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Independence Separation on Square Hexagonal Chessboard

K S P Sowndarya[①*], Y Lakshmi Naidu[①]

① Department of Mathematics and Computer Science, Sri Sathya Sai Institute of Higher Learning, Andhra Pradesh, India
E-mail: kspsowndarya44@gmail.com

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Abstract The *Independence Number*(β) is the number of vertices in a maximum independent dominating set. The *Independent Dominating Set* D is a subset of vertices V where D is a dominating set with non-adjacent vertices. The aim of this paper is to bring the concept of *separation problem* on square hexagonal chessboard (i.e., a square board with hexagonal cells). The *queens independence separation number*, $s_Q(\beta, n+k, n)$ is the placement of maximum of $n+k$ non-attacking queens on an $n \times n$ board using minimum number of pawns k. Here we are interested in finding the queens and bishops independence separation numbers on a square hexagonal board which we denote by $s_{QH}(\beta, \beta(QH)+k, n)$ and $s_{BH}(\beta, \beta(BH)+k, n)$ respectively.

Key Words Square Hexagonal Chessboard, Queens Separation Problem

MSC 2010 05CA10, 05A20

1 Introduction

The *Queens Separation Problem* is the legal placement of fewest number of pawns with the maximum number of independent queens placed on an $n \times n$ board which results in a separated board. Here a legal placement is defined as separation of attacking queens by pawns. Queens independence separation number on a square board is denoted by $s_Q(\beta, n+k, n)$. We denote the queens and bishops independence separation numbers on square hexagonal board by $s_{QH}(\beta, \beta(QH)+k, n)$ and $s_{QH}(\beta, \beta(BH)+k, n)$ respectively. Here β is the independence number, (QH) and (BH) are queens and bishops on square hexagonal board respectively. The square hexagonal chessboard is a square board with hexagonal cells which are taken as vertices, and edges are formed between the cells that are adjacent to each other according to the movement of the chess piece. Here n is the size of the board with n cells in each of the n columns, thus we have n^2 hexagonal cells. Note that in this board the cells are diagonally attached to one another therefore the cells in the rows are not adjacent and the board will have $2n$ rows and n columns.

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The studies on domination parameters started with famous eight queens problem which was first posed in the year 1848 which places 8 non-attacking queens on an 8×8 chessboard. A lot of work has been done in various domination parameters on different chessboard graphs starting with square boards [4], and then this work has been extended to queens separation problems. Chatham et al., in [3] found the queens separation on square chessboard, and in [2] he determined the separation problem of various other chess pieces with independence and domination parameters on square chessboards.

In [5] bounds for queens graph with respect to various domination parameters on square bee-hive(hexagon) were determined. In [1] Berghammer has mentioned the independence number (α or β) of bishops on hexagonal board as

$$\alpha(B_{m,n}) = \begin{cases} \frac{m+n}{2} - 1 & \text{if } m \neq n \text{ and both are even} \\ \left\lfloor \frac{m+n}{2} \right\rfloor & \text{if } m \neq n \text{ and atleast one is odd} \end{cases}$$

Using the above bounds mentioned in [1] we take $m = 2n$, and since we consider the square hexagonal board has $2n$ rows and n columns the independence number of bishops is $\lfloor \frac{3n}{2} \rfloor$ when n is odd and $\frac{3n}{2} - 1$ when n is even.

In Section 3 and 4 we reflect our focus in finding the independence separation number $s_{QH}(\beta, n + k, n) = k$ and $s_{BH}(\beta, n+k, n) = k$, for queens and bishops respectively. Here we follow the following cell placement for square hexagonal chessboard throughout this paper.

2 Cell Placemnet

As mentioned earlier the hexagonal board of order n has $2n$ rows and n columns with n^2 hexagonal cells. Here we number the cells by (r, c) where r and c denotes row and column respectively. We number the rows from bottom to top and columns from left to right.

If c is odd, we number the cells in column c by $(2i + 1, c)$ where $i = 0, \ldots, n - 1$. If c is even, we number the cells in column c by $(2i, c)$ where $i = 1, \ldots, n$. Thus the bottom left most cell is numbered as $(1, 1)$.

We define the sum diagonal s_i to be the the diagonal with cells (r, c) where $r + c = i$, and difference diagonal d_j to be the the diagonal with cells (r, c) where $r - c = j$.

3 Queens Independence Separation

Since the independence number for queens on square hexagonal board $\beta(QH)$ is n in [5], therefore to increase the independence number by k we need at least k pawns i.e., $s_{QH}(\beta, n+k, n) \geqslant k$. For $n = 3$, to increase the independence number by 1 we need 2 pawns i.e., $s_{QH}(\beta, 4, 3) = 2$ as shown in Figure 1.

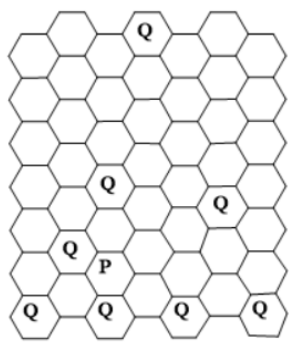
Therefore, we show the placement of queens for $n \geqslant 4$ with $k = 1, 2$ in section 3.1 and then generalise it to k pawns, which increases the independence number of queens by k in Section 3.2 .

1 $s_{QH}(\beta, 3+1, 3) = 2$ which implies $s_{QH}(\beta, n+k, n) \geqslant k$

3.1 Queens separation with 1 pawn and 2 pawns

Lemma 1. For n$\geqslant$ 4, $s_{QH}(\beta, n+1, n) = 1$

Proof. First place the queens in the 1^{st} row. Now place the pawn in the next odd row in the center if number of cells in the row is even, otherwise place the pawn in $\left\lfloor \frac{n-1}{2} \right\rfloor$ cell from left side of the board. Now place two queens diagonal to the pawn one each on either side of it at equal distance in the empty columns. Then, place one queen above the pawn in the cell at which the empty cell in the first column and the last column from bottom intersect in the column at which pawn is placed. Continue placing the queens in the remaining empty columns in the $2n^{th}$ row (since the pawn is placed in an odd row, we place remaining queens in the $2n^{th}$ row). See Figure 2.



2 $s_{QH}(\beta, 7+1, 7) = 1$

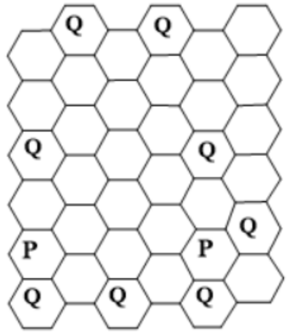
□

Lemma 2. For $n \geqslant 4$, $s_{QH}(\beta, n+2, n) = 2$

Proof. Place the queens in the first row and then place the 2 pawns in the innermost two cells of the row if the row has even number of cells. If the row has odd number of cells then place one pawn each on either side of the center cell. Now place two queens with respect to each pawn placed on each in the sum and difference diagonal. Now place the left over queens in the $2n^{th}$ row if the pawns are placed in odd row, otherwise place queens in the $(2n-1)^{th}$ row. See Figure 3. □

3.2 Queens separation with k pawns

Lemma 3. For even n $\geqslant$ 4, and n $\equiv$ 2 mod 6, with $0 \leqslant k \leqslant \left\lfloor \frac{n}{3} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor$, $s_{QH}(\beta, n+k, n) = k$



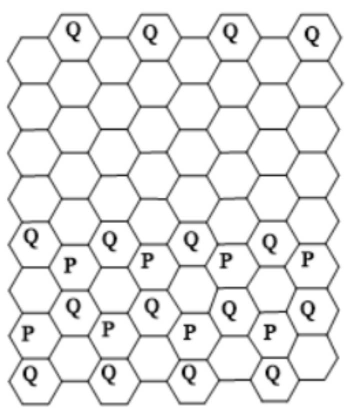
3 $s_{QH}(\beta, 6+2, 6) = 2$

Proof. For boards of the form n $\equiv$ 2 mod 6, we have $\lfloor \frac{n}{2} \rfloor$ even-numbered columns and $\lfloor \frac{n}{2} \rfloor$ odd-numbered columns.

We first define the rows of the form $3m$ ($m = 1, 2, \ldots, \lfloor \frac{n}{3} \rfloor$) to be the rows at which pawns can be placed and rows of the form $3m-2$ ($m = 1, 2, \ldots, \lfloor \frac{n}{3} \rfloor + 1$) to be the rows at which queens can be placed.

Since there are $\lfloor \frac{n}{3} \rfloor$ rows where pawns can be placed and as each row has $\lfloor \frac{n}{2} \rfloor$ cells we first place $\lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{2} \rfloor$ pawns. Next as there are $\lfloor \frac{n}{3} \rfloor + 1$ rows which can be filled with queens, and as there are $\lfloor \frac{n}{2} \rfloor$ cells in each row, we place $(\lfloor \frac{n}{3} \rfloor + 1) \lfloor \frac{n}{2} \rfloor$ queens.

But $(\lfloor \frac{n}{3} \rfloor + 1) \lfloor \frac{n}{2} \rfloor \leqslant n + k = n + \lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{2} \rfloor$ which is the required number of queens as the independence number of queens on hexagonal board is n. Thus we are yet to place $n - \lfloor \frac{n}{2} \rfloor$ queens. If the last pawns row (i.e., the last row filled with pawns from bottom to top) is an even (odd) row the queens will be placed in the $2n^{th}$ ($(2n-1)^{th}$) row. This gives the desired bound at which $s_{QH}(\beta, n+k, n) = k$ as shown in Figure 4.

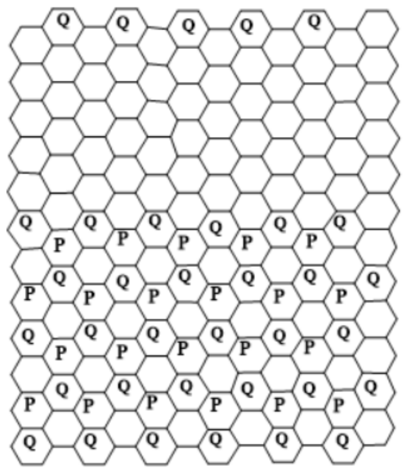


4 $s_{QH}(\beta, 8+8, 8) = 8$

□

Lemma 4. For even n $\geqslant 4$, n $\equiv 0$ mod 6 and n $\equiv 4$ mod 6 with $0 \leqslant k \leqslant \lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{2} \rfloor - 1$, $s_{QH}(\beta, n+k, n) = k$

Proof. We begin the proof by placing the queens and the pawns as mentioned in Lemma 3. Then, we remove the last(first) pawn from the last placed pawns row, and the last(first) queen from $2n^{th}((2n-1)^{th})$ row for the boards of the form n $\equiv 0$ mod 6 (n $\equiv 4$ mod 6). Thus, we place $\lfloor \frac{n}{3} \rfloor \lfloor \frac{n}{2} \rfloor - 1$ pawns as shown in Figure 5 which results in $n+k$ queens separation with k pawns.



5 $s_{QH}(\beta, 12+23, 12) = 23$

□

Lemma 5. For odd n $\geqslant 4$, n $\equiv 1$ mod 6, with $0 \leqslant k \leqslant \lceil \frac{n-2}{6} \rceil (\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor)$, $s_{QH}(\beta, n+k, n) = k$

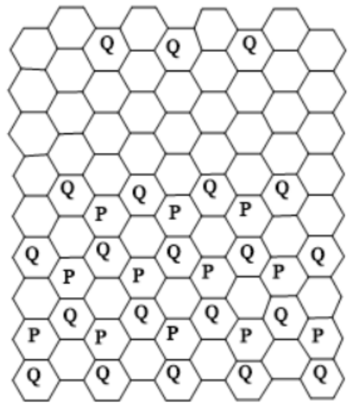
Proof. It should be noted that the board taken is of odd order, therefore we have $\lceil \frac{n}{2} \rceil$ odd-numbered columns and $\lfloor \frac{n}{2} \rfloor$ even-numbered columns. We repeat the same placement used for pawns as used in Lemma 3 and divide $\lfloor \frac{n}{3} \rfloor$ rows equally where pawns can be placed into $\lceil \frac{n-2}{6} \rceil$ rows. Now place $\lceil \frac{n-2}{6} \rceil$ rows with $\lceil \frac{n}{2} \rceil$, and other $\lceil \frac{n-2}{6} \rceil$rows with $\lfloor \frac{n}{2} \rfloor$ pawns. Therefore, it results in placing $\lceil \frac{n-2}{6} \rceil (\lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor)$ pawns in total. Next place the queens as mentioned in Lemma 3. □

Lemma 6. For odd n $\geqslant 4$, n $\equiv 5$ mod 6, with $0 \leqslant k \leqslant \lceil \frac{n-2}{6} \rceil \lceil \frac{n}{2} \rceil + \lfloor \frac{n-2}{6} \rfloor \lfloor \frac{n}{2} \rfloor$, $s_{QH}(\beta, n+k, n) = k$

Proof. We prove this using Lemma 5, by dividing $\lfloor \frac{n}{3} \rfloor$ rows where pawns can be placed equally into $\lceil \frac{n-2}{6} \rceil$ and $\lfloor \frac{n-2}{6} \rfloor$ rows. Since $\lfloor \frac{n}{3} \rfloor$ is odd and as the first pawns row starts with odd-numbered row it also ends with odd-numbered row. Thus, $\lceil \frac{n-2}{6} \rceil$ odd-numbered rows will be occupied by $\lceil \frac{n}{2} \rceil$ odd-numbered columns, and $\lfloor \frac{n-2}{6} \rfloor$ even-numbered rows with $\lfloor \frac{n}{2} \rfloor$ even-numbered columns. Repeat the placement for queens as mentioned in Lemma 3. This gives us $n+k$ queens separation with k pawns. □

Lemma 7. For odd n $\geqslant 4$, n $\equiv 3$ mod 6, with $0 \leqslant k \leqslant \lceil \frac{n-2}{6} \rceil \lceil \frac{n}{2} \rceil + \lfloor \frac{n-2}{6} \rfloor \lfloor \frac{n}{2} \rfloor - 2$, $s_{QH}(\beta, n+k, n) = k$

Proof. We prove this lemma by placing the pawns as discussed earlier in Lemma 6. Now we remove the first and last pawns from the last pawns row, and queens from $2n^{th}((2n-1)^{th})$ queens row if last pawns row is even(odd) respectively as shown in Figure 6. This proves the lemma for separation of $n+k$ queens with k pawns .



6 $s_{QH}(\beta, 9+12, 9) = 12$

□

Theorem 8. For $n \geqslant 4$ and $k \geqslant 0$, $s_{QH}(\beta, n+k, n) = k$

Proof. The proof for this follows from Lemma 3, Lemma 4, Lemma 5, Lemma 6, and Lemma 7 which shows that $s_{QH}(\beta, n+k, n) = k$ for $n \geqslant 4$. □

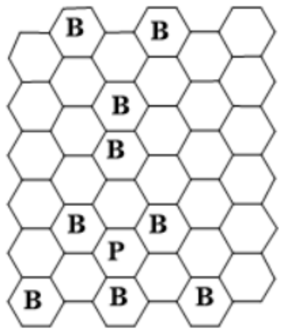
4 Bishops Independence Separation

Since the board of size 2 has only 4 cells and as the independence number is 2, the two bishops must be placed either in the same column or one in the 1st row and other in the last row. Thus, to increase the independence number by 1 either the pawn and queen must be placed in the same column or in different columns which results in bishops attacking each other. Therefore, we prove that $s_{BH}(\beta, n+k, n) = k$ for $n \geqslant 3$ for k=1, 2 in section 4.1 and generalise for k pawns in Section 4.2.

4.1 Bishops Separation for $k = 1, 2$

Lemma 9. For odd $n \geqslant 3$; $s_{BH}(\beta, \lfloor \frac{3n}{2} \rfloor + 1, n) = 1$

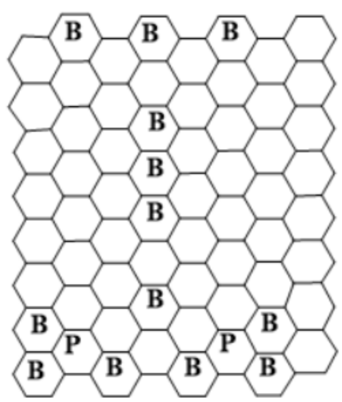
Proof. For n odd, first place bishops in the first row. Here we place pawns on the even columns of the board. If number of cells in the even row from the bottom at which the pawn has to be placed is even, then place a pawn in the odd row just above the center cell at which bishop is placed. If number of cells in the even row is odd then place a pawn in the center cell. Now place two bishops diagonally one each on either side of the pawn just above the diagonal at which bishop in the first and last column from bottom are placed. Next, place the remaining bishops in the column above the pawn where the previously placed bishops on either side of the pawn intersect and below the cell at which d_{n+1} and s_{2n+2} meet. Place the remaining bishops in the $2n^{th}$ row. Similarly we follow the same pattern for even $n \geqslant 3$ by leaving the last column empty which shows that $s_{BH}(\beta, \frac{3n}{2} - 1 + 1, n) = 1$ as shown in Figure 7. □



7 $s_{BH}(\beta, 8 + 1, 6) = 1$

Lemma 10. For odd $n \geqslant 3$; $s_{BH}(\beta, \lfloor \frac{3n}{2} \rfloor + 2, n) = 2$

Proof. Here first we place the bishops in the 1^{st} row and then place the two pawns in the next even row from bottom. Place one pawn each in the 1^{st} and last last cell of the row and then place one bishop in the first and other in the last column diagonal to the pawn. Now place one bishop in the cell at which the two pawns intersect. Place the remaining bishops in the $2n^{th}$ row, and below the cell at which d_{n+1} and s_{2n+2} meet and above the cell at which the bishops with respect to the pawns that are placed on either side of the board meet. Similar pattern follows for even n as mentioned in Lemma 9 which shows that $s_{BH}(\beta, \frac{3n}{2} - 1 + 2, n) = 2$. By removing the last column in Figure 8. Show the placement for odd n with 2 pawns. □



8 $s_{BH}(\beta, 11 + 2, 8) = 2$

4.2 Bishops separation with k pawns

Theorem 11. For odd $n \geqslant 3$ with $0 \leqslant k \leqslant \left\lfloor \frac{n}{2} \right\rfloor (n-1)$; $s_{BH}(\beta, \left\lfloor \frac{3n}{2} \right\rfloor + k, n) = k$

Proof. Here we know that independence number of bishops on square hexagonal board is $\left\lfloor \frac{3n}{2} \right\rfloor$. To increase the independence number by k we need at least k pawns. From this it is clear that $s_{BH}(\beta, \left\lfloor \frac{3n}{2} \right\rfloor + k, n) \geqslant k$.

In this proof we show the placement of bishops, which increases the independence number by k using exactly k pawns.

Since n is odd, it has $\frac{n+1}{2}$ odd columns and $\frac{n-1}{2}$ even columns. Also, the board starts with odd row and hence, it ends with even row. As mentioned earlier, hexagonal board of order n has $2n$ rows and n columns.

Case 1: For $0 \leqslant k < (\frac{n-1}{2})^2$

In this case we first divide the number of pawns k with $\frac{n-1}{2}$ which gives m as the quotient. Now, place $\frac{n-1}{2}$ in each of the m rows from bottom to top as follows:

First place bishops on the first row and on the $(2n)^{th}$ row of the board. Since the board starts with odd row and ends with even row we can place n bishops(i.e.,$(\frac{n+1}{2})$+$(\frac{n-1}{2})$=n). Now start placing the pawns one after the other.

If k is odd, place a pawn in the center column and remaining on either side of it starting from the outermost even columns and moving towards the center, and place the bishops as mentioned below in 1 for the outermost pawns and 3 for the pawns in between innermost pawns.

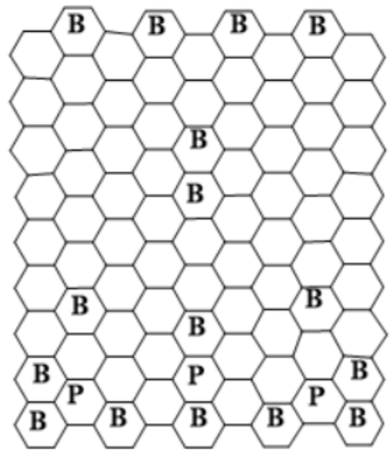
If k is even, place the pawns on either side of the center column from the outermost even columns moving towards the center and place bishops as mentioned below in 1, 2 and 3 .

1. For the outermost pawns on the either side of the row place one bishop diagonally to it just above the diagonal of uppermost bishops in the 1^{st} and last columns.
2. Now for the innermost two pawns in the row place one bishop in the central column at which the two pawns meet diagonally.
3. For the pawns in between the innermost and outermost pawns, place the bishops diagonally just in the above row in between the pawns placed. Thus, if k is even in a particular row, then we can place $k + 1$ bishops.
4. Now place the remaining bishops in the central column just above the cells at which the uppermost bishops from down on either side of the board meet, and below the cell at which d_{n+1} and s_{2n+2} meet.

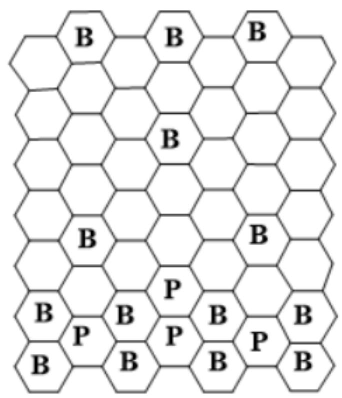
Thus, we have placed $n + [m \times \frac{n+1}{2}]$ bishops as shown in Figure 9 and 10.

Case 2: For $(\frac{n-1}{2})^2 < k \leqslant \frac{n^2-1}{4}$

Start the placement with placing $(\frac{n-1}{2})^2$ pawns and $(\frac{n+1}{2})^2$ bishops in the even and odd columns respectively from bottom to top.



9 $s_{BH}(\beta, 13+3, 9) = 3.$



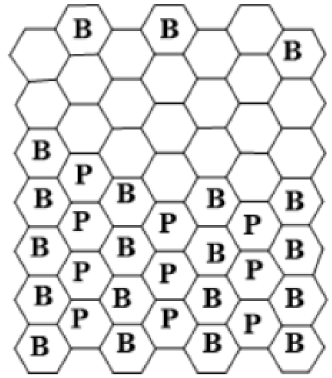
10 $s_{BH}(\beta, 10+4, 7) = 4$

Now continue placing the remaining pawns one after the other in the second column by placing the bishops as mentioned below after placing each pawn.

Place two bishops one in the sum diagonal(i.e., first column) and the other in the difference diagonal(i.e., $(2n-1)^{th}$ row) to the pawn placed. Now place the remaining bishops from right to left in $2n^{th}$ row just above the diagonal at which the uppermost bishop in the 1^{st} column is placed. See Figure 11.

Case 3: For $\frac{n^2-1}{4} < k \leqslant \left\lfloor \frac{n}{2} \right\rfloor (n-1)$

1. Start placing $\frac{n^2-1}{4}$ pawns and $(\frac{n+1}{2})^2$ bishops in the even columns and odd columns respectively from down to top where the last pawns row will be placed at $(n+1)^{th}$ row and last bishops row at n^{th} row.
2. Now place the remaining pawns as follows:



11 $s_{BH}(\beta, 10+10, 7) = 10$

(a) First place the pawns in the second column one at a time, leaving the last cell and place the bishops in all the cells of the first column.

(b) Then, place two bishops one in the difference diagonal and other in the sum diagonal of the uppermost pawn in the 2^{nd} column.

(c) Then, continue placing the bishops from left to right next to the bishop placed in the difference diagonal, and from right to left next to the bishop placed in sum diagonal.

(d) Now remaining bishops are placed below the diagonal at which leftmost bishop in $(2n)^{th}$ row and right most bishop in $(n+2)^{th}$ row intersect.

3. If the second column is filled, then place the remaining pawns one by one in $(n+3)^{rd}$ row from left to right along with the bishops as mentioned below:

   (a) Place the bishops one in the previous column diagonal to the pawn placed and one in the cell at which the innermost pawns in the row intersect.

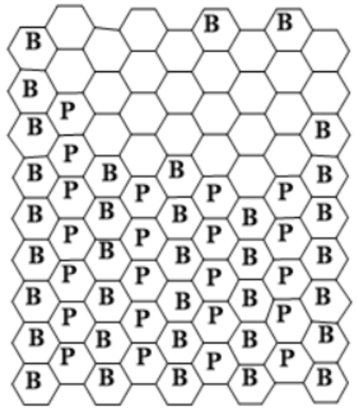
   (b) Now place the remaining pawns in $2n^{th}$ row diagonally to the last pawn present in the inner most pawns column from left.

4. If the $(n+3)^{rd}$ row is filled, then place the pawns in the second last column from bottom to top and place the bishops one diagonally to the right side of each pawn and other diagonally in $(2n-1)^{th}$ row. Now place bishops to the left of the bishop placed in $(2n-1)^{th}$ row and the rest of them in $(2n)^{th}$ row just above the diagonal at which uppermost bishop in the last column is placed.

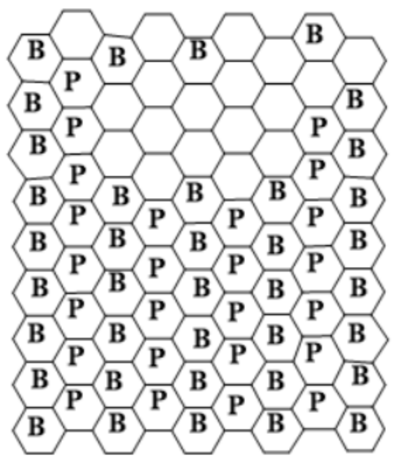
5. If second last column is filled, then place the pawns in the next even column from left and repeat 4 of case 3 by changing left as right for the bishops placemnet in $(2n-1)^{th}$ row.

6. If the next even column is filled, then move to the next row(i.e., $(n+5)^{th}$ row) and repeat as mentioned in step 3 of Case 3 by placing pawns in the row presently chosen instead of $(n+3)^{rd}$ row.

7. Continue the steps 5 and 6 alternatively until the required number of pawns are placed. In Figure 12, 13 and 14.



12 $s_{BH}(\beta, 13+22, 9) = 22$



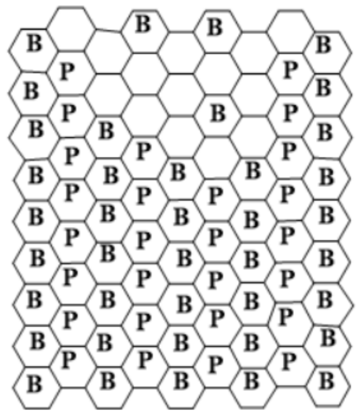
13 $s_{BH}(\beta, 13+25, 9) = 25$

□

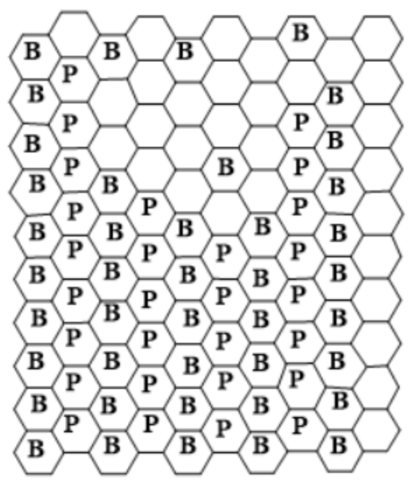
Theorem 12. For even $n \geqslant 4$ with $0 \leqslant k \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor (n-1)$; $s_{BH}(\beta, \frac{3n}{2}+k-1, n) = k$

Proof. For the boards of even order the independence number of bishops on square hexagonal board is $\frac{3n}{2} - 1$. To increase the independence number by k we need at least k pawns. Therefore from this it is clear that $s_{BH}(\beta, \frac{3n}{2}+k-1, n) \geqslant k$.

We follow the same pattern as mentioned in Theorem 11, by leaving the last column of the board without any placement as shown in Figure 15. □



14 $s_{BH}(\beta, 13 + 27, 9) = 27$



15 $s_{BH}(\beta, 14 + 28, 10) = 28$

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