with \( \tilde{f}, \tilde{g} \) are presented in the following two results

Theorem 3.4. Let \( f \) and \( g \) be two functions on \([0, \infty]\), then for \( t > 0, \alpha, \beta > -1 \), we have

\[
\frac{t^{(\beta)}}{\Gamma_q (\beta + 1)} J_q^\alpha \left(\int g \right) (t) \frac{t^{(\alpha)}}{\Gamma_q (\alpha + 1)} J_q^\beta \left(\int g \right) (t) \geq J_q^\alpha \left(\int \right) (t) J_q^\beta \left(\int \right) (t) + J_q^\alpha \left(\int \right) (t) J_q^\beta \left(\int \right) (t) \tag{3.9}
\]

Proof. The proof follows from Theorem 3.1, (3.2), and via Lemma 2.2.

Theorem 3.6. Let \( f \) and \( g \) be two functions on \([0, \infty]\), then for \( t > 0, \alpha, \beta > 0 \) we have

\[
\frac{1}{\Gamma_q (\beta + 1)} J_q^\alpha \left(\int \right) (t) + \frac{1}{\Gamma_q (\alpha + 1)} J_q^\beta \left(\int \right) (t) \geq I_q^\alpha \left(\int \right) (t) J_q^\beta \left(\int \right) (t) + I_q^\alpha \left(\int \right) (t) I_q^\beta \left(\int \right) (t) \tag{3.10}
\]

Proof. Replacing \( f, g \) by \( \tilde{f}, \tilde{g} \) in (3.4), we obtain

\[
\frac{1}{\Gamma_q (\beta + 1)} J_q^\alpha \left(\int \right) (t) + \frac{1}{\Gamma_q (\alpha + 1)} J_q^\beta \left(\int \right) (t) \geq I_q^\alpha \left(\int \right) (t) J_q^\beta \left(\int \right) (t) + I_q^\alpha \left(\int \right) (t) I_q^\beta \left(\int \right) (t).
\]

The result follows via Lemma 2.2. Our last result concerning some bounds

Theorem 3.7. Let \( f \) and \( g \) be two differentiable functions on \([0, \infty] \). Define

\[
\|f'\|_\infty = \sup_{x \in [0,\infty]} |f'(x)| < \infty,
\]

then for \( t > 0, \alpha, \beta > 1 \) we have

\[
\left| \frac{t^{\beta}}{\Gamma (\beta + 1)} J_q^\alpha (fg) (t) + \frac{t^{\alpha}}{\Gamma (\alpha + 1)} J_q^\beta (fg) (t) - J_q^\alpha (f) (t) J_q^\beta (g) (t) - J_q^\alpha (f(t) J_q^\beta (g(t)) \right| \]
\[
\leq t^{\alpha+\beta+2} C_{\alpha,\beta} \|f^{''}\|_{\infty} \|g^{'}\|_{\infty},
\]  
(3.11)

where

\[
C_{\alpha,\beta} = \max \left\{ \frac{1}{\Gamma_q(\alpha+1)\Gamma_q(\beta+3)}, \frac{1}{\Gamma_q(\beta+1)\Gamma_q(\alpha+3)} \right\}.
\]

In particular

\[
\left| \frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha(fg)(t) - I^\alpha f(t) I^\alpha g(t) \right| \leq \frac{\|f^{'}\|_{\infty} \|g^{'}\|_{\infty}}{\Gamma(\alpha+1)\Gamma_q(\alpha+3)} t^{2\alpha+2}.
\]  
(3.12)

Proof.

\[
J = \left| \frac{t^\beta}{\Gamma(\beta+1)} I^\beta(fg)(t) + \frac{t^\alpha}{\Gamma(\alpha+1)} I^\alpha(fg)(t) - I^\alpha f(t) I^\beta g(t) - I^\beta f(t) I^\alpha g(t) \right|
\]

\[
= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left| \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} (f(x)-f(y))(g(x)-g(y)) \, dx \, dy \right|
\]

\[
= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left| \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} \int_x^y f'(u) \, du \int_y^x g'(v) \, dv \, dx \, dy \right|
\]

\[
\leq \frac{\|f^{'}\|_{\infty} \|g^{'}\|_{\infty}}{\Gamma(\alpha)\Gamma(\beta)} \left| \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} \int_y^x \, dv \, dx \right| \|
\]

\[
= \frac{\|f^{'}\|_{\infty} \|g^{'}\|_{\infty}}{\Gamma(\alpha)\Gamma(\beta)} \left| \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta-1} |x-y|^2 \, dx \right|.
\]

If \( x \geq y, |x-y| = x-y \leq t-y \), hence

\[
J \leq \frac{\|f^{'}\|_{\infty} \|g^{'}\|_{\infty}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-x)^{\alpha-1} (t-y)^{\beta+1} \, dx \, dy
\]

\[
= \|f^{'}\|_{\infty} \|g^{'}\|_{\infty} j_q^\alpha(1) j_q^{\beta+2}(1)
\]

\[
= C_{\alpha,\beta} t^{\alpha+\beta+2} \|f^{'}\|_{\infty} \|g^{'}\|_{\infty}
\]

\[
= \frac{\alpha(\beta+2)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\|f^{'}\|_{\infty} \|g^{'}\|_{\infty}}{\Gamma(\alpha)\Gamma(\beta)} \leq \frac{\alpha(\beta+2)}{\Gamma(\alpha)\Gamma(\beta)} C_{\alpha,\beta} \|f^{'}\|_{\infty} \|g^{'}\|_{\infty}.
\]

Similarly, if \( x \leq y \), we have

\[
J \leq \frac{\|f^{'}\|_{\infty} \|g^{'}\|_{\infty}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^t (t-x)^{\alpha+1} (t-y)^{\beta-1} \, dx \, dy \leq C_{\alpha,\beta} t^{\alpha+\beta+2} \|f^{'}\|_{\infty} \|g^{'}\|_{\infty}.
\]

4 Conclusion

Three definitions of fractional q-integral are given in order to introduce generalizations for the results of [1] and [5] as well as other results are presented including some bounds.
References