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Finite groups having the same ratio of cyclic subgroups

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Abstract Let $c(G)$ be the number of cyclic subgroups of the finite group G and $\alpha(G)$ the ratio function $c(G)/|G|$. In this paper we study finite simple groups $PSL(n,q)$ and $PSU(n,q)$ having the same ratio of cyclic subgroups.

Key Words number of cyclic subgroups,2-groups,involutions,diagonal matrix

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1 Introduction

In this paper all groups considered are finite. Let $c(G)$ be the number of cyclic subgroups of the group G and $\alpha(G)$ denotes the function $c(G)/|G|$. Observe that

$$c(G) = \sum_{x\in G} \frac{1}{\phi(o(x))},$$

where ϕ is Euler's totient function and $o(x)$ denote the order of the element x. Garonzi and Lima proved if a finite group G and its quotient group G/N have the same value α and G/N is isomorphic to a symmetric group S_n, then $G \cong N \times S_n$ (see Theorem 1 of [1]). It is an interesting question to ask what can we say about G if $\alpha(G) = \alpha(G/N)$ given some information G/N. In this paper we study finite groups G in case G/N is isomorphic to a non-abelian simple group. For an integer n, denote by n_2 the 2-part of n. Moreover, denote $L_n^\epsilon(q)$ is $PSL(n,q)$ if $\epsilon = 1$, and $PSU(n,q)$ if $\epsilon = -1$.

Main Theorem *Let G be a group and N a normal subgroup of G such that G/N is isomorphic to a non-abelian simple group $L_n^\epsilon(q)$, where $n_2 \neq (q-1)_2 \geqslant 4$ and $(n,q) \neq (3,4),(6,2)$. If $\alpha(G) = \alpha(G/N)$, then $G \cong N \times G/N$.*

2 Some Lemmas

In this section, we give some lemmas. The first six lemmas are cited from the second section of [1].

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Lemma 2.1 *If $\alpha(G) = \alpha(G/N)$, then N is an elementary abelian 2-group.*

Lemma 2.2 *If G is any finite group, then $\alpha(G) = \alpha(G \times Z_2^n)$ for all $n \geqslant 0$.*

Lemma 2.3 *If $\alpha(G) = \alpha(G/N)$ and $L \trianglelefteq G$, $L \subseteq N$, then $\alpha(G) = \alpha(G/L)$.*

Lemma 2.4 *If $\alpha(G) = \alpha(G/N)$ and $K \leqslant G$, then $\alpha(K) = \alpha(K/K \cap N)$.*

Lemma 2.5 *Suppose $\alpha(G) = \alpha(G/N)$. If an element $a \in G$ has order 2 modulo N, then a centralizes N. In particular, if G/N can be generated by elements of order 2 then $N \subseteq Z(G)$.*

Since every finite non-abelian simple group can be generated by its elements of order 2, the following corollary is obtained by Lemma 2.5 directly.

Corollary 2.6 *If $\alpha(G) = \alpha(G/N)$ and G/N is a non-abelian simple group, then N is an elementary 2-group and $N \subseteq Z(G)$.*

Recalled if S is any finite group, we say that G is a covering group of S if $G/Z(G) \cong S$ and $Z(G) \leqslant G'$ (also called central expansion).If the center has order 2, the covering group is referred to a double cover of S. If every finite perfect group S has a unique maximal covering group G, with the property that every other covering group is a quotient of G. This is called the universal cover, and its center is called the Schur multiplier of S (see section 2.7.1, [14]). The Schur multipliers of the non-abelian simple groups were classified by Griess B., ect (see [10],[11], [12]). Also all Schur multipliers of the non-abelian simple groups are collected in Table 4.1 of [13]. So we can give the possible double covers of all non-abelian simple groups.

Lemma 2.7 *Let S be a non-abelian simple group and G the universal cover of S. Suppose that Schur multiplier of S is of even order. Then S and its multiplier are listed in the following table.*

Table 1: Schur Multipliers of Even Order

Simple Group S	Multiplier	Simple Group S	Multiplier
$PSL_l(q), 2\|(l, q-1)$	$Z_{(l,q-1)}$	$PSL(2,4)$	Z_2
$PSU_l(q), 2\|(l, q+1)$	$Z_{(l,q+1)}$	$PSL(2,9)$	$Z_2 \times Z_3$
$PSL(3,2)$	Z_2	$PSL(3,4)$	$Z_3 \times Z_4^2$
$PSL(4,2)$	Z_2	$PSU(4,2)$	Z_2
$PSU(4,3)$	$Z_4 \times Z_3^2$	$PSU(6,2)$	$Z_2^2 \times Z_3$

3 Proof of Main Theorem

Assume that G is a contra-example of the minimum order. By Corollary 2.6, N is an elementary abelian 2-group and N is central in G. Let $R \cong Z_2^l$ be a minimal normal subgroup of G contained in N. By Lemma 2.3 we have $\alpha(G/R) = \alpha(G) = \alpha(S)$, so by induction, since G/N is a quotient of G/R, we have $G/R \cong C_2^l \times S$ for some $l \geqslant 1$. Let $K \trianglelefteq G$ be the normal subgroup of G such that $K/R \cong \{1\} \times S$. Observe that $K \cap N$ contains R, so $K \cap N/R$ is a normal 2-subgroup of $K/R \cong S$. Hence $K \cap N = R$.

If $N \neq R$. As $l \geqslant 1$, $|K| < |G|$ and $\alpha(K) = \alpha(G/R) = \alpha(S)$ by Lemma 2.4 (being $K \cap N = R$). By induction we deduce that $K \cong R \times S$. Set $M := \{1\} \times S \leqslant K$. Since

$$S \cong G/N \geqslant KN/N \cong K/K \cap N = K/R \cong S$$

we obtain $G = KN = MRN = MN$ so being N central in G (by Lemma 2.6) and $N \cap M = N \cap K \cap M = R \cap M = \{1\}$ we deduce $G = N \times M \cong N \times G/N$. Assume now $N = R$, so N is a minimal normal subgroup of G. Since N is central, $|N| = 2$ and actually $N = \langle z \rangle = Z(G)$ is the center of G (being $G/N \cong S$). Suppose by contradiction that G is not a direct product $C_2 \times S$. We claim that N is contained in the derived subgroup of G. Indeed if $G' \neq G$ and N is not contained in G' then $G'N/N$ is a nontrivial normal subgroup of $G/N \cong S$, a contradiction. So G is a double cover of S. Note that if the Schur multiplier listed in Table 1 has a cyclic Sylow 2-subgroup, then S has the unique double cover.

Lemma 3.1 *Suppose $S \cong PSL(n,q)$ and if $n_2 = (q-1)_2 \geqslant 4$ or $S = PSL(3,4)$, then $\alpha(G) \neq \alpha(S)$.*

Proof. First we assume that $(n,q) \neq (2,4), (2,9), (3,2), (3,4), (4,2)$. By Table 1, the universal cover of $PSL(n,q)$ is $SL(n,q)$ and its Schur multiplier is $Z_{(n,q-1)}$. Let c be a generator of a Sylow 2-subgroup of $Z_{(n,q-1)}$ and $o(c) = 2^s$. We want to find a matrix A in $SL(n,q)$ such that $A^2 = diag(c,c,\cdots,c)$. Let $T_i = Z_2 \wr Z_2 \wr Z_2 \wr \cdots \wr Z_2$ be the wreath product of Z_2 i times. If W is a Sylow 2-subgroup of $GL(2,q)$, then $W_r = W \wr T_{r-1}$ is a Sylow 2-subgroup of the corresponding group in dimension 2^r (we define $W_1 := W$) (see Lemma 1 of [15]).

If $4|q-1$, then $W \cong Z_{2^s} \wr Z_2$, and so $W_r \cong Z_{2^s} \wr T_r$. On the other hand, let $T = \{diag(c_1, c_2, \cdots, c_{2^r}) \in GL(2^r,q) | c_i \in \langle c \rangle \ for\ 1 \leqslant i \leqslant 2^r\}$ and P_2 a Sylow 2-subgroup of the symmetric group of degree 2^r. We regard a permutation σ of P_2 as a permutation matrix $E^\sigma := (\epsilon_1, \epsilon_2, \cdots, \epsilon_{2^r})^\sigma = (\epsilon_{1^\sigma}, \epsilon_{2^\sigma}, \cdots, \epsilon_{2^{r\sigma}})$, where ϵ_i is the i-th unit column vector. Note that every element of the set TP_2 is a 2-element in $GL(2^r,q)$. In fact if $o(\sigma) = 2^m$, then $(diag(c_1,c_2,\cdots,c_{2^r})E^\sigma)^{2^m} = diag(\prod_{i=0}^{2^m-1} c_{1^{\sigma^{-i}}}, \prod_{i=0}^{2^m-1} c_{2^{\sigma^{-i}}}, \cdots, \prod_{i=0}^{2^m-1} c_{(2^r)^{\sigma^{-i}}})$, and so $diag(c_1,c_2,\cdots,c_{2^r})E^\sigma$ is a 2-element. Moreover, $|TP_2| = |W_r| = 2^{2^r(s+1)-1}$, then TP_2 is a Sylow 2-subgroup of $GL(2^r,q)$.

Let $2^{r_1} + 2^{r_2} + \cdots + 2^{r_t}, r_1 < r_2 < \cdots < r_t$, be the 2-adic expansion of n. Certainly, 2^{r_1} is the 2-part of n. By Theorem 1 of [15], a Sylow 2-subgroup of $GL(n,q)$ is isomorphic to $W_{r_1} \times W_{r_2} \times \cdots \times W_{r_t}$. So we first need to find a matrix A_i of $GL(2^{r_i},q)$ such that $A_i^2 = diag(c,c,\cdots,c)$ for all i (for $1 \leqslant i \leqslant t$), and then $A = diag(A_{r_1}, A_{r_2}, \cdots, A_{r_t})$. Next we set $\sigma \in Sym(2^{r_i})$ and $A_i = diag(c_1,c_2,\cdots,c_{2^{r_i}})E^\sigma \in GL(2^{r_i},q)$. If $A_i^2 = diag(c,c,\cdots,c)$, then $(diag(c_1,c_2,\cdots,c_{2^{r_i}})E^\sigma)^2 = diag(c_1 c_{1^{\sigma^{-1}}}, c_2 c_{2^{\sigma^{-1}}}, \cdots, c_{2^{r_i}} c_{(2^{r_i})^{\sigma^{-1}}})E^{\sigma^2} = diag(c,c,\cdots,c)$, hence $c_i c_{i^\sigma} = c$ for all i. Let $\sigma = (i_1 i_2)(i_3 i_4)\cdots(i_{2k_i-1} i_{2k_i})$, then σ has $2^{r_i} - 2k_i$ fixed points with $0 \leqslant k_i \leqslant 2^{r_i-1}$. It follows that

$$c_{i_1}c_{i_2} = c_{i_3}c_{i_4} = \cdots = c_{i_{2k_i-1}}c_{i_{2k_i}} = c_{i_{2k_i+1}}^2 = \cdots = c_{i_{2^{r_i}}}^2 = c. \tag{2.1}$$

Now if $n_2 < (q-1)_2$, for the equation (2.1) we chose $\sigma = 1$, then $c_i^2 = c$, which has a solution c_0 in the cyclic group Z_{q-1}. So we can chose $A_i = diag(c_0,c_0,\cdots,c_0)$, and then $A = diag(c_0,c_0,\cdots,c_0)$.

Moreover, since $n_2 = 2^s$, it follows that $2^s < 2^{r_i}$ for all i. So

$$|A| = \prod_{i=1}^{t} |A_i| = \prod_{i=1}^{t} c_0^{2^{r_i}} = \prod_{i=1}^{t} c^{2^{r_i - 1}} = 1,$$

and then $A \in SL(n, q)$.

Next if $n_2 > (q-1)_2$, then $2^s = (q-1)_2$. For the equation (2.1) we can chose $k_i = 2^{r_i - 1}$ and $c_{i_j} = c_{i_{j-1}}^{-1} c$ for $1 \leqslant j \leqslant 2^{r_i - 1}$. So

$$|A| = |diag(A_1, A_2, \cdots, A_t)| = \prod_{i=1}^{t} |A_i| = \prod_{i=1}^{t} c^{2^{r_i - 1}} = c^{\frac{n}{2}} = 1,$$

we have $A \in SL(n, q)$.

Next we assume that $n_2 = (q-1)_2 \geqslant 4$. Since $\langle c \rangle$ is a Sylow 2-subgroup of Z_{q-1}, the equation $c_i^2 = c$ has no solution in Z_{q-1}, so σ has no fixed point, that is $k_i = 2^{r_i - 1}$. By (2.1) similarly it follows that $|A| = c^{\frac{n}{2}}$. Since $o(c) = 2^s = n_2 > (n/2)_2$, we have $|A| \neq 1$, and so $A \notin SL(n, q)$. Therefore there does not exist an element $A \in SL(n, q)$ such that $A^2 = diag(c, c, \cdots, c)$.

In the sequel, we assume that $4 \nmid q - 1$, that is $q \equiv 3(mod\ 4)$. Clearly, the center of $GL(n, q)$ is of order 2. Let $2^{s'}$ be the 2-part of $q+1$. Certainly, $s' \geqslant 2$. Observe that a Sylow 2-subgroup W of $GL(2, q)$ is a so-called semidihedral group, that is $W = \langle a, b | a^{2^{s'+1}} = b^2 = 1, bab = a^{2^{s'} - 1} \rangle$, and

$$a = \begin{pmatrix} 0 & 1 \\ 1 & \epsilon + \epsilon^q \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where ϵ is a primitive $2^{s'+1}$-st root of unity in $GF(q^2)$ (see page 142 of [15]). Note that a matrix $diag(u_1, u_2, \cdots, u_{2^r})E^{\sigma}$ is a 2-element in $GL(2^r, q)$, where $\sigma \in Sym(2^{r-1})$ is a 2-element and $u_i \in W$. Let $A = diag(a^{2^{s'-1}}, a^{2^{s'-1}}, \cdots, a^{2^{s'-1}})$ be a matrix in $GL(n, q)$. Since $a^{2^{s'}} = diag(-1, -1)$ is central of order 2 in $GL(2, q)$, it follows that $A^2 = diag(-1, -1, \cdots, -1)$. Moreover, $|A| = |a|^{2^{s'-2} \cdot n} = (-1)^{2^{s'-2} \cdot n} = 1$, so there exists a matrix $A \in SL(n, q)$ such that $A^2 = diag(c, c, \cdots, c)$.

According to ATLAS of finite groups, if $(n, q) = (3, 4)$, we can get there exists a covering of $PSU(3, 4)$ by the center Z_2 such that $\alpha(G) \neq \alpha(S)$(see page 24 of [16]).

Lemma 3.2 *Suppose $S \cong PSU(n, q)$, and if $n_2 = (q+1)_2 \geqslant 4$ or $S = PSU(4, 3)$, then $\alpha(G) \neq \alpha(S)$.*

Proof. First we assume that $(n, q) \neq (4, 2), (4, 3), (6, 2)$. By Table 1, the universal cover of $PSU(n, q)$ is $SU(n, q)$ and its Schur multiplier is $Z_{(n,q+1)}$. By Table 1, the universal cover of $PSU(n, q)$ is $SU(n, q)$ except $(n, q) \neq (4, 2), (4, 3), (6, 2)$.Note that $SU(n, q) = \{T \in SL(n, q^2) | T'T^{\lambda} = I\}$, where λ: $x \mapsto x^q, x \in F_{q^2}$ is the automorphism of F_{q^2} of order 2.

Let c be a generator of a Sylow 2-subgroup of $Z_{(n,q+1)}$ and $o(c) = 2^s$. We want to find a matrix A in $SU(n, q)$ such that $A^2 = diag(c, c, \cdots, c)$. Let $T_i = Z_2 \wr Z_2 \wr Z_2 \wr \cdots \wr Z_2$ be the wreath product of Z_2 i times. If W is a Sylow 2-subgroup of $GU(2, q)$, then $W_r = W \wr T_{r-1}$ is a Sylow 2-subgroup of the corresponding group in dimension 2^r (we define $W_1 := W$)(see Lemma 1 of [15]).

If $q \equiv 3(mod\ 4)$ and $2^s = (q+1)_2$, then $W \cong Z_{2^s} \wr Z_2$, and so $W_r \cong Z_{2^s} \wr T_r$. On the other hand, let $T = \{diag(c_1, c_2, \cdots, c_{2^r}) \in GL(2^r, q^2) | c_i \in \langle c \rangle\ for\ 1 \leqslant i \leqslant 2^r\}$ and P_2 a Sylow 2-subgroup of the symmetric group of degree 2^r. We regard a permutation σ of P_2 as a permutation matrix $E^\sigma := (\epsilon_1, \epsilon_2, \cdots, \epsilon_{2^r})^\sigma = (\epsilon_{1^\sigma}, \epsilon_{2^\sigma}, \cdots, \epsilon_{2^{r\sigma}})$, where ϵ_i is the i-th unity column vector. Note that every element of the set TP_2 is a 2-element in $GL(2^r, q)$. In fact if $o(\sigma) = 2^m$, then

$$(diag(c_1, c_2, \cdots, c_{2^r})E^\sigma)^{2^m} = diag(\prod_{i=0}^{2^m-1} c_{1^{\sigma^{-i}}}, \prod_{i=0}^{2^m-1} c_{2^{\sigma^{-i}}}, \cdots, \prod_{i=0}^{2^m-1} c_{(2^r)^{\sigma^{-i}}}),$$

and so $diag(c_1, c_2, \cdots, c_{2^r})E^\sigma$ is a 2-element. Moreover, $|TP_2| = |W_r| = 2^{2^r(s+1)-1}$, then TP_2 is a Sylow 2-subgroup of $GL(2^r, q)$.

Let $2^{r_1} + 2^{r_2} + \cdots + 2^{r_t}, r_1 < r_2 < \cdots < r_t$, be the 2-adic expansion of n. Certainly, 2^{r_1} is the 2-part of n. By Theorem 1 of [15], a Sylow 2-subgroup of $GU(n, q)$ is isomorphic to $W_{r_1} \times W_{r_2} \times \cdots \times W_{r_t}$. So we first need to find a matrix A_i of $GU(2^{r_i}, q)$ such that $A_i^2 = diag(c, c, \cdots, c)$ for all i (for $1 \leqslant i \leqslant t$), and then we can construct a block diagonal matrix $A = diag(A_{r_1}, A_{r_2}, \cdots, A_{r_t})$. Next we set $\sigma \in Sym(2^{r_i})$ and $A_i = diag(c_1, c_2, \cdots, c_{2^{r_i}})E^\sigma \in GL(2^{r_i}, q)$. If $A_i^2 = diag(c, c, \cdots, c)$, then $(diag(c_1, c_2, \cdots, c_{2^{r_i}})E^\sigma)^2 = diag(c_1 c_{1^{\sigma^{-1}}}, c_2 c_{2^{\sigma^{-1}}}, \cdots, c_{2^{r_i}} c_{(2^{r_i})^{\sigma^{-1}}})E^{\sigma^2} = diag(c, c, \cdots, c)$, hence $c_i c_{i^\sigma} = c$ for all i. Let $\sigma = (i_1 i_2)(i_3 i_4) \cdots (i_{2k_i-1} i_{2k_i})$, then σ has $2^{r_i} - 2k_i$ fixed points with $0 \leqslant k_i \leqslant 2^{r_i - 1}$. It follows that

$$c_{i_1} c_{i_2} = c_{i_3} c_{i_4} = \cdots = c_{i_{2k_i-1}} c_{i_{2k_i}} = c^2_{i_{2k_i+1}} = \cdots = c^2_{i_{2^{r_i}}} = c. \tag{2.2}$$

Now if $n_2 < (q+1)_2$, for the equation (2.2) we chose $\sigma = 1$, then $c_i^2 = c$, which has a solution c_0 in the cyclic group Z_{q^2-1} since $(q+1)_2 < (q^2-1)_2$. So we can chose $A_i = diag(c_0, c_0, \cdots, c_0)$, and then $A = diag(c_0, c_0, \cdots, c_0)$. Moreover, since $n_2 = 2^s$, it follows that $2^s < 2^{r_i}$ for all i. So

$$|A| = \prod_{i=1}^{t} |A_i| = \prod_{i=1}^{t} c_0^{2^{r_i}} = \prod_{i=1}^{t} c^{2^{r_i-1}} = 1,$$

and then $A \in SL(n, q^2)$. Also since $c_0 c_0^q = c_0^{q+1} = 1$, it follows that $A \in SU(n, q)$.

Next if $n_2 > (q+1)_2$, then $2^s = (q+1)_2$. For the equation (2.2) we can chose $k_i = 2^{r_i-1}$ and $c_{i_j} = c_{i_{j-1}}^{-1} c$ for $1 \leqslant j \leqslant 2^{r_i-1}$. Without loss generalization, we assume $\sigma = (1,2)(3,4)\cdots(2^{r_i}-1, 2^{r_i})$. Then

$$A_i = diag(c_1, c_1^{-1}c, c_3, c_3^{-1}c, \cdots, c_{2^{r_i}-1}, c_{2^{r_i}-1}^{-1}c).$$

Note that $A_i' A_i^\lambda = diag(c_1^{q+1}, (c_1^{-1}c)^{q+1}, c_3^{q+1}, (c_3^{-1}c)^{q+1}, \cdots, (c_{2^{r_i}-1})^{q+1}, (c_{2^{r_i}-1}^{-1}c)^{q+1}) = E$ if we choose every $c_i \in \langle c \rangle$. Moreover,

$$|A| = |diag(A_1, A_2, \cdots, A_t)| = \prod_{i=1}^{t} |A_i| = \prod_{i=1}^{t} c^{2^{r_i-1}} = c^{\frac{n}{2}} = 1,$$

we have $A \in SU(n, q)$.

Next we assume that $n_2 = (q+1)_2$. Observe that $A_i = diag(c_1, c_2, \cdots, c_{2^{r_i}})E^\sigma$ satisfied (2.2). Since $A_i \in GU(2^{r_i, q})$, we have $A_i' A_i^\lambda = E$, and then

$$E^{\sigma^{-1}} diag(c_1, c_2, \cdots, c_{2^{r_i}}) diag(c_1^q, c_2^q, \cdots, c_{2^{r_i}}^q) E^\sigma = E.$$

So $c_i^{q+1} = 1$ for all $1 \leqslant i \leqslant 2^{r_i}$. Then $c_i \in \langle c \rangle$. Since $\langle c \rangle$ is a Sylow 2-subgroup of Z_{q+1}, the equation $c_i^2 = c$ has no solution in Z_{q^2-1}. Hence σ has no fixed point, that is $k_i = 2^{r_i-1}$. By (2.2) similarly it follows that $|A| = c^{\frac{n}{2}}$. Since $o(c) = 2^s = n_2 > (n/2)_2$, we have $|A| \neq 1$, and so $A \notin SU(n,q)$. Therefore there does not exist an element $A \in SU(n,q)$ such that $A^2 = diag(c, c, \cdots, c)$.

In the sequel, we assume that $4 \mid q-1$, that is $q \equiv 1(\mod 4)$. Clearly, the center of $GL(n,q)$ is of order 2. Let 2^s be the 2-part of $q-1$. Certainly, $s \geqslant 2$. Let ϵ be a primitive 2^{s+1}-st root of unity in $GF(q^2)$. Then $\epsilon^q = -\epsilon$. Set $\gamma = (\epsilon+\epsilon^{-1})/2$, $\delta = (\epsilon-\epsilon^{-1})/2$. Set $\alpha, \beta \in GF(q)$ so that $\alpha^2+\beta^2 = \gamma^2$. Observe that a Sylow 2-subgroup W of $GU(2,q)$ is a semidihedral group, that is $W = \langle a, b | a^{2^{s+1}} = b^2 = 1, bab = a^{2^s-1}\rangle$, and

$$a = \begin{pmatrix} \alpha+\delta & \beta \\ \beta & -\alpha+\delta \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(see page 143 of [15]). Note that a matrix $diag(u_1, u_2, \cdots, u_{2^r})E^{\sigma}$ is a 2-element in $GU(2^r, q)$, where $\sigma \in Sym(2^{r-1})$ is a 2-element and $u_i \in W$. Let $A = diag(a^{2^{s-1}}, a^{2^{s-1}}, \cdots, a^{2^{s-1}})$ be a matrix in $GU(n,q)$. Since $a^{2^s} = diag(-1,-1)$ is central of order 2 in $GU(2,q)$, it follows that $A^2 = diag(-1,-1,\cdots,-1)$. Moreover, $|A| = |a|^{2^{s-2}\cdot n} = (-1)^{2^{s-2}\cdot n} = 1$, so there exists a matrix $A \in SU(n,q)$ such that $A^2 = diag(c, c, \cdots, c)$.

According to ATLAS of finite groups, if $(n,q) = (6,2)$, we can get there exists a covering of $PSU(6,2)$ by the center Z_2 such that $\alpha(G) \neq \alpha(S)$(see page 116 of [16]).

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