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R-Weakly Commuting Mappings for Generalized-ψ-$\emptyset$-Weak Contraction in Metric Spaces

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Abstract In this paper, we introduce generalized ψ-$\emptyset$-weak contraction condition that involves cubic and quadratic terms of distance function $d(x, y)$ and prove common fixed point theorems for weakly commuting and its variants point wise R-weakly commuting and reciprocal continuous mappings, R-weakly commuting mappings of type (P).

Key Words ψ-$\emptyset$-Weak Contraction, R-weakly commuting, R-weakly commuting of type (P)

MSC 2010 47H10, 54H25

1 Introduction

Metric fixed point theory involves the study of fixed points depending on the mapping conditions on the spaces under consideration. There is a revolution in metric fixed point theory from the time when Banach contraction principle was introduced in fixed point theory literature. This principle is popularly known as the contraction principle, which states that "every contraction mappings on a complete metric has a unique fixed point". This principle gives the existence and uniqueness of fixed points and provides a technique for obtaining approximate fixed points.

This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Most of the problems of applied mathematics reduce to inequality which in turn their solutions give rise to the fixed points of certain mappings.

The Banach Contraction principle has been generalized in various ways such as by relaxing continuity, extending the number of mappings, using control functions and soothing minimal commutative conditions and various properties. Banach's contraction principle is the primary tool for finding fixed points of all contraction type maps.

It was a turning point in fixed point theory literature when the notion of commutative mappings was introduced. On the other hand, Sessa [16] coined the notion of weak commutativity of mappings.

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2 Preliminaries

Banach fixed point theorem states that every contraction mapping on a complete metric space has a unique fixed point. Let (X, d) be a complete metric space. If $T : X \to X$ satisfies $d(T(x), T(y)) \leqslant k(d(x, y))$ for all $x, y \in X$, $0 \leqslant k < 1$, then it has a unique fixed point.

In 1969, Boyd and Wong [2] replaced the constant k in Banach contraction principle by a control function ψ as follows:

Let (X, d) be a complete metric space and $\psi : [0\ ,\infty) \to [0, \infty)$ be upper semi continuous from the right such that $0 \leqslant \psi(t) < t$ for all $t > 0$.

If $T : X \to X$ satisfies $d(T(x), T(y)) \leqslant \psi(d(x, y))$ for all $x, y \in X$, then it has a unique fixed point.

In 1997, Alber and Gueree-Delabriere [1] introduced the concept of weak contraction as follows:

A map $T : X \to X$ is said to be weak contraction if for each $x, y \in X$, there exists a function $\emptyset : [0, \infty) \to [0, \infty)$, $\emptyset\ (t) > 0$ for all $t > 0$ and $\emptyset(0) = 0$ such that

$$d(Tx, Ty) \leqslant d(x, y) - \emptyset(d(x, y)).$$

A pair of self-mappings f and g on a metric space (X, d) is called commuting if $fgu = gfu$ for all u in X.

***Definition* 2.1.** Two self mappings f and g of a metric space (X, d) are said to be commuting if $fgx = gfx$ for all x in X.

The first ever attempt to relax the commutativity of mappings to weak commutative was initiated by Sessa [16] in 1982 as follows:

***Definition* 2.2** ([16])**.** Two self mappings f and g of a metric space (X, d) are said to be weakly commuting if $d(fgx, gfx) = d(gx, fx)$ for all x in X.

***Remark* 2.1** ([16])**.** Commutative mappings are weak commutative mappings, but converse may not be true.

In 1994, Pant [13] introduced the notion of R-weakly commuting mappings in metric spaces to widen the scope of finding fixed point for mappings from class of compatible maps to a wider class of R-weakly commuting mappings. These maps are not necessarily continuous at fixed point.

***Definition* 2.3** ([13])**.** A pair (f, g) of self-mappings of a metric space (X, d) is said to be R-weakly commuting if there exists some real number $R > 0$ such that

$$d(fgx, gfx) \leqslant Rd(fx, gx), \quad \text{for all } x \in X.$$

***Remark* 2.2** ([13])**.**

1. For $R = 1$, every R-weakly commuting pair is weakly commuting.
2. Weak commutativity implies R-weak commutativity. However, R-weak commutativity implies weak commutativity only when $R \leqslant 1$.

In 1998, Pant [14] investigated the existence of common fixed points for non-compatible mappings and point wise R-weak commutativity.

***Definition* 2.4** ([14])**.** A pair (f, g) of self-mappings of a metric space (X, d) is said to be point wise R-weakly commuting on X, iff given $x \in X$, there exists $R > 0$ such that

$$d(fgx, gfx) \leqslant Rd(fx, gx).$$

***Remark* 2.3** ([14])**.** It is clear from the Definition 2.5 that f and g can fail to be pointwise R-weakly commuting only if there exists some x in X such that $fx = gx$ but $fgx \neq gfx$, i.e., only if they possess a coincidence point at which they do not commute.

In 1999, Pant [11] introduced a new type of continuity condition, known as reciprocal continuity defined as follows:

***Definition* 2.5** ([11])**.** A pair (f, g) of self-mappings of a metric space (X, d) is said to be reciprocally continuous if $\lim_{n\infty} fgx_n = fz$ and $\lim_{n\infty} gfx_n = gz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} d(fx_n, z) = \lim_{n\to\infty} d(gx_n, z) = 0$ for some $z \in X$.

In 1986, Jungck [7] generalized and extend the notion of weak commutativity to compatible mappings.

***Definition* 2.6** ([7])**.** Two self mappings f and g of a metric space (X, d) are called compatible if $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_n fx_n = \lim_n gx_n = t, \quad \text{for some } t \text{ in } X.$$

Now we state some properties for compatible mappings that are fruitful for further study.

Proposition 2.1 ([7])**.** *Let S and T be compatible mappings of a metric space (X, d) into itself. If $St = Tt$ for some $t \in X$, then $STt = SSt = TTt = TSt$.*

Proposition 2.2 ([7])**.** *Let S and T be compatible mappings of a metric space (X, d) into itself. Suppose that $\lim_n Sx_n = \lim_n Tx_n = t$ for some t in X. Then the following holds:*

1. *$\lim_n TSx_n = St$ if S is continuous at t;*
2. *$\lim_n STx_n = Tt$ if T is continuous at t;*
3. *$STt = TSt$ and $St = Tt$ if S and T are continuous at t.*

3 Main Results

In 2013, Murthy and Prasad [10] introduced a new type of inequality for a map that involves cubic terms of metric function $d(x, y)$ that extended and generalized the results of Alber and Gueree-Delabriere [1] and many others cited in the literature of fixed point theory.

Now we introduce the generalized ψ-$\emptyset$-weak contraction of Murthy and Prasad [12] for a pairs of mappings in the following way:

Let A, B, S and T are self mappings on a metric space (X, d) satisfying the following conditions:

1. $S(X) \subset B(X)$, $T(X) \subset A(X)$;

2. $d^3(Sx,Ty) \leqslant \psi\{d^2(Ax,Sx)d(By,Ty), d(Ax,Sx)d^2(By,Ty),$
$$d(Ax,Sx)d(Ax,Ty)d(By,Sx), d(Ax,Ty)d(By,Sx)d(By,Ty)\} - \emptyset\{m(Ax,By)\}$$
$$m(Ax,By) = \max\Big\{d^2(Ax,By), d(Ax,Sx)d(By,Ty), d(Ax,Ty)d(By,Sx), \frac{1}{2}[d(Ax,Sx)d(Ax,Ty) + d(By,Sx)d(By,Ty)]\Big\}$$

for all $x, y \in X$, where $\psi : [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function with $\psi(t) < t$ for each $t > 0$ and $\emptyset : [0,\infty) \to [0,\infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset\ (t) > 0$ for each $t > 0$.

Now, we prove fixed point theorems for a pair of weakly commuting mappings involves cubic terms and product of quadratic and linear terms of distance function $d(x,y)$.

Theorem 3.1. *Let (X,d) be a complete metric space. Let S,T,A and B be four self mappings of a complete metric space (X,d) satisfying (C1), (C2) and the following conditions:*

1. *one of S,T,A and B is continuous.*

Further, assume that the pairs (A,S) and (B,T) are weakly commuting. Then S,T,A and B have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. From (C1) we can find x_1 such that $S(x_0) = B(x_1) = y_0$ for this x_1 one can find $x_2 \in X$ such that $T(x_1) = A(x_2) = y_1$. Continuing in this way, one can construct a sequence $\{y_n\}$ in X such that

$$y_{2n} = S(x_{2n}) = B(x_{2n+1}),$$
$$y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2}), \quad \text{for each } n \geqslant 0.$$

For brevity, one can denote $\alpha_{2n} = d(y_{2n}, y_{2n+1})$.

From [9], we can easily prove that $\{y_n\}$ is a Cauchy sequence in X.

Since (X,d) is a complete metric space, therefore, $\{y_n\}$ converges to a point z as $n \to \infty$. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converges to the same point z.

Case 1. Suppose that A is continuous. Then $\{AAx_{2n}\}$ and $\{ASx_{2n}\}$ converges to Az as $n \to \infty$. Since the mappings, A and S are weakly commuting on X, therefore,

$$d(ASx_{2n}, SAx_{2n}) = d(Sx_{2n}, Ax_{2n})\,.$$

Proceeding limit as $n \to \infty$, we get

$$\lim_{n\to\infty} d(SAx_{2n}, Az) = d(z,z) = 0,$$

i.e.,

$$\lim_{n\to\infty} SAx_{2n} = Az.$$

Now, we show that $z = Az$. On putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (C2) we get

$$\begin{aligned} d^3(SAx_{2n}, Tx_{2n+1}) \leqslant \psi\{ & d^2(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\ & d(AAx_{2n}, SAx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1}), \\ & d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}), \\ & d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\} \\ & - \emptyset(m(AAx_{2n}, Bx_{2n+1})), \end{aligned}$$

where

$$\begin{aligned} m(AAx_{2n}, Bx_{2n+1}) = \max\Big\{ & d^2(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}), \\ & d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}), \\ & \frac{1}{2}\Big[d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1}) \\ & + d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\Big]\Big\}. \end{aligned}$$

Proceeding limit as $n \to \infty$, we get

$$d^3(Az, z) \leqslant \psi\{0, 0, 0, 0\} - \emptyset(d^2(Az, z)),$$

or

$$d^3(Az, z) \leqslant -\emptyset(d^2(Az, z)).$$

On simplification, and using the properties of ψ and $\emptyset$, we get $d^2(Az, z) = 0$.

This implies that $Az = z$.

Next, we will show that $Sz = z$.

On putting $x = z$ and $y = x_{2n+1}$ in (C2) we have

$$\begin{aligned} d^3(Sz, Tx_{2n+1}) \leqslant \psi\{ & d^2(Az, Sz)d(z, z), d(Az, Sz)d^2(z, z), d(Az, Sz)d(Az, z)d(z, Sz), \\ & (Az, z)d(z, Sz)d(z, z)\} - \emptyset(m(Az, z)), \end{aligned}$$

where

$$\begin{aligned} m(Az, z) = \max\Big\{ & d^2(Az, z), d(Az, Sz)d(z, z), d(Az, z)d(z, Sz), \\ & \frac{1}{2}[d(Az, Sz)d(Az, z) + d(z, Sz)d(z, z)]\Big\} = 0. \end{aligned}$$

Passing limit as $n \to \infty$, and after simplification, using the properties of ψ and $\emptyset$, we have

$$d^3(Sz, z) \leqslant \psi\{0, 0, 0, 0\} - \emptyset(0).$$

Thus we get $d^3(Sz, z) = 0$. This implies that $Sz = z$. Since $S(X) \subset B(X)$, therefore, there exists a point $u \in X$ such that $z = Sz = Bu$.

Now we show that $z = Tu$.

For this we put $x = z$ and $y = u$ in (C2) we get

$$\begin{aligned} d^3(Sz, Tu) \leqslant \psi\{&d^2(Az, Sz)d(Bu, Tu), d(Az, Sz)d^2(Bu, Tu) \\ &d(Az, Sz)d(Az, Tu)d(Bu, Sz), \\ &d(Az, Tu)d(Bu, Sz)d(Bu, Tu)\} - \emptyset(m(Az, Bu)), \end{aligned}$$

where

$$\begin{aligned} m(Az, Bu) = \max\Big\{&d^2(Az, Bu), d(Az, Sz)d(Bu, Tu), d(Az, Tu)d(Bu, Sz), \\ &\frac{1}{2}[d(Az, Sz)d(Az, Tu) + d(Bu, Sz)d(Bu, Tu)]\Big\}, \end{aligned}$$

i.e.,

$$\begin{aligned} m(Az, Bu) = \max\Big\{&d^2(z, z), d(z, z)d(z, Tu), d(z, Tu)d(z, z), \\ &\frac{1}{2}[d(z, z)d(Az, Tu) + d(z, z)d(z, Tu)]\Big\} = 0. \end{aligned}$$

On solving, we get

$$\begin{aligned} d^3(z, Tu) \leqslant \psi\{&d^2(z, z)d(z, Tu), d(z, z)d^2(z, Tu), \\ &d(z, z)d(z, Tu)d(z, z), d(z, Tu)d(z, z)d(z, Tu)\} - \emptyset(0). \end{aligned}$$

This implies that $z = Tu$. Since the pair (B, T) is weak commutative, therefore, we have

$$d(Bz, Tz) = d(BTu, TBu) = d(Bu, Tu) = d(z, z) = 0.$$

Thus $Bz = Tz$.

From (C2) we have

$$\begin{aligned} d^3(Sz, Tz) \leqslant \psi\{&d^2(Az, Sz)d(Bz, Tz), d(Az, Sz)d^2(Bz, Tz) \\ &d(Az, Sz)d(Az, Tz)d(Bz, Sz), d(Az, Tz)d(Bz, Sz)d(Bz, Tz)\} \\ &- \emptyset(m(Az, Bz)), \end{aligned}$$

where

$$\begin{aligned} m(Az, Bz) = \max\Big\{&d^2(Az, Bz), d(Az, Sz)d(Bz, Tz), d(Az, Tz)d(Bz, Sz), \\ &\frac{1}{2}[d(Az, Sz)d(Az, Tz) + d(Bz, Sz)d(Bz, Tz)]\Big\} = d^2(z, Tz). \end{aligned}$$

Therefore, we get

$$d^3(z, Tz) \leqslant \psi\{0, 0, 0, 0\} - \emptyset(d^2(z, Tz)).$$

Using the properties of ψ and $\emptyset$, we have $z = Tz$.

Case 2. Suppose that B is continuous; we can obtain the same result by way of Case 1.

Case 3. Suppose that S is continuous.

Then $\{SSx_{2n}\}$ and $\{SAx_{2n}\}$ converges to Sz as $n \to \infty$. Since the mappings A and S are weakly commuting on X, therefore, we have

$$d(ASx_{2n}, SAx_{2n}) \;=\; d(Sx_{2n}, Ax_{2n})$$

Proceeding limit as $n \to \infty$, we get

$$\lim_{n\to\infty} d(ASx_{2n}, Sz) = d(z, z) = 0,$$

i.e.,

$$\lim_{n\to\infty} ASx_{2n} = Sz.$$

Now, we prove that $z = Sz$.

For this put $x = Sx_{2n}$ and $y = x_{2n+1}$ in (C2), we get

$$\begin{aligned} d^3(SSx_{2n}, Tx_{2n+1}) \leqslant \psi\{&d^2(ASx_{2n}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1}),\\ &d(ASx_{2n}, SSx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1})\\ &d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n}),\\ &d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\}\\ &-\emptyset(m(ASx_{2n}, Bx_{2n+1})), \end{aligned}$$

where

$$\begin{aligned} m(ASx_{2n}, Bx_{2n+1}) = \max\Bigg\{&d^2(ASx_{2n}, Bx_{2n+1}), d(ASx_{2n}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1}),\\ &d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n}),\\ &\frac{1}{2}\Bigg[d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tx_{2n+1})\\ &+ d(Bx_{2n+1}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\Bigg]\Bigg\}. \end{aligned}$$

Taking limit as $n \to \infty$, and using the properties of ψ and $\emptyset$, we have

$$d^3(Sz, z) \leqslant \psi\{0, 0, 0, 0\} - \emptyset(d^2(Sz, z))$$

or

$$d^3(Sz, z) \leqslant -\emptyset(d^2(Sz, z)).$$

Thus we get $d^2(Sz, z) = 0$. This implies that $Sz = z$.

Since $S(X) \subset B(X)$ and hence there exists a point $v \in X$ such that $z = Sz = Bv$.

Now we claim that $z = Tv$.

For this we put $x = Sx_{2n}$ and $y = v$ in (C2) we get

$$d^3(SSx_{2n}, Tv) \leqslant \psi\{d^2(ASx_{2n}, SSx_{2n})d(Bv, Tv), d(ASx_{2n}, SSx_{2n})d^2(Bv, Tv),$$

$$d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tv)d(Bv, SSx_{2n}),$$
$$d(ASx_{2n}, Tv)d(Bv, SSx_{2n})d(Bv, Tv)\} - \emptyset(m(ASx_{2n}, Bv)),$$

where

$$m(ASx_{2n}, Bv) = \max\Big\{d^2(ASx_{2n}, Bv), d(ASx_{2n}, SSx_{2n})d(Bv, Tv), d(ASx_{2n}, Tv)d(Bv, SSx_{2n}), \frac{1}{2}\Big[d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tv) + d(Bv, SSx_{2n})d(Bv, Tv)\Big]\Big\}.$$

Taking limit as $n \to \infty$, and

$$d^3(z, Tv) \leqslant \psi\{d^2(z, z)d(z, Tv), d(z, z)d^2(z, Tv), d(z, z)d(z, Tv)d(z, z), d(z, Tv)d(z, z)d(z, Tv)\} - \emptyset(0).$$

Using the properties of ψ and $\emptyset$, we have $z = Tv$. Since the pair (B, T) is weakly commuting on X, so we have $d(TBv, BTv) = d(Tv, Bv) = d(z, z) = 0$.

So, $Bz = Tz$.

Now put $x = x_{2n}$ and $y = z$ in (C2), we get

$$d^3(Sx_{2n}, Tz) \leqslant \psi\{d^2(Ax_{2n}, Sx_{2n})d(Bz, Tz), d(Ax_{2n}, Sx_{2n})d^2(Bz, Tz), d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz)d(Bz, Sx_{2n}), d(Ax_{2n}, Tz)d(Bz, Sx_{2n})d(Bz, Tz)\} - \emptyset(m(Ax_{2n}, Bz)),$$

where

$$m(Ax_{2n}, Bz) = \max\Big\{d^2(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n})d(Bz, Tz), d(Ax_{2n}, Tz)d(Bz, Sx_{2n}), \frac{1}{2}\Big[d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz) + d(Bz, Sx_{2n})d(Bz, Tz)\Big]\Big\} = d^2(z, Tz).$$

Passing limit as $n \to \infty$, we get

$$d^3(z, Tz) \leqslant \psi\{0, 0, 0, 0\} - \emptyset(d^2(z, Tz)),$$

using the properties of ψ and $\emptyset$, we have $z = Tz$.

Since $T(X) \subset A(X)$, therefore, there exists a point $w \in X$ such that $z = Tz = Aw$.

We claim that $z = Sw$.

For this, we put $x = w$ and $y = z$ in (C2) we have

$$d^3(Sw, Tz) \leqslant \psi\{d^2(Aw, Sw)d(Bz, Tz), d(Aw, Sw)d^2(Bz, Tz),$$

$$d(Aw,Sw)d(Aw,Tz)d(Bz,Sw), d(Aw,Tz)d(Bz,Sw)d(Bz,Tz)\}$$
$$-\emptyset(m(Aw,Bz)),$$

where

$$m(Aw,Bz) = \max\Big\{d^2(Aw,Bz), d(Aw,Sw)d(Bz,Tz), d(Aw,Tz)d(Bz,Sw),$$
$$\frac{1}{2}[d(Aw,Sw)d(Aw,Tz) + d(Bz,Sw)d(Bz,Tz)]\Big\}.$$

On simplification $m(Aw,Bz)$ give rise to

$$m(Aw,Bz) = \max\Big\{d^2(z,z), d(z,Sw)d(Tz,Tz), d(z,z)d(z,Sw),$$
$$\frac{1}{2}[d(z,Sw)d(z,z) + d(z,Sw)d(Tz,Tz)]\Big\} = 0.$$

On solving, we have

$$d^3(Sw,z) \leqslant \psi\{d^2(z,Sw)d(z,z), d(z,Sw)d^2(z,z),$$
$$d(z,Sw)d(z,z)d(z,Sw), d(z,z)d(z,Sw)d(z,z)\} - \emptyset(0).$$

This implies that $Sw = z$. Since the pair (S,A) is weakly commuting on X, therefore,

$$d(ASw, SAw) = d(Sw, Aw) = d(z,z) = 0,$$

therefore,

$$Az = Sz.$$

Hence $z = Az = Sz = Bz = Tz$. Therefore, z is a common fixed point of S, T, A and B.

Case 4. Suppose that T is continuous, we can obtain a similar result by way of Case 3.

Uniqueness: Suppose $z \neq w$ be two common fixed points of S, T, A and B.

On putting $x = z$ and $y = w$ in (C2), we have

$$d^3(Sz,Tw) \leqslant \psi\{0,0,0,0\} - \emptyset(m(Az,Bw)).$$

On solving we have $d^2(z,w) = 0$. This implies $z = w$.

This completes the proof. □

Pointwise R-Weakly Commuting and Reciprocal Continuous Mappings

Now, we prove a common fixed point theorem using the notion of point wise R-weakly commuting mappings along with the notion of compatible and reciprocal continuous mappings.

Theorem 3.2. *Let S,T, A and B be four mappings of a complete metric space (X,d) into itself satisfying (C1), (C2) and the following conditions:*

6. (A,S) and (B,T) are point wise R-weakly commuting pairs,

7. *(A,S) and (B,T) are compatible pairs of reciprocally continuous mappings.*

Then S,T,A and B have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. From (C1), we can find x_1 such that $S(x_0) = B(x_1) = y_0$. For this, x_1 one can find $x_2 \in X$ such that $T(x_1) = A(x_2) = y_1$. Continuing in this way, one can construct a sequence $\{y_n\}$ such that

$$y_{2n} = S(x_{2n}) = B(x_{2n+1}),\ y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2}), \quad \text{for each } n \geqslant 0.$$

From [9], we can easily prove that $\{y_n\}$ is a Cauchy sequence in X.

From the completeness of X, there exists a $z \in X$ such that $y_n \to z$ as $n \to \infty$.

Moreover, since

$$y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2}) \quad \text{and} \quad y_{2n} = S(x_{2n}) = B(x_{2n+1})$$

are subsequences of $\{y_n\}$, we obtain

$$\lim_{n\to\infty} T(x_{2n+1}) = \lim_{n\to\infty} A(x_{2n+2}) = \lim_{n\to\infty} S(x_{2n}) = \lim_{n\to\infty} B(x_{2n+1}) = z.$$

If B and T are compatible, then

$$\lim_{n\to\infty} d(BTx_n, TBx_n) = 0$$

that is, $Bz = Tz$. Also by the reciprocal continuity of B and T, we have

$$\lim_{n\to\infty} BTx_{2n} = Bz \quad \text{and} \quad \lim_{n\to\infty} TBx_{2n} = Tz.$$

Since $T(X) \subset A(X)$, there exists a point w in X such that $Tz = Aw$.

Setting $x = w$ and $y = z$ in (C2), we get

$$\begin{aligned} d^3(Sw,Tz) \leqslant \psi\{&d^2(Aw,Sw)d(Bz,Tz), d(Aw,Sw)d^2(Bz,Tz),\\ &d(Aw,Sw)d(Aw,Tz)d(Bz,Sw), d(Aw,Tz)d(Bz,Sw)d(Bz,Tz)\}\\ &-\emptyset(m(Aw,Bz)),\end{aligned}$$

where

$$\begin{aligned} m(Aw,Bz) = \max\Big\{&d^2(Aw,Bz), d(Aw,Sw)d(Bz,Tz), d(Aw,Tz)d(Bz,Sw),\\ &\frac{1}{2}[d(Aw,Sw)d(Aw,Tz) + d(Bz,Sw)d(Bz,Tz)]\Big\} = 0.\end{aligned}$$

This implies that

$$\begin{aligned} d^3(Sw,Tz) \leqslant \psi\{&d^2(Tz,Sw)d(Tz,Tz) + d(Tz,Sw)d^2(Tz,Tz),\\ &d(Tz,Sw)d(Tz,Tz)d(Tz,Sw), d(Tz,Tz)d(Tz,Sw)d(Tz,Tz)\} - \emptyset(0),\end{aligned}$$

i.e.,

$$d^3(Sw,Tz) \leqslant \psi\{0,0,0,0\} - \emptyset(0),$$

using the properties of ψ and $\emptyset$, we have $Sw = Tz$, and hence $Sw = Tz = Aw = Bz$.

The point wise R-weak commutativity of B and T implies that there exists an $R > 0$ such that $d(BTz, TBz) \leqslant Rd(Bz, Tz)$, which implies that $BTz = TBz$ and $TTz = TBz = BTz = BBz$.

Similarly, the point wise R-weak commutativity of A and S implies that there exists an $R > 0$ such that $d(ASw, SAw) \leqslant Rd(Aw, Sw)$, which implies that $ASw = SAw$ and $AAw = ASw = SAw = SSw$.

Again substituting $x = w$ and $y = Tz$ in (C2), we get

$$\begin{aligned} d^3(Sw, TTz) \leqslant \psi\{&d^2(Aw, Sw)d(BTz, TTz), d(Aw, Sw)d^2(BTz, TTz), \\ &d(Aw, Sw)d(Aw, TTz)d(BTz, Sw), d(Aw, TTz)d(BTz, Sw)d(BTz, TTz)\} \\ &- \emptyset(m(Aw, BTz)), \end{aligned}$$

where

$$\begin{aligned} m(Aw, BTz) = \max\Big\{&d^2(Aw, BTz), d(Aw, Sw)d(BTz, TTz), d(Aw, TTz)d(BTz, Sw), \\ &\frac{1}{2}\Big[d(Aw, Sw)d(Aw, TTz) + d(BTz, Sw)d(BTz, TTz)\Big]\Big\}. \end{aligned}$$

On simplification and using the properties of ψ and $\emptyset$, we have

$$d^3(Tz, TTz) \leqslant \psi\{0, 0, 0, 0\} - \emptyset(d^2(Tz, TTz)).$$

Hence $Tz = TTz$. Thus $Tz = TTz = BTz$.

Therefore, Tz is a common fixed point of B and T.

Taking $x = Sw$ and $y = z$ in (C2), we get

$$\begin{aligned} d^3(SSw, Tz) \leqslant \psi\{&d^2(ASw, SSw)d(Bz, Tz), d(ASw, SSw)d^2(Bz, Tz), \\ &d(ASw, SSw)d(ASw, Tz)d(Bz, SSw), d(ASw, Tz)d(Bz, SSw)d(Bz, Tz)\} \\ &- \emptyset(m(ASw, Bz)), \end{aligned}$$

where

$$\begin{aligned} m(ASw, Bz) = \max\Big\{&d^2(ASw, Bz), d(ASw, SSw)d(Bz, Tz), d(ASw, Tz)d(Bz, SSw), \\ &\frac{1}{2}\Big[d(ASw, SSw)d(ASw, Tz) + d(Bz, SSw)d(Bz, Tz)\Big]\Big\}. \end{aligned}$$

On solving, we have

$$d^3(SSw, Sw) \leqslant \psi\{0, 0, 0, 0\} - \emptyset(d^2(Sw, SSw)).$$

Using the properties of ψ and $\emptyset$, we have

Hence $Sw = SSw$. Thus $Sw = SSw = AAw$,

Thus Sw is a common fixed point of A and S.

If $Sw = Tz = u$, then $Tu = Bu = Su = Au = u$. Hence u is a common fixed point of A, B, S and T.

Uniqueness: Suppose that $v \neq u$ are two common fixed points of S, T, A and B.

On putting $x = u$ and $y = v$ in (C2), we have

$$d^3(Su, Tv) \leqslant \psi\{0, 0, 0\} - \emptyset(m(Au, Bv))$$

i.e.,

$$d^3(u, v) \leqslant \psi\{0, 0, 0\} - \emptyset(d^2(u, v))$$

i.e., $d^2(u, v) = 0$, This implies $u = v$.

This completes the proof. □

R-Weakly Commuting Mappings of Type (P)

In 2009, Kumar et al. [8] defined the concept of R-weakly commuting mappings of type (P) in metric spaces and proved a common fixed point theorem using these mappings.

Now, we prove a common fixed point theorem for pairs of R-weakly commuting mappings of type (P).

Theorem 3.3. *Let S, T, A and B be four mappings of a complete metric space (X, d) into itself satisfying (C1), (C2), 1 and the following conditions:*

8. (A, S) and (B, T) are R-weakly commuting of type (P).

Then S, T, A and B have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. From (C2) we can find an x_1 such that $S(x_0) = B(x_1) = y_0$. For this x_1 one can find an $x_2 \in X$ such that $T(x_1) = A(x_2) = y_1$. Continuing in this way, one can construct a sequence such that

$$y_{2n} = S(x_{2n}) = B(x_{2n+1}),\ y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2}), \quad \text{for each } n \geqslant 0.$$

From [9], we can easily prove that $\{y_n\}$ is a Cauchy sequence in X.

By the completeness of X, there exists a $z \in X$ such that $y_n \to z$ as $n \to \infty$.

Moreover, since

$$y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2}) \quad \text{and} \quad y_{2n} = S(x_{2n}) = B(x_{2n+1})$$

are subsequences of $\{y_n\}$, so these subsequences

$$T(x_{2n+1}) = A(x_{2n+2}) = S(x_{2n}) = B(x_{2n+1})$$

also converges to the same limit as $n \to \infty$.

Case 1: Suppose that A is continuous. Then $\{AAx_{2n}\}$ and $\{ASx_{2n}\}$ converges to Az as $n \to \infty$. Since the mappings A and S are R-weakly commuting of type (P), we have

$$d(SSx_{2n}, AAx_{2n}) \leqslant Rd(Ax_{2n}, Sx_{2n}).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} SAx_{2n} = Az$.

On putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (C2), we get

$$\begin{aligned} d^3(SAx_{2n}, Tx_{2n+1}) \leqslant \psi\{ & d^2(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}), \\ & d(AAx_{2n}, SAx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1}), \\ & d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}), \\ & d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\} \\ & - \emptyset(m(AAx_{2n}, Bx_{2n+1})), \end{aligned}$$

where

$$\begin{aligned} m(AAx_{2n}, Bx_{2n+1}) = \max \Big\{ & d^2(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}), \\ & d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}), \\ & \frac{1}{2}\Big[d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1}) \\ & + d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\Big]\Big\}. \end{aligned}$$

Proceeding limit as $n \to \infty$, we get

$$d^3(Az, z) \leqslant \psi\{0, 0, 0, 0\} - \emptyset(d^2(Az, z)),$$

i.e.,

$$d^3(Az, z) \leqslant -\emptyset(d^2(Az, z)).$$

Using the properties of ψ and $\emptyset$, we have $d^2(Az, z) = 0$, i.e., $Az = z$.

Next, we shall show that $Sz = z$.

For this, putting $x = z$ and $y = x_{2n+1}$ in (C2), we get

$$\begin{aligned} d^3(Sz, Tx_{2n+1}) \leqslant \psi\{ & d^2(Az, Sz)d(z, z), d(Az, Sz)d^2(z, z), \\ & d(Az, Sz)d(Az, z)d(z, Sz), d(Az, z)d(z, Sz)d(z, z)\} - \emptyset(m(Az, z)), \end{aligned}$$

where

$$\begin{aligned} m(Az, z) = \max \Big\{ & d^2(Az, z), d(Az, Sz)d(z, z), d(Az, z)d(z, Sz), \\ & \frac{1}{2}\Big[d(Az, Sz)d(Az, z) + d(z, Sz)d(z, z)\Big]\Big\} = 0. \end{aligned}$$

Therefore,

$$d^3(Sz, z) \leqslant \psi\{0, 0, 0, 0\} - \emptyset(0).$$

Using the properties of ψ and $\emptyset$, we have

Thus, $d^2(Sz, z) = 0$, implies $Sz = z$.

Since $S(X) \subset B(X)$, therefore, there exists a point $u \in X$ such that $z = Sz = Bu$.

We claim that $z = Tu$.

For this, on putting $x = z$ and $y = u$ in (C2), we get

$$\begin{aligned} d^3(Sz, Tu) \leqslant \psi\{&d^2(Az, Sz)d(Bu, Tu), d(Az, Sz)d^2(Bu, Tu)\\ &d(Az, Sz)d(Az, Tu)d(Bu, Sz), d(Az, Tu)d(Bu, Sz)d(Bu, Tu)\}\\ &- \emptyset(m(Az, Bu)), \end{aligned}$$

where

$$\begin{aligned} m(Az, Bu) = \max\Big\{&d^2(Az, Bu), d(Az, Sz)d(Bu, Tu), d(Az, Tu)d(Bu, Sz),\\ &\frac{1}{2}[d(Az, Sz)d(Az, Tu) + d(Bu, Sz)d(Bu, Tu)]\Big\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} d^3(z, Tu) \leqslant \psi\{&d^2(z, z)d(z, Tu), d(z, z)d^2(z, Tu)\\ &d(z, z)d(z, Tu)d(z, z), d(z, Tu)d(z, z)d(z, Tu)\} - \emptyset(0). \end{aligned}$$

Using the properties of ψ and $\emptyset$, we have $z = Tu$. Since (B, T) is R-weakly commuting of type (P), we have $d(Bz, Tz) = d(BBu, TTu) \leqslant Rd(Tu, Bu) = Rd(z, z) = 0$.

Hence $Bz = Tz$.

Finally, we have

$$\begin{aligned} d^3(Sz, Tz) \leqslant \psi\{&d^2(Az, Sz)d(Bz, Tz), d(Az, Sz)d^2(Bz, Tz),\\ &d(Az, Sz)d(Az, Tz)d(Bz, Sz), d(Az, Tz)d(Bz, Sz)d(Bz, Tz)\}\\ &- \emptyset(m(Az, Bz)), \end{aligned}$$

where

$$\begin{aligned} m(Az, Bz) &= \max\Big\{d^2(Az, Bz), d(Az, Sz)d(Bz, Tz), d(Az, Tz)d(Bz, Sz),\\ &\qquad \frac{1}{2}[d(Az, Sz)d(Az, Tz) + d(Bz, Sz)d(Bz, Tz)]\Big\}\\ &= d^2(z, Bz). \end{aligned}$$

On simplification, we have

$$d^2(z, Tz) \leqslant \psi\{0, 0, 0, 0\} - \emptyset(d^2(z, Tz)).$$

Using the properties of ψ and $\emptyset$, we have $z = Tz$. Hence $z = Bz = Tz = Az = Sz$. Therefore, z is a common fixed point of S, T, A and B.

Case 2: Suppose that B is continuous. Then we can obtain the same result by using Case 1.

Case 3: Suppose that S is continuous.

Then $\{SSx_{2n}\}$ and $\{SAx_{2n}\}$ converge to Sz as $n \to \infty$.

Since the mappings A and S are R-weakly commuting of type (P), we have

$$d(AAx_{2n}, SSx_{2n}) \leqslant Rd(Sx_{2n}, Ax_{2?D}).$$

Letting $n \to \infty$, we get $\lim_{n\to\infty} ASx_{2n} = Sz$.

On putting $x = Sx_{2n}$ and $y = x_{2n+1}$ in (C2), we get

$$\begin{aligned} d^3(SSx_{2n}, Tx_{2n+1}) \leqslant \psi\{&d^2(ASx_{2n}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1}),\\ &d(ASx_{2n}, SSx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1}),\\ &d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n}),\\ &d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\}\\ &-\emptyset(m(ASx_{2n}, Bx_{2n+1})), \end{aligned}$$

where

$$\begin{aligned} m(ASx_{2n}, Bx_{2n+1}) = \max\Big\{&d^2(ASx_{2n}, Bx_{2n+1}), d(ASx_{2n}, SSx_{2n})\\ &d(Bx_{2n+1}, Tx_{2n+1}), d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n}),\\ &\frac{1}{2}[d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tx_{2n+1})\\ &+ d(Bx_{2n+1}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1})]\Big\}. \end{aligned}$$

Proceeding limit as $n \to \infty$, we get

$$d^3(Sz, z) \leqslant \psi\{0, 0, 0, 0\} - \emptyset(d^2(Sz, z)),$$

i.e.,

$$d^3(Sz, z) \leqslant -\emptyset(d^2(Sz, z)).$$

Thus we get $d^3(Sz, z) = 0$, which implies that $Sz = z$.

Since $S(X) \subset B(X)$, there exists a point $v \in X$ such that $z = Sz = Bv$.

We claim that $z = Tv$.

For this, putting $x = Sx_{2n}$ and $y = v$ in (C2), we get

$$\begin{aligned} d^2(SSx_{2n}, Tv) \leqslant \psi\Big\{&\frac{1}{2}[d^2(ASx_{2n}, SSx_{2n})d(Bv, Tv) + d(ASx_{2n}, SSx_{2n})d^2(Bv, Tv)],\\ &d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tv)d(Bv, SSx_{2n}),\\ &d(ASx_{2n}, Tv)d(Bv, SSx_{2n})d(Bv, Tv)\Big\} - \emptyset(m(ASx_{2n}, Bv)), \end{aligned}$$

where

$$\begin{aligned} m(ASx_{2n}, Bv) = \max\Big\{&d^2(ASx_{2n}, Bv), d(ASx_{2n}, SSx_{2n})d(Bv, Tv),\\ &d(ASx_{2n}, Tv)d(Bv, SSx_{2n}), \end{aligned}$$

$$\left.\begin{aligned}&\frac{1}{2}[d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tv)\\&+ d(Bv, SSx_{2n})d(Bv, Tv)]\end{aligned}\right\}.$$

On simplification, we get

$$\begin{aligned}d^3(z, Tv) \leqslant \psi\{&d^2(z, z)d(z, Tv), d(z, z)d^2(z, Tv)\\&d(z, z)d(z, Tv)d(z, z), d(z, Tv)d(z, z)d(z, Tv)\} - \emptyset(0).\end{aligned}$$

Using the properties of ψ and $\emptyset$, we have $z = Tv$.

Since (B, T) is R-weakly commuting of type (P), we have

$$d(Tz, Bz) = d(TTv, BBv) \leqslant Rd(Bv, Tv) = Rd(z, z) = 0.$$

This gives $Bz = Tz$, for $R > 0$.

Finally, from (C2) we have

$$\begin{aligned}d^3(Sx_{2n}, Tz) \leqslant \psi\{&d^2(Ax_{2n}, Sx_{2n})d(Bz, Tz), d(Ax_{2n}, Sx_{2n})d^2(Bz, Tz),\\&d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz)d(Bz, Sx_{2n}),\\&d(Ax_{2n}, Tz)d(Bz, Sx_{2n})d(Bz, Tz)\} - \emptyset(m(Ax_{2n}, Bz)),\end{aligned}$$

where

$$\begin{aligned}m(Ax_{2n}, Bz) = \max\Big\{&d^2(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n})d(Bz, Tz), d(Ax_{2n}, Tz)d(Bz, Sx_{2n}),\\&\frac{1}{2}[d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz) + d(Bz, Sx_{2n})d(Bz, Tz)]\}.\end{aligned}$$

Therefore, we have

$$d^3(z, Tz) \leqslant \psi\{0, 0, 0, 0\} - \emptyset(d^2(z, Tz)).$$

Using the properties of ψ and $\emptyset$, we have

$$z = Tz.$$

Since $T(X) \subset A(X)$, therefore, there exists a point $w \in X$ such that $z = Tz = Aw$.

We claim that $z = Sw$.

To establish this, on putting $x = w$ and $y = z$ in (C2), we get

$$\begin{aligned}d^3(Sw, Tz) \leqslant \psi\{&d^2(Aw, Sw)d(Bz, Tz), d(Aw, Sw)d^2(Bz, Tz)\\&d(Aw, Sw)d(Aw, Tz)d(Bz, Sw), d(Aw, Tz)d(Bz, Sw)d(Bz, Tz)\}\\&- \emptyset(m(Aw, Bz)),\end{aligned}$$

where

$$\begin{aligned}m(Aw, Bz) = \max\Big\{&d^2(Aw, Bz), d(Aw, Sw)d(Bz, Tz), d(Aw, Tz)d(Bz, Sw),\\&\frac{1}{2}[d(Aw, Sw)d(Aw, Tz) + d(Bz, Sw)d(Bz, Tz)]\},\end{aligned}$$

i.e.,

$$m(Aw, Bz) = \max\Big\{d^2(z,z), d(z,Sw)d(Tz,Tz), d(z,z)d(z,Sw), \frac{1}{2}[d(z,Sw)d(z,z) + d(z,Sw)d(Tz,Tz)]\Big\} = 0.$$

Hence we get

$$d^3(Sw,z) \leqslant \psi\{d^2(z,Sw)d(z,z), d(z,Sw)d^2(z,z) \; d(z,Sw)d(z,z)d(z,Sw), d(z,z)d(z,Sw)d(z,z)\} - \emptyset(0).$$

Properties of ψ and $\emptyset$, implies that $Sw = z$.

Since (S, A) is R-weakly commuting of type (P), we have

$$d(Az, Sz) = d(AAw, SSw) \leqslant Rd(Sw, Aw) = Rd(z,z) = 0.$$

Hence $Az = Sz$.

Hence $z = Az = Sz = Bz = Tz$, and z is a common fixed point of S, T, A and B.

Case 4: Suppose that T is continuous. We can obtain the same result by using Case 3.

Uniqueness: Suppose that $z \neq w$ are two common fixed points of S, T, A and B.

On putting $x = z$ and $y = w$ in (C2), we get

$$d^3(Sz, Tw) \leqslant \psi\{0,0,0,0\} - \emptyset(m(Az, Bw)),$$

i.e.,

$$d^2(z,w) = 0 \;\text{ implies }\; z = w.$$

This completes the proof. □

***Example* 3.1.** Let $X = [2, 20]$ and d be a usual metric. Define the self mappings A, B, S and T on X by

$$Ax = \begin{cases} 12 & \text{if } 2 < x \leqslant 5 \\ x-3 & \text{if } x > 5 \\ 2 & \text{if } x = 2, \end{cases} \qquad Bx = \begin{cases} 2 & \text{if } x = 2 \\ 6 & \text{if } x > 2, \end{cases}$$

$$Sx = \begin{cases} 6 & \text{if } 2 < x \leqslant 5 \\ x & \text{if } x = 2 \\ 2 & \text{if } x > 5 \end{cases} \quad \text{and} \quad Tx = \begin{cases} x & \text{if } x = 2 \\ 3 & \text{if } x > 2. \end{cases}$$

Let us consider a sequence $\{x_n\}$ with $x_n = 2$. It is easy to verify that all the conditions of Theorem 3.1 are satisfied. In fact, 2 is the unique common fixed point of S, T, A and B.

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