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On Binary α-Open Sets and Binary α-ω-Open Sets in binary topological spaces

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Abstract As a generalization of binary open sets in binary topological, we used the notions of α-open sets in topological spaces to introduce and study the notions of binary α-open sets and binary α-ω-open sets in binary topological spaces. Furthermore, we develop some properties on binary α-ω-compact spaces and binary α-ω-connected spaces. Moreover, we define and discuss the concepts of binary α-continuous functions and binary α-ω-continuous functions.

Key Words Binary topological spaces, binary α-open sets, binary α-ω-open sets, binary α-ω-compact spaces
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1 Introduction and preliminaries

The notion of binary topological space was introduced by Nithyanantha and Thangavelu in 2011 [5], A binary topology from X to Y is a binary structure which satisfies the conditions of the Definition 1.1 which are analogous to the axioms of topology. This concept has been studied by many mathematicians in different fields of the general topology. In 2017, Mustafa [4] introduced the concept of binary generalized closed set in binary topological spaces in which he showed some properties in well known open sets such that semi-open, pre-open, b-open, etc., in the next year, Chacko and Susha [1] used the notion of binary topological space to introduce the concept of binary linear topology in metric spaces. In 2019, Mehmood et.al. [3] used the notions of soft semi open sets to introduce the notions of soft binary topological spaces in which showed and proved some applications and characterizations in separation axioms. On the other hand, Hdeib [2] introduced the concept of ω-closed set as generalized of closed sets. A point $x \in X$ is said to be a condensation point of C, if for each $V \in \tau$ with $x \in V$, the set $V \cap C$ is uncountable. C is said to be ω-closed [2], if it contains all its condensation points. The complement of a ω-closed set is called ω-open. The collection of all ω-open sets of (X, τ) is denoted by τ_ω in which is finer than τ. In this paper, as a generalization of binary open sets in binary topological spaces, we used the notions of α-open

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sets [6] to introduce and study the notions of binary α-open sets and binary α-ω-open sets, as well as, some of their properties and characterizations are shown. Besides, we introduce the notions of binary α-continuous functions, binary α-ω-continuous functions, strongly binary α-ω-continuous functions and perfectly binary α-ω-continuous functions.

Throughout this paper, $P(X)$ and $P(Y)$ are the power sets of X and Y, respectively. Now, we show some definitions which are useful for the developing of this paper.

Definition 1.1. *[5] Let X, Y be any two empty sets and $A \subseteq X$ and $B \subseteq Y$. A binary topology from X to Y is a binary structure $M \subseteq P(X) \times P(Y)$ that satisfies the following conditions:*

1. *$(\emptyset, \emptyset)$ and $(X, Y) \in M$.*
2. *$(A_1 \cap A_2, B_1 \cap B_2) \in M$, for any $(A_1, B_1) \in M$ and $(A_2, B_2) \in M$.*
3. *If $\{(A_\delta, B_\delta) : \delta \in \Delta\}$ is a family of members of M, then $(\bigcup_{\delta \in \Delta} A_\delta, \bigcup_{\delta \in \Delta} B_\delta) \in M$.*

Definition 1.2. *[5] If M is a binary topology from X to Y, then the triplet (X, Y, M) is said to be a binary topological space and the members of M are called binary open sets of (X, Y, M). The elements of $X \times Y$ are said to be the binary points of the binary topological space (X, Y, M).*

Definition 1.3. *[5]Let (X, Y, M) be a binary topological space and let $(x, y) \in (X, Y)$. The binary open set (A, B) is said to be a binary neighbourhood of (x, y) if $x \in A$ and $y \in B$.*

Proposition 1.4. *[5] Let $(A, B) \subseteq (C, D) \subseteq (X, Y)$ and (X, Y, M) be a binary topological space. Then, the following statements hold:*

1. *$Int(A, B) \subseteq (A, B)$.*
2. *If (A, B) is binary open, then $Int(A, B) = (A, B)$.*
3. *$Int(A, B) \subseteq Int(C, D)$.*
4. *$Int(Int(A, B)) = Int(A, B)$.*
5. *$(A, B) \subseteq Cl(A, B)$.*
6. *If (A, B) is binary closed, then $Cl(A, B) = (A, B)$.*
7. *$Cl(A, B) \subseteq Cl(C, D)$.*
8. *$Cl(Cl(A, B)) = Cl(A, B)$.*

Definition 1.5. *[5] Let $f : Z \to X \times Y$ be a function. Let $A \subseteq X$ and $B \subseteq Y$. We define $f^{-1}(A, B) = \{z \in Z : f(z) = (x, y) \in (A, B)\}$.*

Definition 1.6. *[5] Let (X, Y, M) be a binary topological space and let (Z, τ) be a topological space. Now, let $f : (Z, \tau) \to X \times Y$ be a function, then f is said to be binary continuous if $f^{-1}(A, B)$ is open in (Z, τ) for every binary open set (A, B) in $X \times Y$.*

2 Binary α-open sets and binary α-continuous functions

In this section, we used the notions of binary open sets to introduce and study the notions of binary α-open sets.

Definition 2.1. *Let (A, B) be a subset of a binary topological space (X, Y, M). Then, (A, B) is said to be binary α-open if $(A, B) \subseteq Int(Cl(Int(A, B)))$. The complement of a binary α-open set is called binary α-closed.*

Remark 2.2. *The collection of all binary α-open sets and binary α-closed sets are denoted by $B\alpha O(X, Y, M)$ and $B\alpha C(X, Y, M)$, respectively.*

Proposition 2.3. *Every binary open set is binary α-open.*

Proof. Let (A, B) be a binary open set in (X, Y, M), then $(A, B) \subseteq Int(Cl(Int(A, B)))$, since (A, B) is a binary open set, $Int(A, B) = (A, B)$, then $(A, B) \subseteq Int(Cl(A, B))$, now either $Cl(A, B) = (X, Y)$ or $Cl(A, B) = (C, D)$ where $A \subseteq C$ and $B \subseteq D$, therefore (A, B) is binary α-open. □

The converse of the above Proposition need not be true as can be seen in the following example.

Example 2.4. *Let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$ and $M = \{(\emptyset, \emptyset), (\{a\}, \{1\}), (X, Y)\}$. Then, $(\{a, b\}, \{1, 2\})$ is a binary α-open set, but it is not a binary open set.*

Theorem 2.5. *Let (X, τ) and (Y, σ) be two any topological spaces. If A and B are α-open in X and Y, respectively. Then, (A, B) is binary α-open in (X, Y, M).*

Proof. Let $A \neq \emptyset$ and $B \neq \emptyset$ be α-open sets in X and Y, respectively. Suppose that (A, B) is not binary α-open in (X, Y, M), this implies that $(A, B) \nsubseteq Int(Cl(Int(A, B)))$, indeed we can assume that $Int(A, B) = (\emptyset, \emptyset)$. Thus, $Int(A) = \emptyset$ and $Int(B) = \emptyset$, therefore $A \nsubseteq Int(Cl(Int(A)))$ and $B \nsubseteq Int(Cl(Int(B)))$ and hence A and B are not α-open sets and this is a contradiction. Therefore, (A, B) is binary α-open. □

Lemma 2.6. *Let $A \subseteq X$ and $B \subseteq Y$. If (A, B) is binary α-open in (X, Y, M), then A is α-open in (X, τ) and B is α-open in (Y, σ).*

Proof. The proof is followed by the Theorem 2.5. □

Theorem 2.7. *Let (A, B) and (C, D) be a binary open set in a binary topological space (X, Y, M). Then, the following statements hold:*

1. $Cl((A, B) \cup (C, D)) \supseteq Cl(A, B) \cup Cl(C, D)$.
2. $Cl((A, B) \cap (C, D)) \subseteq Cl(A, B) \cap Cl(C, D)$.
3. $Int((A, B) \cup (C, D)) \supseteq Int(A, B) \cup Int(C, D)$.
4. $Int(A, B) \cap Int(C, D) \supseteq Int((A, B) \cap (C, D))$.

Proof. 1. Let $(A, B) \subseteq (A, B) \cup (C, D)$ and $(C, D) \subseteq (A, B) \cup (C, D)$. Then, $Cl(A, B) \subseteq Cl((A, B) \cup (C, D))$ and $Cl(C, D) \subseteq Cl((A, B) \cup (C, D))$ and hence $Cl(A, B) \cup Cl(C, D) \subseteq Cl((A, B) \cup (C, D))$.

2. Let $(A,B)\cap(C,D)\subseteq(A,B)$ and $(A,B)\cap(C,D)\subseteq(C,D)$. Then, $Cl((A,B)\cap(C,D))\subseteq Cl(A,B)$ and $Cl((A,B)\cap(C,D))\subseteq Cl(C,D)$ and hence $Cl((A,B)\cap(C,D))\subseteq Cl(A,B)\cap Cl(C,D)$.

3. Let $(A,B)\subseteq(A,B)\cup(C,D)$ and $(C,D)\subseteq(A,B)\cup(C,D)$. Then, $Int(A,B)\subseteq Int((A,B)\cup(C,D))$ and $Int(C,D)\subseteq Int((A,B)\cup(C,D))$ and hence $Int(A,B)\cup Int(C,D)\subseteq Int((A,B)\cup(C,D))$.

4. Let $(A,B)\cap(C,D)\subseteq(A,B)$ and $(A,B)\cap(C,D)\subseteq(C,D)$. Then, $Int((A,B)\cap(C,D))\subseteq Int(A,B)$ and $Int((A,B)\cap(C,D))\subseteq Int(C,D)$ and hence $Int((A,B)\cap(C,D))\subseteq Int(A,B)\cap Int(C,D)$.

□

The equality in part (1), (2), (3) and (4) in the above Theorem need not be true as can be seen in the following example.

Example 2.8. *Let $X=\{1,2,3\}$, $Y=\{4,5\}$ and $M=\{(\emptyset,\emptyset),(X,Y),(\{1\},\{5\}),(\{2\},Y),(\{1,2\},Y)\}$. Then, $Cl(\{1\},\emptyset)\cup Cl(\{2\},\emptyset)=(\{1,3\},\emptyset)\cup(\{2,3\},\{4\})=(X,\{4\})$, but $Cl((\{1\},\emptyset)\cup(\{2\},\emptyset))=(X,Y)$. Besides, $Cl((\{1,2\},\{4\})\cap(\{3\},\{5\}))=(\{3\},\emptyset)$, but $Cl(\{1,2\},\{4\})\cap Cl(\{3\},\{5\})=(X,Y)$.*
Now, $Int(\{1\},\{4\})\cup Int(\{2\},\{5\})=(\emptyset,\emptyset)$, but $Int((\{1\},\{4\})\cup(\{2\},\{4\}))=(\{1,2\},Y)$. Besides, $Int((\{1\},\{3\})\cap(\{2\},Y))=(\emptyset,\emptyset)$, but $Int(\{1\},\{3\})\cap Int(\{2\},Y)=(\emptyset,\{2\})$.

Theorem 2.9. *Arbitrary union of binary α-open sets is binary α-open.*

Proof. Let $\{(A,B)_\delta:\delta\in\Delta\}$ be a collection of family of binary α-open sets of (X,Y,M), then $(A,B)_\delta\subseteq Int(Cl(Int((A,B_\delta))))$. Now, let $\bigcup_{\delta\in\Delta}(A,B)_\delta\subseteq\bigcup_{\delta\in\Delta}Int(Cl(Int((A,B)_\delta))))$, by the Theorem 2.7 parts (1) and (3), we have that $\bigcup_{\delta\in\Delta}(A,B)_\delta\subseteq Int(Cl(Int(\bigcup_{\delta\in\Delta}(A,B)_\delta))))$. Therefore, $\bigcup_{\delta\in\Delta}(A,B)_\delta$ is a binary α-open set. □

The following example shows that arbitrary intersection of binary α-open sets need not be a binary α-open set.

Example 2.10. *Let $X=\{a,b,c\}$, $Y=\{1,2,3\}$ and $M=\{(\emptyset,\emptyset),(\{a\},\{1\}),(\{b\},\{2\}),(\{a,b\},\{1,2\}),(X,Y)\}$. Then, $(\{b,c\},\{2,3\})$ and $(\{a,c\},\{1,3\})$ are binary α-open sets, but $(\{b,c\},\{2,3\})\cap(\{a,c\},\{1,3\})=(\{c\},\{3\})$ is not a binary α-open set.*

Definition 2.11. *Let (X,Y,M) be a binary topological space and let (Z,τ) be a topological space. Now, let $f:(Z,\tau)\to X\times Y$ be a function, then f is said to be binary α-continuous if $f^{-1}(A,B)$ is open in (Z,τ) for every binary α-open set (A,B) in $X\times Y$.*

Theorem 2.12. *Every binary continuous function is binary α-continuous.*

Proof. It follows form the fact that every binary open is binary α-open . □

The converse of the above Theorem need not be true as can be seen in the following example.

Example 2.13. *Let $X=\{a,b,c\}$, $Y=\{0,1\}$, $M=\{(\emptyset,\emptyset),(\{a\},\{1\}),(\{b\},\{0,1\}),(X,Y)\}$, $Z=\{q,w,e\}$ and $\tau=\{\emptyset,Z,\{q\},\{w\},\{q,w\}\}$. Define the function $f:(Z,\tau)\to X\times Y$ by $f(q)=(a,1)$, $f(w)=(b,0)$ and $f(e)=(c,1)$. Then, f is a binary α-continuous function, but it is not a binary continuous function, because $f^{-1}(\{b,c\},\{2\})=\{w,e\}$ is not an open set.*

3 Binary α-ω-open sets and binary α-ω-continuous functions

In this section, we used the notions of binary open sets and binary semi-open sets to introduce and study the notions of binary α-ω-open sets.

Definition 3.1. *Let (X, Y, M) be a binary topological space and $(A, B) \subseteq X \times Y$. Then (A, B) is said to be binary α-ω-open if for each $(x, y) \in (A, B)$ there exits a binary α-open set $(V, U)_{(x,y)}$ containing (x, y) such that $(V, U)_{(x,y)} - (A, B)$ is a countable set. The complement of a binary α-ω-open set is called binary α-ω-closed set.*

Remark 3.2. *The collection of all binary α-ω-open sets and binary α-ω-closed sets are denoted by $B\alpha\omega O(X, Y, M)$ and $B\alpha\omega C(X, Y, M)$.*

Lemma 3.3. *Every binary α-open set is binary α-ω-open.*

Proof. The proof is followed by the Definition 3.1. □

The converse of the above Lemma need not be true as can be seen in the following example.

Example 3.4. *Let $X = \{a, b, c, d\}$, $Y = \{q, w, e, r\}$ and $M = \{(\emptyset, \emptyset), (\{a\}, \{w\}), (\{c\}, \{r\}), (X, Y), (\{a, c\}, \{w, r\})\}$. Then, $(\{c, d\}, \{e, r\})$ is a binary α-ω-open set, but it is not a binary α-open set.*

Lemma 3.5. *Let (X, Y, M) and (X_1, Y_1, M) be two binary topological spaces such that $(A, B) \subseteq (X_1, Y_1)$ and $X_1 \subseteq X$ and $Y_1 \subseteq Y$. If (A, B) is a binary α-ω-open set of (X, Y, M), then (A, B) is a binary α-ω-open set of (X_1, Y_1, M).*

Proof. Let (A, B) be a binary α-ω-open set of (X, Y, M). Then, for every $(x, y) \in (A, B)$, there exits a α-ω-open set (U, V) of (X, Y, M) containing (x, y) such that $(U, V) - (A, B)$ is a countable. In consequence, we have that (U, V) is a α-ω-open set of (X_1, Y_1, M) containing (x, y). This proofs that (A, B) is a α-ω-open set of (X_1, Y_1, M). □

Theorem 3.6. *Let (X, Y, M) be a binary topological space and $(A, B) \subseteq (X, Y)$. Then (A, B) is said to be binary α-ω-open if and only if for every $(x, y) \in (A, B)$, there exits a binary α-open set $(U, V)_{(x,y)}$ containing (x, y) and a countable subset (A, B) such that $(U, V)_{(x,y)} - (C, D) \subseteq (A, B)$.*

Proof. **Necessary**: Let (A, B) be a binary α-ω-open set and $(x, y) \in (A, B)$, then there exits a binary α-open set $(U, V)_{(x,y)}$ containing (x, y) such that $(U, V)_{(x,y)} - (A, B)$ is countable. Now, let $(C, D) = (U, V)_{(x,y)} - (A, B) = (U, V)_{(x,y)} \cap ((X - A), X - B))$. Then, $(U, V)_{(x,y)} - (C, D) \subseteq (A, B)$.
Sufficiency: Let $(x, y) \in (A, B)$. Then, there exits a binary α-ω-open sets $(U, V)_{(x,y)}$ containing (x, y) and a countable set (C, D) such that $(U, V)_{(x,y)} - (C, D) \subseteq (A, B)$. Therefore, $(U, V)_{(x,y)} - (A, B) \subseteq (C, D)$ and $(U, V)_{(x,y)} - (A, B)$ is countable. □

Definition 3.7. *Let $\{\psi_\delta : \delta \in \Delta\}$ be a collection of binary α-open sets in a binary topological space (X, Y, M) is said to be a binary α-open cover of a subset (A, B) of (X, Y) if $(A, B) \subseteq \bigcup_{\delta \in \Delta} \psi_\delta$.*

Definition 3.8. *Let (X, Y, M) be a binary topological space. Then, (X, Y) is said to be binary-α-Lindeloff, if every binary α-open cover of (X, Y) has a countable sub-cover.*

Theorem 3.9. *Let (X, Y, M) be a binary topological space. Then, the following statements are equivalent:*

1. *(X, Y) is binary-α-Lindeloff.*
2. *Every countable cover of (X, Y) by binary α-open sets has a countable sub-cover.*

Proof. (2) $\Rightarrow$ (1): Since every binary α-open set is binary α-ω-open set, the proof follows.
(1) $\Rightarrow$ (2) : Let $\{\psi_\delta : \delta \in \Delta\}$ be a cover of (X, Y) by binary α-ω-open sets of (X, Y). Now, for each $(x, y) \in (X, Y)$ there exits a $\delta_{(x,y)} \in \Delta$ such that $(x, y) \in \psi_{\delta_{(x,y)}}$. Since $(U, V)_{\delta_{(x,y)}}$ is a binary α-ω-open. Then, there exits a binary α-open set $(N, M)_{\delta_{(x,y)}}$ such that $(x, y) \in (N, M)_{\delta_{(x,y)}}$ and $(N, M)_{\delta_{(x,y)}} - (U, V)_{\delta_{(x,y)}}$ is countable. Then, the family $\{(N, M)_\delta : \delta \in \Delta\}$ is a binary cover of (X, Y) and (X, Y) is binary-α-Lindeloff. Therefore, there exits a countable sub-cover $\delta_{(x,y)_i}$ with $i \in I$ such that $(X, Y) = \bigcup_{i\in I}(N, M)_{\delta_{(x,y)_i}}$. Since $(X, Y) = \bigcup_{i\in I}[(N, M)_{\delta_{(x,y)_i}} - (U, V)_{\delta_{(x,y)_i}}) \cup (U, V)_{\delta_{(x,y)_i}}] = \bigcup_{i\in I}[(N, M)_{\delta_{(x,y)_i}} - (U, V)_{\delta_{(x,y)_i}}) \cup \bigcup_{i\in I}(U, V)_{\delta_{(x,y)_i}}]$. Since $(N, M)_{\delta_{(x,y)_i}} - (U, V)_{\delta_{(x,y)_i}}$ is a countable set, for each $\delta((x, y)_i)$, there exits a countable subset $\Delta_{\delta((x,y)_i)}$ of Δ such that $(N, M)_{\delta_{(x,y)_i}} - (U, V)_{\delta_{(x,y)_i}} \subseteq \bigcup_{\Delta_{\delta((x,y)_i)}} (U, V)_\delta$ and therefore $(X, Y) \subseteq \bigcup_{i\in I}(\bigcup_{\delta\in\Delta_{\delta((x,y)_i)}} (U, V)_\delta) \cup (\bigcup_{i\in I}(U, V)_{\delta_{(x,y)_i}})$. □

Theorem 3.10. *Let (X, Y, M) be a binary topological space and $(C, D) \subseteq (X, Y)$. If (A, B) is a binary α-ω-closed set. Then, $(C, D) \subseteq (J, K) \cup (A, B)$, for some binary α-ω-closed set (J, K) and a countable set (A, B).*

Proof. If (C, D) is a binary α-ω-closed set. Then, $(X - C, X - D)$ is a binary α-ω-open set and hence by Theorem 3.6, for every $(x, y) \in (X - C, X - D)$, there exits a binary α-ω-open set (U, V) containing (x, y) and a countable set (A, B) such that $(U - A, V - B) \subseteq (X - C, X - D)$. Thus, $(C, D) \subseteq ((X - (U - A), (X - (V - B))) = X - ((U, V) \cap ((X - A), (X - B))) = ((X - U), (X - V)) \cup (A, B)$, let $(J, K) = (X - U, X - V)$. Then, (J, K) is a binary α-ω-closed set such that $(C, D) \subseteq (J, K) \cup (A, B)$. □

Theorem 3.11. *The union of any family of binary α-ω-open sets is binary α-ω-open set.*

Proof. Let $\{(A, B)_\delta : \delta \in \Delta\}$ is a collection of binary α-ω-open subsets of (X, Y). Then, for every $(x, y) \in \bigcup_{\delta\in\Delta} (A, B)_\delta$, $(x, y) \in (A, B)_\delta$, for some $\delta \in \Delta$. Hence, there exits a binary α-ω-open subset (U, V) containing (x, y), such that $(U - A, V - B)_\delta$ is countable. Now, as $((U - (\bigcup_{\delta\in\Delta} A_\delta), (V - (\bigcup_{\delta\in\Delta} B_\delta)) \subseteq (U - A, V - B)_\delta$, and thus $((U - (\bigcup_{\delta\in\Delta} A_\delta)), (V - (\bigcup_{\delta\in\Delta} B_\delta)))$ is countable. Therefore, $\bigcup_{\delta\in\Delta} (A, B)_\delta$ is a binary α-ω-open set. □

Definition 3.12. *The union of all binary α-ω-open sets contained in $(A, B) \subseteq (X, Y)$ is called binary α-ω-interior of (A, B) and is denoted by $Int_{\alpha\omega}(A, B)$.*

Definition 3.13. *The intersection of all binary α-ω-closed sets of (X, Y) containing (A, B) is called binary α-ω-closure of (A, B) and is denoted by $Cl_{\alpha\omega}(A, B)$.*

The $Int_{\alpha\omega}(A, B)$ is a binary α-ω-open set and the $Cl_{\alpha\omega}(A, B)$ is a binary α-ω-closed set.

Theorem 3.14. *Let (X, Y, M) be a binary topological space and $(A, B), (C, D) \subseteq (X, Y)$. Then, the following statements hold:*

1. $Int_{\alpha\omega}(Int_{\alpha\omega}(A, B)) = Int_{\alpha\omega}(A, B)$.
2. *if* $(A, B) \subset (C, D)$*, then* $Int_{\alpha\omega}(A, B) \subset Int_{\alpha\omega}(C, D)$.
3. $Int_{\alpha\omega}((A, B) \cap (C, D)) \subset Int_{\alpha\omega}(A, B) \cap Int_{\alpha\omega}(C, D)$.
4. $Int_{\alpha\omega}(A, B) \cup Int_{\alpha\omega}(C, D) \subset Int_{\alpha\omega}((A, B) \cup (C, D))$.
5. $Int_{\alpha\omega}(A, B)$ *is the largest binary* α-ω*-open subset of* (X, Y). *contained in* (A, B).
6. (A, B) *is binary* α-ω*-open if and only if* $(A, B) = Int_{\alpha\omega}(A, B)$.
7. $Cl_{\alpha\omega}(Cl_{\alpha\omega}(A, B)) = Cl_{\alpha\omega}(A, B)$.
8. *If* $(A, B) \subset (C, D)$*, then* $Cl_{\alpha\omega}(A, B) \subset Cl_{\alpha\omega}(C, D)$.
9. $Cl_{\alpha\omega}(A, B) \cup Cl_{\alpha\omega}(C, D) \subset Cl_{\alpha\omega}((A, B) \cup (C, D))$.
10. $Cl_{\alpha\omega}((A, B) \cap (C, D)) \subset Cl_{\alpha\omega}((A, B)) \cap Cl_{\alpha\omega}(C, D)$.

Proof. (1), (2), (6), (7) and (8) are follow directly from the Definition 3.1. (3), (4) and (5) are follow from part (2) of this Theorem. (9) and (10) are follow by applying part (8) of this Theorem. □

Theorem 3.15. *Let (X, Y, M) be a binary topological space and $(A, B) \subset (X, Y)$. Then, the following statements hold:*

1. $Cl_{\alpha\omega}((X - A), Y - B)) = (X, Y) - Cl_{\alpha\omega}(A, B)$.
2. $Int_{\alpha\omega}((X - A), (Y - B)) = (X, Y) - Int_{\alpha\omega}(A, B)$.

Proof. We will prove (1) and (2), then:

1. Let $(x, y) \in (X, Y) - Cl_{\alpha\omega}(A, B)$. Then, there exits $(U, V) \in B\alpha\omega O(X, Y, M)$ such that $(U, V) \cap (A, B) = \emptyset$ and hence it has $(x, y) \in Int_{\alpha\omega}(A, B)$. This shows that $(X, Y) - Cl_{\alpha\omega}(A, B) \subset Int_{\alpha\omega}((X - A), (X - B))$. Now, take $(x, y) \in Int_{\alpha\omega}((X - A), (X - B))$. Since $Int_{\alpha\omega}((X - A), (X - B)) \cap (A, B) = \emptyset$, it gets that $(x, y) \notin Cl_{\alpha\omega}(A, B)$. In consequence, $Cl_{\alpha\omega}((X - A), (X - B)) = (X, Y) - Int_{s\omega}(A, B)$.
2. Let $(x, y) \in (X, Y) - Int_{\alpha\omega}((X - A), (X - B))$. Since $Int_{\alpha\omega}((X - A), (X - B)) \cap (A, B) = \emptyset$, we have that $(x, y) \notin Cl_{\alpha\omega}(A, B)$ and this implies that $(x, y) \in (X, Y) - Cl_{\alpha\omega}(A, B)$. Now, take $(x, y) \in (X, Y) - Cl_{\alpha\omega}(A, B)$. Then, there exist $(U, V) \in B\alpha\omega O(X, Y, M)$ such that $(U, V) \cap (A, B) = \emptyset$. Therefore, $Int_{\alpha\omega}((X - A), (X - B)) = (X, Y) - Cl_{\alpha\omega}(A, B)$.

□

Definition 3.16. *Let (X, Y, M) be a binary topological space and $(A, B) \subseteq (X, Y)$. Then (A, B) is said to be binary α-ω-neighbourhood of a point $(x, y) \in (X, Y)$ if there exists a binary α-ω-open set (J, K) such that $(x, y) \in (J, K) \subset (A, B)$.*

Theorem 3.17. *Let $(X, Y, M))$ be a binary topological space and $(A, B) \subseteq (X, Y)$. Then, (A, B) is binary α-ω-open set if and only if it is a binary α-ω-neighbourhood of each of its points.*

Proof. **Necessary**: Let (A, B) be a binary α-ω-open set of (X, Y). Then by the Definition 3.16 (A, B) is a binary α-ω-neighbourhood of each of its points.
Sufficiency: If (A, B) is a binary α-ω-neighbourhood of each of its points. Then, for each $(x, y) \in (A, B)$, there exits $(C, D)_{(x,y)} \in B\alpha\omega O(X, Y, M)$ such that $(C, D)_{(x,y)} \subset (A, B)$. In consequence, $(A, B) = \bigcup\{(C, D)_{(x,y)} : (x, y) \in (A, B)\}$. Since, each $(C, D)_{(x,y)}$ is a binary α-ω-open and arbitrary union of binary α-ω-open sets is a binary α-ω-open set. therefore, (A, B) is a binary α-ω-open set of (X, Y). □

Definition 3.18. *Let (X, Y, M) be a binary topological space, then (X, Y, M) is said to be binary α-ω-compact if every cover of (X, Y) by binary α-ω-open sets has a finite subcover.*

Theorem 3.19. *Let (X, Y, M) be a binary topological space, then (X, Y, M) is binary α-ω-compact if and only if for every collection $\{(A, B)_\alpha : \alpha \in \Delta\}$ of binary α-ω-closed sets in (X, Y, M) satisfying $\bigcap\{(A, B)_\alpha : \alpha \in \Delta\} = \emptyset$, there is a finite subcollection $(A, B)_{\alpha_1}, (A, B)_{\alpha_2}, \ldots, (A, B)_{\alpha_n}$ with $\bigcap\{(A, B)_{\alpha_k} : k = 1, \ldots, n\} = \emptyset$.*

Proof. Let $\{(A, B)_\alpha : \alpha \in \Delta\}$ be a collection of binary α-ω-closed sets such that $\bigcap\{(A, B)_\alpha : \alpha \in \Delta\} = \emptyset$, then $\{(X - A, Y - B)_\alpha : \alpha \in \Delta\}$ is a collection of binary α-ω-open sets such that

$$(X, Y) = (X, Y) - \emptyset = (X, Y) - \bigcap\{(A, B)_\alpha : \alpha \in \Delta\} = \bigcup\{(X - A, Y - B)_\alpha : \alpha \in \Delta\},$$

that is, $\{(X - A, Y - B)_\alpha : \alpha \in \Delta\}$ is a cover of (X, Y) by binary α-ω-open sets. Since (X, Y, M) is binary α-ω-compact, there exists a finite subcollection $(X - A, Y - B)_{\alpha_1}, (X - A, Y - B)_{\alpha_2}, \ldots, (X - A, Y - B)_{\alpha_n}$ such that

$$(X, Y) = \bigcup\{(X - A, Y - B)_{\alpha_k} : k = 1, \ldots, n\} = (X, Y) - \bigcap\{(A, B)_{\alpha_k} : k = 1, \ldots, n\}.$$

This shows that $\bigcap\{(A, B)_{\alpha_k} : k = 1, \ldots, n\} = \emptyset$. Conversely, suppose that $\{(U, V)_\alpha : \alpha \in \Delta\}$ is a cover of (X, Y) by binary α-ω-open sets, then $\{(X - U, Y - V)_\alpha : \alpha \in \Delta\}$ is a collection of binary α-ω-closed sets such that $\bigcap\{(X - U, Y - V)_\alpha : \alpha \in \Delta\} = (X, Y) - \bigcup\{(U, V)_\alpha : \alpha \in \Delta\} = (X, Y) - (X, Y) = \emptyset$. By hypothesis, there exists a finite subcollection $(X-U, Y-V)_{\alpha_1}, (X-U, Y-V)_{\alpha_2}, \ldots, (X-U, Y-V)_{\alpha_n}$ such that $\bigcap\{(X-U, Y-V)_{\alpha_k} : k = 1, \ldots, n\} = \emptyset$. Follows $(X, Y) = (X, Y)-\emptyset = (X, Y)-\bigcap\{(X-U, Y-V)_{\alpha_k} : k = 1, \ldots, n\} = (X, Y) - ((X, Y) - \bigcup\{(U, V)_{\alpha_k} : k = 1, \ldots, n\}) = \bigcup\{(U, V)_{\alpha_k} : k = 1, \ldots, n\}$. This shows that (X, Y, M) is binary α-ω-compact. □

Definition 3.20. *Let (X, Y, M) be a binary topological space, then (X, Y, M) is said to be binary α-connected if (X, Y) cannot be written as a disjoint union of two non-empty binary α-open sets.*

Definition 3.21. *Let (X, Y, M) be a binary topological space, then (X, Y, M) is said to be binary α-ω-connected if (X, Y) cannot be written as a disjoint union of two non-empty binary α-ω-open sets.*

Theorem 3.22. *Let (X, Y, M) be a binary topological space. If (X, Y, M) is binary α-ω-connected, then (X, Y, M) is binary α-connected.*

Proof. Let (X, Y, M) be binary α-ω-connected. Now, Suppose that (X, Y, M) is not binary α-connected, then there exist non-empty binary α-open sets (A, B) and (C, D) such that $(A, B) \cap (C, D) = \emptyset$ and $(A, B) \cup (C, D) = (X, Y)$. Then, by the Proposition 3.3, we have that (A, B) and (C, D) are binary α-ω-open sets and so, (X, Y, M) is not binary α-ω-connected and this is a contradiction, therefore (X, Y, M) is binary α-connected. □

Theorem 3.23. *For a binary topological space (X, Y, M), the following statements are equivalent:*

1. *(X, Y, M) is binary α-ω-connected.*

2. *$(\emptyset, \emptyset)$ and (X, Y) are the only subsets of (X, Y) which are both binary α-ω-open and binary α-ω-closed.*

Proof. (1)⇒(2) Let (V, U) be a subset of (X, Y) which is both binary α-ω-open and binary α-ω-closed, then $(X - V, X - U)$ is both binary α-ω-open and binary α-ω-closed, so $(X, Y) = (V, U) \cup (X - V, X - U)$. Since (X, Y, M) is binary α-ω-connected, then one of those sets is $(\emptyset, \emptyset)$. Therefore, $(V, U) = (\emptyset, \emptyset)$ or $(V, U) = (X, Y)$.
(2)⇒(1) Suppose that (X, Y, M) is not binary α-ω-connected and let $(X, Y) = (U, N) \cup (V, M)$, where (U, N) and (V, M) are disjoint non-empty binary α-ω-open sets in (X, Y, M), then $(U, N) = (X, Y) - (V, M)$ is both binary α-ω-open and binary α-ω-closed. By hypothesis, $(U, N) = (\emptyset, \emptyset)$ or $(U, N) = (X, Y)$, which is a contradiction. Therefore, $(X, Y, M))$ is binary α-ω-connected.

□

Now, in this part, we define the concepts of binary α-ω-continuous functions. Moreover, we prove some of their properties.

Definition 3.24. *Let (X, Y, M) be a binary topological space and let (Z, τ) be a topological space. Now, let $f : (Z, \tau) \to X \times Y$ be a function, then f is said to be binary α-ω-continuous if $f^{-1}(A, B)$ is open in (Z, τ) for every binary α-ω-open set (A, B) in $X \times Y$.*

Theorem 3.25. *Every binary α-continuous function is binary α-ω-continuous.*

Proof. It follows form the fact that every binary α-open is binary α-ω-open . □

The converse of the above Theorem need not be true as can be seen in the following example.

Example 3.26. *Let $X = \{a, b, c\}$, $Y = \{0, 1\}$, $M = \{(\emptyset, \emptyset), (\{a\}, \{1\}), (\{b\}, \{0, 1\}), (X, Y)\}$, $Z = \{q, w, e\}$ and $\tau = \{\emptyset, Z, \{q\}, \{w\}, \{q, w\}\}$. Define the function $f : (Z, \tau) \to X \times Y$ by $f(q) = (a, 1)$, $f(w) = (b, 0)$ and $f(e) = (c, 1)$. Then, f is a binary α-ω-continuous function, but it is not a binary α-continuous function, because $f^{-1}(\{a, c\}, \{1\}) = \{q, e\}$ is not a α-open set.*

Theorem 3.27. *For a function $f : (Z, \tau) \to X \times Y$, the following statements are equivalent:*

1. *f is binary α-ω-continuous.*
2. *$f^{-1}(A, B)$ is a closed set in (Z, τ) for each binary α-ω-closed set (A, B) in $X \times Y$.*
3. *For each $(x, y) \in (X, Y)$ and each binary α-ω-open set (V, U) in $X \times Y$ containing $(f(x), f(y))$ there exists an open set (N, M) in (Z, τ) containing (x, y) such that $f(N, M) \subset (V, U)$.*

Proof. (1)$\Rightarrow$(2) Let (A, B) be any binary α-ω-closed set in $X \times Y$, then $(V, U) = (X, Y) \setminus (A, B)$ is a binary α-ω-open set in $X \times Y$ and since f is binary α-ω-continuous, $f^{-1}((V, U))$ is an open subset in (Z, τ), but $f^{-1}((V, U)) = f^{-1}((X, Y) \setminus (A, B)) = f^{-1}((X, Y)) \setminus f^{-1}((A, B)) = (X, Y) \setminus f^{-1}((A, B))$ and hence, $f^{-1}((A, B))$ is a closed set in (Z, τ).
(2) $\Rightarrow$ (1) Let (V, U) be any binary α-ω-open set in $X \times Y$, then $(A, B) = (X, Y) \setminus (V, U)$ is a binary α-ω-closed set in $X \times Y$. By hypothesis, we have $f^{-1}((A, B))$ is a closed set in (Z, τ), but $f^{-1}((A, B)) = f^{-1}((X, Y) \setminus (V, U)) = f^{-1}((X, Y)) \setminus f^{-1}((V, U)) = (X, Y) \setminus f^{-1}((V, U))$ and so, $f^{-1}((V, U))$ is an open set in (Z, τ). This shows that f is binary α-ω-continuous.
(1) $\Rightarrow$ (3) Let $(x, y) \in (X, Y)$ and (V, U) any binary α-ω-open set in $X \times Y$ such that $(f(x), f(y)) \in (V, U)$, then $(x, y) \in f^{-1}((V, U))$ and since f is a binary α-ω-continuous function, $f^{-1}((V, U))$ is an open set in (Z, τ). If $(N, M) = f^{-1}((V, U))$, then (N, M) is an open set in (Z, τ) containing (x, y) such that $f((N, M)) = f(f^{-1}((V, U))) \subset (V, U)$.
(3) $\Rightarrow$ (1) Let (V, U) be any binary α-ω-open set in $X \times Y$ and $(x, y) \in f^{-1}((V, U))$, then $(f(x), f(y)) \in (V, U)$ and by (3) there exists an open set $(N, M)_x$ in (Z, τ) such that $(x, y) \in (N, M)_(x, y)$ and $f((N, M)_{(x,y)}) \subset (V, U)$. Thus, $(x, y) \in (N, M)_{(x,y)} \subset f^{-1}(f(N, M)_{(x,y)})) \subset f^{-1}((V, U))$ and hence $f^{-1}((V, U)) = \bigcup\{(N, M)_{(x,y)} : (x, y) \in f^{-1}((V, U))\}$. Then, we have $f^{-1}((V, U))$ is an open set in (Z, τ) and so f is a binary α-ω-continuous function. □

Proposition 3.28. *Let $f : (Z, \tau) \to X \times Y$ be a binary α-ω-continuous if and only if for each $A \subseteq X$ and $B \subseteq Y$, $f^{-1}(Int_{\alpha\omega}(A, B)) \subseteq Int_{\alpha\omega}(f^{-1}(A, B))$*

Proof. **Necessary**: Let $f : (Z, \tau) \to X \times Y$ be a binary α-ω-continuous and let $A \subseteq X$ and $B \subseteq Y$. Then, $Int_{\alpha\omega}(A, B)$ is a binary α-ω-open set of (X, Y, M) and contained in (A, B). Hence, $f^{-1}(Int_{\alpha\omega}(A, B))$ is a pen set of (Z, τ). Now,

$$Int_{\alpha\omega}(A, B) \subseteq (A, B)$$

$$\Rightarrow f^{-1}(Int_{\alpha\omega}(A, B)) \subseteq f^{-1}(A, B)$$

$$\Rightarrow Int_{\alpha\omega}(f^{-1}(Int_{\alpha\omega}(A, B))) \subseteq Int_{\alpha\omega}(f^{-1}(A, B))$$

$$\Rightarrow f^{-1}(Int_{\alpha\omega}(A, B)) \subseteq Int_{\alpha\omega}(f^{-1}(A, B)).$$

Sufficiency: Suppose that $f^{-1}(Int_{\alpha\omega}(A, B)) \subseteq Int_{\alpha\omega}(f^{-1}(A, B))$ for each $A \subseteq X$ and $B \subseteq Y$. Now, let $(A, B) \in (X, Y, M)$, this implies that $Int_{\alpha\omega}(A, B) = (A, B)$. Hence, $f^{-1}(A, B) = f^{-1}(Int_{\alpha\omega}(A, B)) \subseteq Int_{\alpha\omega}(f^{-1}(A, B))$. Therefore, $Int_{\alpha\omega}(f^{-1}(A, B))$ is an open set of (Z, τ).

□

Definition 3.29. *Let $f : (Z, \tau) \to X \times Y$ be a function, then f is said to be:*

1. *Strongly binary α-ω-continuous if the inverse image of every binary α-ω-closed set in $X \times Y$ is closed set in (Z, τ).*
2. *Perfectly binary α-ω-continuous if the inverse image of every binary α-ω-closed set in $X \times Y$ is both open a closed in (Z, τ).*

Theorem 3.30. *Let $f : (Z, \tau) \to X \times Y$ be strongly binary α-ω-continuous, then f is binary continuous.*

Proof. Let (V, U) be any binary closed set in $X \times Y$, since every binary closed set is binary α-open and it is well known that every binary α-open is binary α-ω-closed set, (V, U) is binary α-ω-closed set in $X \times Y$. Since f is strongly binary α-ω-continuous, $f^{-1}(V, U)$ is closed set in (Z, τ). Therefore, f is binary continuous. □

Theorem 3.31. *Let $f : (Z, \tau) \to X \times Y$ be perfectly binary α-ω-continuous, then f is strongly binary α-ω-continuous.*

Proof. Let (V, U) be any binary α-ω-closed set in $X \times Y$. Since f is perfectly binary α-ω-continuous, $f^{-1}(V, U)$ is closed in (Z, τ). Therefore f is strongly binary α-ω-continuous. □

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