South Asian Journal of Mathematics 2020, Vol. 10 (2): 77 ∼ 89
www.sajm-online.com ISSN 2251-1512 RESEARCH ARTICLE

Eisenhart Problem to Almost η-Ricci Solitons on f-Kenmotsu manifolds

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Received: June-2-2020; Accepted: Dec-10-2020 *Corresponding author

Abstract We apply the Eisenhart problem to discuss the f-kenmotsu manifolds conceding an almost η-Ricci soliton and almost Ricci soliton with respect to a semi-symmetric non metric connection. It is shown that Eisenhart problem of adopting parallel tensors is solved for the symmetric case in the regular f-Kenmotsu manifolds with a semi-symmetric non metric connection.

Key Words Almost η-Ricci Solitons, f-kenmotsu manifolds, Einstein manifold

MSC 2010 53C15, 53C20

1 Introduction

Eisenhart Problem. In 1923, Eisenhart [11] proposed that if a positive definite Riemannian manifold conceding a second order parallel symmetric covaraint tensor other than a constant multiple of the metric tensor, then it is reducible.

In [15], Levy demonstrated that a second order parallel symmetric non-degenerated tensor h in a space form is proportional to the metric tensor.

On the other hand, Hamilton [12] and Perelman [18] examine the solution of the Poincare conjecture in dimension 3 have produced a flourishing activity in the research of self similar solutions, or solitons, of the Ricci flow. The study of the geometry of solitons, in particular their classification in dimension 3, has been essential in providing a positive answer to the conjecture; however, in higher dimension and in the complete, possibly non-compact case, the understanding of the geometry and the classification of solitons seems to remain a desired goal for a not too proximate future. In the generic case a soliton structure on the Riemannian manifold (M, g) is the choice of a smooth vector field X on M and a real constant λ satisfying the structural requirement

$$S + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

where S is the Ricci tensor of the metric g and $\mathcal{L}_X g$ is the Lie derivative in the direction of vector field X. In what follows we shall refer to λ as to the soliton constant. The soliton is called expanding, steady or shrinking if, respectively, $\lambda > 0$, $\lambda = 0$ or $\lambda > 0$.

Citation: Mohd Danish Siddiqi, Mobin Ahmad, Eisenhart Problem to Almost η-Ricci Solitons on f-Kenmotsu manifolds, South Asian J Math, 2020, 10(2), 77-89.

In 1924, Friedmann and Schouten [10] proposed the idea of a semi-symmetric linear connection. A linear connection ∇ is said to be a semi-symmetric connection if its torsion tensor T is of the type

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1.2}$$

where η is a 1-form.

The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise, it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. Some properties of semi-symmetric non-metric connection were studied by Ahmad et al. and Siddiqi et al. in ([1, 2, 22]) respectively.

On the other hand, Sharma [21] initiated the study of this Eisenhart problem in terms of Ricci solitons in contact Riemannian geometry . After that, many authors extensively discussed about this concept (for more details see [3, 5, 16, 20, 23]).

In 2009, Cho and Kimura established the conception of η-Ricci soliton [7]. Calin and Crasmareanu have discussed the η-Ricci soliton on Hopf hypersurfaces in complex space forms [8].

A Riemannian manifold (M, g) is called a η-Ricci soliton if there exist a smooth vector field ξ such that the Ricci tensor satisfies the following equation [7]

$$2S + \mathcal{L}_\xi g + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.3}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor and λ, μ are real constants. If $\mu = 0$, then η-Ricci soliton becomes Ricci soliton.

The concept of an almost Ricci soliton was first introduced by Pigola et al., in 2010 [19]. A Riemannian manifold (M^n, g) is an almost η-Ricci soliton if λ and μ are consider the smooth soliton functions on M.

η-Ricci solitons in para-Kenmotsu manifolds [3] and Lorentzian para-Sasakian manifolds [5] have been studied by Blaga et al. In [9] Calin and Crasmareanu evaluate Eisenhart problem in terms of Ricci soliton on f-Kenmotsu manifolds. Moreover, in [26] Yildiz et al. examined 3-dimensional f-Kenmotsu with Ricci-soliton. In [6] Chakraborty et al. investigated the Ricci soliton on 3-dimensional β-Kenmotsu manifold with respect to Schouten-van Kampen connection. Recently, Siddiqi also discussed some axioms of η-Ricci solitons with certain connections which is closely related to this paper [24, 25]. Motivated by above these studies in the present paper, we solve the Eisenhart problem to almost η-Ricci solitons in f-Kenmotsu manifold with a semi-symmetric non-metric connection.

2 Preliminaries

Let M be a 3-dimensional differentiable manifold with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and Riemannian metric g such that

$$\phi^2 = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad , \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{2.2}$$

for all $X, Y \in \chi(M)$. Also, for an almost contact manifold M, it follows that [14]

$$\nabla_X \phi Y = (\nabla_X \phi)Y + \phi(\nabla_X Y), \tag{2.3}$$

$$(\nabla_X \eta)Y = \nabla_X \eta(Y) - \eta(\nabla_X Y). \tag{2.4}$$

Let R be Riemannian curvature tensor, S Ricci curvature tensor, Q Ricci operator and $\{e_1, e_2, ...e_n\}$ be orthonormal basis of M. For all $X, Y \in \chi(M)$ it follow that

$$S(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i), \tag{2.5}$$

$$QX = -\sum_{i=1}^{n} (R(e_i, X)e_i \tag{2.6}$$

and

$$S(X, Y) = g(QX, Y). \tag{2.7}$$

If the Ricci tensor S of a f-kenmotsu manifold M satisfies the condition

$$S(X, Y) = ag(X, X)Y + b\eta(X)\eta(Y), \tag{2.8}$$

where a, b are scalars, then M is said to be η-*Einstein* manifold. If $b = 0$, then M is called *Einstein* manifold. In a 3-dimensional Riemannian manifold the curvature tensor R is defined as

$$R(X, Y)Z = S(Y, Z)X - g(X, Z)QY + g(Y, Z)QX - S(X, Z)Y \tag{2.9}$$

$$-\frac{r}{2}[g(Y, Z)X - g(X, Z)Y]$$

where S is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature for 3-dimensional manifold M.

On the other hand, let M be an n-dimensional Riemannian manifold with the Riemannian connection ∇. A linear connection $\bar{\nabla}$ on M is said to be a semi-symmetric metric connection if its torsion tensor $\bar{T}$ of the connection $\bar{\nabla}$ satisfies

$$\bar{T}(X, Y) = \eta(Y)X - \eta(X)Y \tag{2.10}$$

where η is non-zero 1-form and $T \neq 0$.

Moreover, $\bar{\nabla}g = 0$ then the connection is called a semi-symmetric metric connection. If $\bar{\nabla}g \neq 0$ then the connection is called a semi-symmetric non-metric connection [10].

3 f-Kenmotsu manifolds

The concept of f-Kenmotsu manifold, where f is a real constant, appears for the first time in the paper of Jannsens and Vanhecke [13]. More recently, Olszak and Rosca [17] defined and studied the f-Kenmotsu manifold by the following formula (3.1), where f is a function on M such that $df \wedge \eta = 0$. Here, η is the dual 1-form corresponding to the characteristic vector field ξ of an almost contact metric structure on M. The condition $df \wedge \eta = 0$ follows in fact following from (3.1) if $dimM \geqslant 5$. This does not hold in general if $dimM = 3$.

Let M be a 3-dimensional almost contact manifold. (M, ϕ, ξ, η, g) is an f-Kenmotsu manifold if the covariant differentiation of ϕ satisfies [13],

$$(\nabla_X \phi)Y = f(g(\phi X, Y) - \eta(Y)\phi X) \tag{3.1}$$

where $f \in C^{\infty}(M)$ such that $df \wedge \eta = 0$. If $f = \beta = constant \neq 0$, the manifold is said to be an β-Kenmotsu. If $f = 1$, then 1-Kenmotsu manifold is also called Kenmotsu manifold. If $f^2 + f^{'} \neq 0$, then f-Kenmotsu manifold is said to be regular, where $f^{'} = \xi f$ [13]. By using (2.1) and (2.2), it can be shown that

$$(\nabla_X \eta)Y = fg(\phi X, \phi Y). \tag{3.2}$$

From (3.1), we have

$$\nabla_X \xi = f(X - \eta(X)\xi). \tag{3.3}$$

Also from (2.8), in 3-dimensional f-Kenmotsu we have

$$R(X,Y)Z = (\frac{r}{2} + 2f^2 + 2f^{'})(X \wedge Y) \tag{3.4}$$

$$-(\frac{r}{2} + 3f^2 + 3f^{'})[\eta(X)(\xi \wedge Y)Z + \eta(Y)(X \wedge \xi)Z]$$

and

$$S(X,Y) = (\frac{r}{2} + 2f^2 + 2f^{'})g(X,Y) - (\frac{r}{2} + 3f^2 + 3f^{'})\eta(X)\eta(Y). \tag{3.5}$$

Thus from (3.5), we get

$$S(X,\xi) = -2(f^2 + f^{'})\eta(X), \tag{3.6}$$

where r is the scalar curvature of M and $f^{'}\xi$.
Using (3.4) and (3.5), we obtain

$$R(X,Y)\xi = -(f^2 + f^{'})[\eta(Y)X - \eta(X)Y], \tag{3.7}$$

$$R(\xi,X)\xi = -(f^2 + f^{'})[\eta(X)\xi - X], \tag{3.8}$$

$$QX = (\frac{r}{2} + 2f^2 + 2f^{'})X - (\frac{r}{2} + 3f^2 + 3f^{'})\eta(X)\xi. \tag{3.9}$$

4 f-Kenmotsu manifolds with a semi-symmetric non-metric connection

Let $\bar{\nabla}$ be a linear connection and ∇ be a Riemannian connection of an f-Kenmotsu manifold M. This $\bar{\nabla}$ linear connection defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X \tag{4.1}$$

where η-1-form and any vector fields $X, Y \in \chi(M)$, denotes the semi-symmetric non-metric connection [10].

For f-Kenmotsu manifold with the semi-symmetric non-metric connection, using (2.2), (3.1) and (4.1) we have

$$\bar{\nabla}_X \phi)Y = f(g(\phi X, \phi Y)\xi - 2\eta(X)\phi X) \tag{4.2}$$

for any vector fields $X, Y \in \chi(M)$, where ϕ is $(1,1)$ tensor filed, ξ is a vector filed, η is a 1-form and f^{∞} such that $df \wedge \eta = 0$. As consequence of $df \wedge \eta = 0$, we get

$$df = f^{'} and \quad X(f) = f^{'}\eta(X) \tag{4.3}$$

where $f^{'} = \xi f$. If $f = \beta = constant \neq 0$, then the manifold is a β-Kenmotsu [14]. If $f = 0$, then the manifold is cosymplectic manifold. An f-Kenmotsu manifold with a semi-symmetric non-metric connection is said to be regular if $f^2 + f + 2f^{'} \neq 0$.

By using (2.1) and (4.2), we get

$$\bar{\nabla}_X \xi = f(2X - \eta(X)\xi). \tag{4.4}$$

From (2.2), (4.1) and (4.2), we have

$$(\bar{\nabla}_X \eta)Y = fg(\phi X, \phi Y). \tag{4.5}$$

The curvature tensor $\bar{R}$ of an f-Kenmotsu manifold M with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X, Y)\xi = \bar{\nabla}_X \bar{\nabla}_Y \xi - \bar{\nabla}_Y \bar{\nabla}_X \xi - \bar{\nabla}_{[X,Y]}\xi. \tag{4.6}$$

With the help of (4.1), (4.4) and (3.3), we get

$$\bar{\nabla}_X \bar{\nabla}_Y \xi = X(f)2Y - X(f)\eta(Y)\xi + 2f\nabla_X Y - fX\eta(Y)\xi - \eta(Y)f^2 X \tag{4.7}$$

$$+\eta(Y)\eta(X)f^2\xi + f\eta(Y)X$$

and

$$-\bar{\nabla}_{[X,Y]}\xi = -2f\nabla_X Y - 2f\eta(Y)X + 2f\nabla_Y X \tag{4.8}$$

$$+2f\eta(X)Y + fX\eta(Y)\xi - fY\eta(X)\xi.$$

Using (4.7) and (4.8) in (4.5), we get

$$\bar{R}(X,Y)\xi = X(f)2Y - X(F)\eta(Y)\xi - Y(f)2X + Y(f)\eta(X)\xi + f^2\eta(X)Y \tag{4.9}$$
$$-f^2\eta(Y)X + f\eta(X)Y - f\eta(Y)X.$$

using (4.3) in (4.9), it follows that

$$\bar{R}(X,Y)\xi = -(f^2 + f + 2f^{'})[\eta(Y)X - \eta(X)Y]. \tag{4.10}$$

From (4.10), we have

$$\bar{R}(\xi,Y)\xi = -(f^2 + f + 2f^{'})[\eta(Y)\xi - Y], \tag{4.11}$$

and

$$\bar{R}(X,\xi)\xi = -(f^2 + f + 2f^{'})[X - \eta(X)\xi]. \tag{4.12}$$

Taking the inner product with Z in (4.10), we have

$$g(\bar{R}(X,Y)\xi, Z) = -(f^2 + f + 2f^{'})[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)] \tag{4.13}$$

which is used in the proof of the following lemma.

Lemma 4.1. *Let M be a 3-dimensional f-Kenmotsu manifold with a semi-symmetric non-metric connection, $\bar{S}$ Ricci curvature tensor and $\bar{Q}$ Ricci operator. Then*

$$\bar{S}(X,\xi) = -2(f^2 + f + 2f^{'})\eta(X), \tag{4.14}$$

$$\bar{Q}\xi = -2(f^2 + f + 2f^{'})\xi. \tag{4.15}$$

Proof. Contracting with Y and Z in (4.13) and taking summation over $i = 1, 2....n$, from (2.5) expression the proof (4.14) is completed. then also using (2.7) and (2.1) in (4.14), the proof of (4.15) is completed. □

Lemma 4.2. *Let M be a 3-dimensional f-Kenmotsu manifold with a semi-symmetric non-metric connection, r scalar curvature tensor, $\bar{S}(X,Y)$ Ricci curvature tensor and $\bar{Q}X$ Ricci operator. Then it follows that*

$$\bar{S}(X,Y) = (\frac{r}{2} + f^2 + f + 2f^{'})g(X,Y) - (\frac{r}{2} + 3f^2 + 3f + 6f^{'})\eta(X)\eta(Y) \tag{4.16}$$

and

$$\bar{Q}X = (\frac{r}{2} + f^2 + f^{'})X - (\frac{r}{2} + 3f^2 + 3f^{'})\eta(Y)\xi. \tag{4.17}$$

Proof. Contracting (4.12) with Y, we get

$$g(\bar{R}(X,\xi)\xi, Y) = -(f^2 + f + 2f^{'})(g(X,Y) - \eta(X)\eta(Y)) \tag{4.18}$$

Using (4.14), putting $X = \xi, Y = X, Z = Y$ in (2.9) and contracting with ξ, we obtain

$$\bar{R}(\xi, X, Y, \xi) = \bar{S}(X,Y) - 2(f^2 + f + 2f^{'})g(X,Y) - \frac{r}{2}(g(X,Y) - \eta(X)\eta(Y)) \tag{4.19}$$

$$+2(f^2+f+2f^{'})\eta(X)\eta(Y)+2f^2+f+2f^{'})\eta(X)\eta(Y).$$

With the help of (4.18) and (4.19) proof of (4.16) is completed.
Using (4.16) and (2.7), its verified that

$$g(\bar{Q}X-[(\frac{r}{2}+f^2+f+2f^{'})X-(\frac{r}{2}+3f^2+3f+6f^{'})\eta(X)\xi,Y]=0. \tag{4.20}$$

Since $Y\neq 0$ in (4.20), the proof of (4.17) is completed. □

5 Parallel symmetric second order tensors and an almost η-Ricci solitons on f-Kenmotsu manifolds with a semi-symmetric non-metric connection

Fix h a symmetric tensor field of $(0,2)$-type which we suppose to be parallel with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ that is $\bar{\nabla}h=0$. Applying the Ricci commutation identity [10].

$$\bar{\nabla}^2h(X,Y;Z,W)-\bar{\nabla}^2h(X,Y;W,Z)=0, \tag{5.1}$$

we obtain the relation

$$h(\bar{R}(X,Y)Z,W)+h(Z,\bar{R}(X,Y)W)=0. \tag{5.2}$$

Realacing $Z=W=\xi$ in (5.2) and using (4.10) and also use the symmetry of h, we have

$$-(f^2+f+2f^{'})[\eta(Y)h(X,\xi)-\eta(X)h(Y,\xi)]-(f^2+f+2f^{'})[\eta(Y)h(\xi,\xi)-h(Y,\xi)] \tag{5.3}$$

Putting $X=\xi$ in (5.3) and by virtue of (2.1), we obtain By using regularity condition in (5.4), we have

$$-(f^2+f+2f^{'})[\eta(Y)h(\xi,\xi)-h(Y,\xi)]=0. \tag{5.4}$$

Suppose $-(f^2+f+2f^{'})\neq 0$, it results

$$h(Y,\xi)=\eta(Y)h(\xi,\xi). \tag{5.5}$$

Now, we can call a regular f-Kenmotsu manifold with semi-symmetric non-metric connection with $-(f^2+f+2f^{'})\neq 0$, where regularity, means the non-vanishing of the Ricci curvature with respect to the generator of f-Kenmotsu manifold with semi-symmetric non-metric connection.

Differentiating (5.5) covariantly with respect to X, we have

$$(\bar{\nabla}_Xh)(Y,\xi)+h(\bar{\nabla}_XY,\xi)+h(Y,\bar{\nabla}_X\xi)=[g(\bar{\nabla}_XY,\xi)+g(Y,\bar{\nabla}_X\xi)]h(\xi,\xi) \tag{5.6}$$
$$+\eta(Y)[(\bar{\nabla}_Xh)(Y,\xi)+2h((\bar{\nabla}_X\xi,\xi)].$$

By using the parallel condition $\bar{\nabla}h=0$, $\eta(\bar{\nabla}_X\xi)=0$ and by the virtue of (5.5) in (5.6), we get

$$h(Y,\nabla_X\xi)=g(Y,\nabla_X\xi)h(\xi,\xi).$$

Now using (4.4) in the above equation, we get

$$h(X,Y)=g(X,Y)h(\xi,\xi), \tag{5.7}$$

which together with the standard fact that the parallelism of h implies that $h(\xi,\xi)$ is a constant, via (5.6). Now by considering the above equations, we can gives the conclusion:

Theorem 5.1. *Let (M, ϕ, ξ, η, g) be an f-Kenmotsu manifold with semi-symmetric non-metric connection with non-vanishing ξ-sectional curvature and endowed with a tensor field $h \in (T_2^0(M))$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to $\bar{\nabla}$ then it is a constant multiple of the metric tensor g.*

Corollary 5.2. *A locally Ricci symmetric regular f-Kenmotsu manifold with semi-symmetric non-metric connection is an quasi-Einstein manifold.*

Corollary 5.3. *A locally Ricci semi-symmetric regular f-Kenmotsu manifold with semi-symmetric non-metric connection is an quasi-Einstein manifold.*

Definition 5.4. *Let (M, ϕ, ξ, η, g) be an be an f-Kenmotsu manifold with semi-symmetric non-metric connection . Consider the equation [7]*

$$\mathcal{L}_\xi g + 2\bar{S} + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{5.8}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are smooth functions. Writing $\mathcal{L}_\xi g$ in terms of semi-symmetric non-metric connection ∇, we obtain:

$$2\bar{S}(X,Y) = -g(\bar{\nabla}_X\xi, Y) - g(X, \bar{\nabla}_Y\xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{5.9}$$

for any $X, Y \in \chi(M)$.

The data (g, ξ, λ, μ) which satisfy the equation (5.8) is said to be η- Ricci soliton on M [6] ; in particular if $\mu = 0$ (g, ξ, λ) is almost Ricci soliton [6] and its called shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively.

Now, from (4.4) , the equation (5.9) becomes:

$$\bar{S}(X,Y) = -(2f + \lambda)g(X,Y) + (f - \mu)\eta(X)\eta(Y). \tag{5.10}$$

The above equations yields

$$\bar{S}(X,\xi) = -(f + \lambda + \mu)\eta(X) \tag{5.11}$$

$$\bar{Q}X = -(2f + \lambda)X + (f - \mu)\xi \tag{5.12}$$

$$\bar{Q}\xi = -(f + \lambda + \mu)\xi \tag{5.13}$$

$$\bar{r} = -\lambda n - (n-1)f - \mu, \tag{5.14}$$

where r is the scalar curvature. Off the two natural situations regrading the vector field V: $V \in Span\xi$ and $V \perp \xi$, we investigate only the case $V = \xi$.

Our interest is in the expression for $\mathcal{L}_\xi g + 2\bar{S} + 2\mu\eta \otimes \eta$. A direct computation gives

$$\mathcal{L}_\xi g(X,Y) = 2f[2g(X,Y) - \eta(X)\eta(Y)]. \tag{5.15}$$

In 3-dimensional f-Kenmotsu manifold with a semi-symmetric non-metric connection the Riemannian curvature tensor is given by

$$\bar{R}(X,Y)Z = g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y + \bar{S}(Y,Z)X - \bar{S}(X,Z)Y \tag{5.16}$$

$$-\frac{r}{2}[g(Y,Z)X-g(X,Z)Y],$$

Putting $Z=\xi$ in (5.16) and using (4.10), (4.14) and (4.17) for 3-dimensional f-Kenmotsu manifold with a semi-symmetric non-metric connection, we get

$$-(f^2+f+2f^{'})[\eta(Y)X-\eta(X)Y]=\eta(Y)[(\frac{r}{2}+f+2f^{'})X-(\frac{r}{2}+3f^2+3f^{'})\eta(X)\xi] \tag{5.17}$$

$$-\eta(X)[(\frac{r}{2}+f+2f^{'})Y-(\frac{r}{2}+3f^2+3f^{'})\eta(Y)\xi]-2(f^2+f+2f^{'})\eta(Y)X$$

$$+2(f^2+f+2f^{'})\eta(X)Y-\frac{r}{2}[\eta(Y)X-\eta(X)Y]$$

Again, putting $Y=\xi$ in the (5.17) and using (2.2) and condition of regularity we obtain

$$\bar{Q}X=\left[\frac{r}{2}+(\frac{r}{2}+f^2+f^{'})-(f^2+f+2f^{'})\right]X \tag{5.18}$$

$$+\left[\frac{r}{2}+(\frac{r}{2}+f^2+f^{'})-3(f^2+f+2f^{'})\right]\eta(X)\xi.$$

From (5.18), we have

$$\bar{S}(X,Y)=\left[\frac{r}{2}+(\frac{r}{2}+f^2+f^{'})-(f^2+f+2f^{'})\right]g(X,Y) \tag{5.19}$$

$$+\left[\frac{r}{2}+(\frac{r}{2}+f^2+f^{'})-3(f^2+f+2f^{'})\right]\eta(X)\eta(Y).$$

Equation (5.19) shows that a 3-dimensional f-Kenmotsu manifold with a semi-symmetric non-metric connections η-*Einstein*.

Next, we consider the equation

$$h(X,Y)=(\mathcal{L}_\xi g)(X,Y)+2S(X,Y)+2\mu\eta(X)\eta(Y). \tag{5.20}$$

By Using (5.16) and (5.19) in (5.20), we have

$$h(X,Y)=(r-4f-f^{'})g(X,Y)+(r-4f+5f^{'}-2f^2)\eta(X)\eta(Y)+2\mu\eta(X)\eta(Y) \tag{5.21}$$

Putting $X=Y=\xi$ in (5.21), we get

$$h(\xi,\xi)=2[r+2f^{'}-f^2+\mu] \tag{5.22}$$

Now, (5.7) becomes

$$h(X,Y)=2[r+2f^{'}-f^2+\mu]g(X,Y). \tag{5.23}$$

From (5.20) and (5.23), it follows that g is an η-Ricci soliton.

Therefore, we can state as:

Theorem 5.5. *Let (M,ϕ,ξ,η,g) be a 3-dimensional f-Kenmotsu manifold with a semi-symmetric non-metric connection, then (g,ξ,μ) yields an almost η-Ricci soliton on M.*

Let V be pointwise collinear with ξ. i.e., $V = b\xi$, where b is a function on the 3-dimensional f-Kenmotsu manifold with semi-symmetric non-metric connection. Then

$$g(\bar{\nabla}_X b\xi, Y) + g(\bar{\nabla}_Y b\xi, X) + 2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

or

$$bg((\bar{\nabla}_X \xi, Y) + (Xb)\eta(Y) + bg(\bar{\nabla}_Y \xi, X) + (Yb)\eta(X)$$
$$+2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Using (4.4), we obtain

$$bg(f(2X - \eta(X)\xi, Y) + (Xb)\eta(Y) + bg(f(2Y - \eta(Y)\xi, X)$$
$$+(Yb)(X) + 2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

which yields

$$4bfg(X,Y) - 2bf\eta(X)\eta(Y) + (Xb)\eta(Y) \tag{5.24}$$
$$+(Yb)\eta(X) + 2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Replacing Y by ξ in (5.24), we obtain

$$(Xb) + (\xi b)\eta(X) + 2bf\eta(X) - 4(f^2 + f + 2f^{'})\eta(X) + 2\lambda\eta(X) + 2\mu\eta(X)\eta(Y). \tag{5.25}$$

Again putting $X = \xi$ in (5.25), we obtain

$$\xi b = 2(f^2 + f + 2f^{'}) - bf - \lambda - \mu.$$

Plugging this in (5.25), we get

$$(Xb) + 2[2(f^2 + f + 2f^{'}) - bf - \lambda + \mu]\eta(X) = 0,$$

or

$$db = \Big\{2(f^2 + f + 2f^{'}) - bf - \lambda - \mu\Big\}\eta. \tag{5.26}$$

Applying d on (5.26), we get $\Big\{2(f^2 + f + 2f^{'}) - bf - \lambda - \mu\Big\}d\eta$. Since $d\eta \neq 0$ we have

$$\Big\{2(f^2 + f + 2f^{'}) - bf - \lambda - \mu\Big\} = 0. \tag{5.27}$$

Equation(5.27) in (5.26) yields b as a constant. Therefore from (5.24), it follows that

$$\bar{S}(X,Y) = -(\lambda + 2bf)g(X,Y) + (bf - \mu)\eta(X)\eta(Y),$$

which implies that M is of constant scalar curvature for constant f. This leads to the following:

Theorem 5.6. *If in a 3-dimensional f-Kenmotsu manifold with a semi-symmetric non-metric connection the metric g is an almost η-Ricci soliton and V is positive collinear with ξ, then V is a constant multiple of ξ and g is an η quasi-Einstein manifold and constant scalar curvature provided bf is a constant.*

6 Example of a 3-dimensional f-Kenmotsu manifold with a semi-symmetric-non-metric connection:

Consider the three dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 \,|z \neq 0\}$, where (x, y, z) are the Cartesian coordinates in $\mathbb{R}^3$ and let the vector fields are

$$e_1 = z^2\frac{\partial}{\partial x}, \qquad e_2 = z^2\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z},$$

where e_1, e_2, e_3 are linearly independent at each point of M. Let g be the Riemannain metric defined by $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$, $g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$,
Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for any vector field X on M,
and ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then by using the linearity of ϕ and g, we have $\phi^2 X = -X + \eta(X)\xi$, with $\xi = e_3$. Further $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields X and Y on M. Hence for $e_3 = \xi$, the structure defines an (δ)-almost contact structure in R^3.

Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ -g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is know as Koszul's formula.

$$\nabla_{e_i} e_i = -\frac{2}{z}e_3, \qquad \nabla_{ei} e_3 = -\frac{2}{z}e_i, i = 1, 2 \tag{6.1}$$
$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \nabla_{e_3} e_1 = \nabla_{e_3} e_2 = \nabla_{e_3} e_3 = 0$$

Here ∇ be the Levi-Civita connection with respect to the metric g , then we have
$[e_1, e_2] = 0$, $[e_1, e_3] = -\frac{2}{z}e_1$, $[e_2, e_3] = -\frac{2}{z}e_2$.

Now consider at this example for semi-symmetric non-metric connection from (4.1) and (5.28),

$$\bar{\nabla}_{ei} e_i = -\frac{2}{z}e_3, \nabla_{e_i} e_3 = -\frac{2}{z}e_i, i = 1, 2 \tag{6.2}$$
$$\nabla_{ei} e_j = \bar{\nabla}_{e3} e_i = 0, \qquad \bar{\nabla}_{e3} e_3 = e_3$$

where $i \neq j$. We know that

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{6.3}$$

By using (5.29) and (5.30) we obtain the components of the Riemann and the Ricci curvature tensor fields are computed as follows:

$$\bar{R}(e_i, e_3)e_3 = (1 - \frac{6}{Z^2})e_i, \bar{R}(e_i, e_j)e_3 = 0 \tag{6.4}$$

$$\bar{R}(e_i, e_j)e_j = (\frac{2}{z} - \frac{4}{z^2})e_i, \quad \bar{R}(e_i, e_3)e_j = 0, \bar{R}(e_3, e_i)e_i = (\frac{2}{z} - \frac{6}{z^2})e_3$$

where $i \neq j = 1, 2$.

From the equation (5.31) we can also obtain

$$\bar{R}(e_1, e_3)e_3 = (1 - \frac{6}{Z^2})e_1, \ \bar{R}(e_2, e_3)e_3 = (1 - \frac{6}{Z^2})e_1, \ \bar{R}(e_1, e_2)e_2 = (\frac{2}{z} - \frac{4}{Z^2})e_1, \tag{6.5}$$

$$\bar{R}(e_3, e_1)e_1 = (\frac{2}{z} - \frac{6}{Z^2})e_3, \ \bar{R}(e_3, e_2)e_2 = (\frac{2}{z} - \frac{6}{Z^2})e_3, \ \bar{R}(e_2, e_1)e_1 = (\frac{2}{z} - \frac{4}{Z^2})e_3,$$

Therefore, we have

$$\bar{S}(e_i, e_i) = \bar{S}(e_2, e_2) = -\frac{10}{z^2} + \frac{2}{z} + 1, i = 1, 2, S(e_3, e_3) = -\frac{12}{z^2} + \frac{4}{z} \tag{6.6}$$

$$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = -\frac{10}{z^2} + \frac{2}{z} + 1 \tag{6.7}$$

for $i = 1, 2$. Hence M is also an *Einstein* manifold. In this case, from (5.9) (e_i, e_i) follows, for

$$f[2g(e_i, e_i) - \eta(e_i)\eta(e_i)] + 2\bar{S}(e_i, e_i) + 2\lambda g(e_i, e_i) + 2\mu\eta(e_i)\eta(e_i)$$

$$2f(2 - \delta_{ij}) + 2(-\frac{10}{z^2} + \frac{2}{z} + 1) + 2\lambda + 2\mu\delta_{ij}$$

for all $i \in \{1, 2, 3\}$. Therefore, we have $\lambda = (2f - \frac{2}{z} - \frac{5}{z^2} + 1)$ and $\mu = (3f - \frac{4}{z} - \frac{5}{z^2} + 1)$, the he data (g, ξ, λ, μ) is an almost η-Ricci soliton on (M, ϕ, ξ, η, g) with respect to a semi-symmetric non-metric connection is expanding.

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