On a connection between max-min identity and GCD-LCM one over partially ordered set

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Abstract In this paper we discuss two partially ordered sets and give a connection between max-min identity and GCD-LCM one.

Key Words partially ordered set, max-min identity, GCD-LCM identity, Inclusion-Exclusion principle

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1 Introduction

We know the fundamental theorem of arithmetic is an important result in number theory. The fact that every positive integer has a unique factorization into primes is special property in the set of integers. We discuss a connection between max-min functions and GCD-LCM by the fundamental theorem of arithmetic. The fundamental theorem of arithmetic asserts that every positive integer $n$ greater than 1 can be written uniquely as

$$n = p_1^{α_1}p_2^{α_2}p_3^{α_3}...p_k^{α_k},$$

where $p_i$ is prime, $p_1 < p_2 < p_3 < ... < p_k$ and $α_i ≥ 0, i = 1, 2, · · · , k$. Called the standard form of the prime-factorization of $n$. If this factorization is given, we can immediately deduce whether a prime $p$ divides $n$ since $p$ divides $n$ if and only if it is appears in this factorization. Denote $max(a_1, a_2, · · · , a_k)$ ($min(a_1, a_2, · · · , a_k)$) the maximal (minimal) real number of the set $\{a_1, a_2, · · · , a_k\}$. Note that GCD (Greatest Common Divisor) and LCM (Least Common Multiple) functions are

$$[a, b] = p_1^{max(α_1, β_1)}p_2^{max(α_2, β_2)}p_3^{max(α_3, β_3)}...p_k^{max(α_k, β_k)}$$

and

$$(a, b) = p_1^{min(α_1, β_1)}p_2^{min(α_2, β_2)}p_3^{min(α_3, β_3)}...p_k^{min(α_k, β_k)}.$$

Also, we denote $(a_1, a_2, a_3, · · · , a_k) = (a_1, (a_2, ..., a_k))$ and $[a_1, a_2, a_3, · · · , a_k] = [a_1, [a_2, ..., a_k]]$ for $k > 2$. 

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In this paper we discuss two partially ordered sets and give a connection between max-min identity and GCD-LCM identity over partially ordered set. We discuss the Equivalence and uniformity of inclusion-exclusion principle in combinatorics, the total probability formula in probability theory, max-min identity and GCD-LCM one in number theory. Then some examples are given.

2 Max-Min functions and Partially ordered set

In this section, we first state some results of Max and Min functions needed later in the paper.

Lemma 2.1 ([1]). If $x$ and $y$ are real numbers, then

$$\max(x, y) + \min(x, y) = x + y.$$ 

Proof. If $x \geq y$, then $\min(x, y) = y$ and $\max(x, y) = x$, so that $\max(x, y) + \min(x, y) = x + y$. If $x < y$, then $\min(x, y) = x$ and $\max(x, y) = y$, we find again that $\max(x, y) + \min(x, y) = x + y$. □

Lemma 2.2. If $x_1, x_2, \ldots, x_k$ are real numbers, then $\max(x_1, x_2, \ldots, x_k) = \max(x_1, \max(x_2, \ldots, x_k))$.

Lemma 2.3. If $x_1, x_2, \ldots, x_k$ are real numbers, then $\min(x_1, x_2, \ldots, x_k) = \min(x_1, \min(x_2, \ldots, x_k))$.

A partially ordered set (poset) $P$ is set, together with a binary relation denoted $\leq_P$, satisfying the following three axioms:

1. For all $x \in P$, $x \leq_P x$; (reflexivity)
2. If $x \leq_P y$ and $y \leq_P x$, then $x = y$; (antisymmetry)
3. If $x \leq_P y$ and $y \leq_P z$, then $x \leq_P z$. (transitivity).

Definition 2.4 ([1]). Two poset $P$ and $Q$ are isomorphic if exists an order-preserving bijection $\varphi : P \rightarrow Q$ whose inverse is order-preserving, that is, $x \leq y$ in $P \Leftrightarrow \varphi(x) \leq \varphi(y)$ in $Q$.

Definition 2.5. A chain (linear ordered set) is a poset in which any two elements are comparable.

Sets are partially ordered by containment. More generally, in a lattice $L$, we can define a partial order $\leq_L$ compatible with the lattice operations on $L$ by $x \leq_L y$ if and only if $x \wedge y = x$. It is easy to prove that $x \wedge y = x$ if and only if $x \vee y = y$. Thus the following three conditions are equivalent:

$$x \leq_L y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y.$$ 

Example 2.6. Let $A$ and $B$ are nonempty sets. Then $A \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B$.

Example 2.7. Let $x$ and $y$ are positive integers. Then $x \leq y \Leftrightarrow \min(x, y) = x \Leftrightarrow \max(x, y) = y$.

Example 2.8. Let $a$ and $b$ are positive integers. Then $a \mid b \Leftrightarrow (a, b) = a \Leftrightarrow [a, b] = b$.

Lemma 2.9 ([1]). If $a$ and $b$ are positive integers, then $\lfloor a, b \rfloor (a, b) = ab$. 

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3 Main results

Theorem 3.1. Let \(x_1, x_2, x_3, \ldots, x_n\) are positive integers and \(n_1 = x_1, n_2 = \max(x_1, x_2), n_3 = \max(x_1, x_2, x_3), \ldots, n_k = \max(x_1, x_2, x_3, \ldots, x_k)\), then \([n] = \{n_1, n_2, \ldots, n_k\}\) is a chain.

Proof. Since \(x_1 \leq \max(x_1, x_2) \leq \max(x_1, x_2, x_3) \leq \cdots \leq \max(x_1, x_2, x_3, \ldots, x_k)\), then \(n_1 \leq n_2 \leq \cdots \leq n_k\), that is \([n] = \{n_1, n_2, \ldots, n_k\}\) is a chain.

Theorem 3.2. Let \(m_1 = x_1, m_2 = \min(x_1, x_2), m_3 = \min(x_1, x_2, x_3), \ldots, m_k = \min(x_1, \ldots, x_k)\), then \([m] = \{m_1, m_2, \ldots, m_k\}\) is a chain.

Proof. Since \(x_1 \geq \min(x_1, x_2) \geq \min(x_1, x_2, x_3) \geq \cdots \geq \min(x_1, x_2, x_3, \ldots, x_k)\), then \(m_1 \geq m_2 \geq \cdots \geq m_k\), that is \([m] = \{m_1, m_2, \ldots, m_k\}\) is a chain.

Theorem 3.3. Let \(p_1 = a_1, p_2 = a_1, a_2, p_3 = a_1, a_2, a_3, \ldots, p_k = a_1, a_2, a_3, \ldots, a_k\), then \([p] = \{p_1, p_2, \ldots, p_k\}\) is a chain.

Proof. Since \(a_1 \leq_{[p]} a_1, a_2, \leq_{[p]} a_1, a_2, a_3, \leq_{[p]} \cdots \leq_{[p]} a_1, a_2, a_3, \ldots, a_k, p_1 \leq_{[p]} p_2 \leq_{[p]} \cdots \leq_{[p]} p_k\) where \(\leq_{[p]} = \{p_1, p_2, \ldots, p_k\}\) is a chain.

Theorem 3.4. Let \(q_1 = b_1, q_2 = (b_1, b_2), q_3 = (b_1, b_2, b_3), \ldots, q_k = (b_1, b_2, b_3, \ldots, b_k)\), then \([q] = \{q_1, q_2, \ldots, q_k\}\) is a chain.

Proof. Since \(b_1 \geq_{[q]} (b_1, b_2) \geq_{[q]} (b_1, b_2, b_3) \geq_{[q]} \cdots \geq_{[q]} (b_1, b_2, b_3, \ldots, b_k), q_1 \geq_{[q]} q_2 \geq_{[q]} \cdots \geq_{[q]} q_k\) where \(\geq_{[q]} = \{q_1, q_2, \ldots, q_k\}\) is a chain.

Theorem 3.5. Let \([n] = \{n_1, n_2, \ldots, n_k\}\) and \([p] = \{p_1, p_2, \ldots, p_k\}\), then there is an isomorphic between \([n]\) and \([p]\), where order preserving bijection \(\varphi : [n] \to [p]\), that is \(\varphi : n_i \leq n_j \to p_i \leq p_j\).

Proof. Since \(\varphi : n_i \leq n_j \to p_i \leq p_j\), then there is an isomorphic between \([n]\) and \([p]\).

Theorem 3.6. Let \([m] = \{m_1, m_2, \ldots, m_k\}\) and \([q] = \{q_1, q_2, \ldots, q_k\}\), then there is an isomorphic between \([m]\) and \([q]\), where order preserving bijection \(\varphi : [m] \to [q]\), that is \(\varphi : m_j \leq m_j' \to q_j \leq q_j\).

Proof. Since \(\varphi : m_j \leq m_j' \to q_j \leq q_j\), then there is an isomorphic between \([m]\) and \([q]\).

Every chain is a lattice. Let \(X = \{x_1, x_2, x_3, \ldots, x_n\}\) is positive integers set, here we have \(x \land y = \min(x, y)\) and \(x \lor y = \max(x, y)\). Then \((X, \max, \min, \leq)\) is a lattice. Let \(A = \{a_1, a_2, a_3, \ldots, a_n\}\) is positive integers set and we have \(a \land b = (a, b)\) and \(a \lor b = [a, b]\), then \((A, (.), [[]])\) is a lattice. Let \(P(A) = \{A_1, A_2, A_3, \ldots, A_n\}\) is power set of set \(A\), we also have \(A \land B = A \cap B\) and \(A \lor B = A \cup B\), then \((P(A), \cap, \cup, \subseteq)\) is a lattice.

Definition 3.7([4]). If \(L\) and \(M\) are \(\lor\)-semilattices then \(f : L \to M\) is said to be a \(\lor\)-morphism if \(f(x \lor y) = f(x) \lor f(y)\) for all \(x, y \in L\).

Theorem 3.8([4]). Lattices \(L, M\) are isomorphism if and only if there is a bijection \(f : L \to M\) that is a \(\lor\)-morphism.

Theorem 3.9. Let \(X\) and \(Y\) are integer sets and \(|X| = |Y|\), then \((X, \max, \min, \leq)\sim(Y, (.), [[]])\).

Next we give two examples.
Further topics and examples

In the theory of combinatorics, the inclusion-exclusion principle(also known as the sieve principle) is an equation related the size of two sets and their intersection. It states that if $A$ and $B$ are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$  

For the general case of principle, in [6], let $\{A_1, A_2, \ldots, A_n\}$ be finite sets. Then

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|.$$  

Example 3.10([1]). If $x, y, z$ are positive integers, then $\max(x, y, z) = x + y + z - \min(x, y) - \min(x, z) - \min(y, z) + \min(x, y, z)$.

Proof. Without loss of generality, suppose that $x \leq y \leq z$, then $\max(x, y, z) = z, \min(x, y) = x, \min(x, z) = y, \min(x, y, z) = x$. Hence $x + y + z - \min(x, y) - \min(x, z) - \min(y, z) + \min(x, y, z) = x + y + z - x - y - x + z = \max(x, y, z)$. $\square$

Example 3.11. If $a, b, c$ are positive integers, then

$$[a, b, c] = \frac{abc(a, b, c)}{(a, b)(a, c)(b, c)}.$$  

Proof. By Theorem 3.8 and Example 3.11, $\max(x, y, z) \mapsto [a, b, c], x + y + z \mapsto abc, -\min(x, y) - \min(x, z) - \min(y, z) \mapsto (a, b)^{-1}(a, c)^{-1}(b, c)^{-1}, \min(x, y, z) \mapsto (a, b, c)$. Then $[a, b, c] = \frac{abc(a, b, c)}{(a, b)(a, c)(b, c)}$. $\square$

In the following we give a max-min identity. The max-min identity is a relation between the maximum element of a set $S$ of $n$ numbers and the minima of the $2^n - 1$ nonempty subsets of $S$.

Theorem 3.12([5]). Let $S = \{x_1, x_2, x_3, \ldots, x_n\}$ is a real number set, then $\max(x_1, x_2, x_3, \ldots, x_n) = \sum_{i=1}^{n} x_i - \sum_{i < j} \min(x_i, x_j) + \sum_{i < j < k} \min(x_i, x_j, x_k) - \cdots - (-1)^{n+1} \min(x_1, x_2, x_3, \ldots, x_n)$.

Conversely, we have

Theorem 3.13. Let $S = \{x_1, x_2, x_3, \ldots, x_n\}$ is a real number set, then $\min(x_1, x_2, x_3, \ldots, x_n) = \sum_{i=1}^{n} x_i - \sum_{i < j} \max(x_i, x_j) + \sum_{i < j < k} \max(x_i, x_j, x_k) - \cdots + (-1)^{n+1} \max(x_1, x_2, x_3, \ldots, x_n)$.

Corollary 3.14([5]). For any $n$ Random Variables $X_1, X_2, \ldots, X_n$, then $E[\max(x_1, \ldots, x_n)] = \sum_{i=1}^{n} E[x_i] - \sum_{i < j} E[\min(x_i, x_j)] + \sum_{i < j < k} E[\min(x_i, x_j, x_k)] - \cdots - (-1)^{n+1} E[\min(x_1, x_2, \ldots, x_n)]$.

Theorem 3.15. Let $a_1, a_2, a_3, \ldots, a_n$ are positive integers, then

$$[a_1, a_2, a_3, \ldots, a_n] = a_1a_2a_3 \cdots a_n(a_1, a_2)^{-1} \cdots (a_{n-1}, a_n)^{-1}(a_1, a_2, a_3) \cdots (a_1, a_2, \ldots, a_n)(-1)^{n+1}.$$  

Corollary 3.15. If $a_1, a_2, a_3, \ldots, a_n$ are pairwise relatively primes integers, then $[a_1, a_2, a_3, \ldots, a_n] = a_1a_2a_3 \cdots a_n$.

Theorem 3.16. Let $a_1, a_2, a_3, \ldots, a_n$ are positive integers, then

$$(a_1, a_2, a_3, \ldots, a_n) = a_1a_2a_3 \cdots a_n[a_1, a_2]^{-1} \cdots [a_{n-1}, a_n]^{-1}[a_1, a_2, a_3] \cdots [a_1, a_2, \ldots, a_n](-1)^{n+1}.$$  

4 Further topics and examples

In the theory of combinatorics, the inclusion-exclusion principle(also known as the sieve principle) is an equation related the size of two sets and their intersection. It states that if $A$ and $B$ are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$  

For the general case of principle, in [6], let $\{A_1, A_2, \ldots, A_n\}$ be finite sets. Then

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|.$$  

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where $|A|$ denotes the cardinality of the set $A$.

**Example 4.1** (Prime approximately formula). Let $N$ is a positive integer, $p_1, p_2, \cdots, p_m$ are all prime integers less than $\sqrt{N}$, then

$$
\pi(N) = m - 1 + N - \sum_{1 \leq i_1 \leq m} \left\lfloor \frac{N}{p_{i_1}} \right\rfloor + \sum_{1 \leq i_1, i_2 \leq m} \left\lfloor \frac{N}{p_{i_1} p_{i_2}} \right\rfloor - \cdots + (-1)^m \left\lfloor \frac{N}{p_{i_1} \cdots p_{i_m}} \right\rfloor.
$$

**Example 4.2** (Euler function formula). Let $N$ is a positive integer, $\varphi(n)$ denotes numbers of relatively prime with $n$ in $1, 2, \ldots, N$, then

$$
\varphi(n) = N \prod_{p|N} (1 - \frac{1}{p}).
$$

In the theory of probability (see [5]), for events $A_1, A_2, \ldots, A_n$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the inclusion-exclusion principle becomes in general the following form:

$$
P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n),
$$

where $P(A_i)$ denotes the probability of the event $A_i$.

**Example 4.3** (The derangement problem). How many permutation $\pi \in \mathfrak{S}_n$ have no fixed points? Such a permutation is called a derangement. Call this number $D(n)$, then

$$
D(n) = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)
$$

and indeed it is to show that $D(n)$ is nearest integer to $\frac{1}{e}$. Thus the probability of an order for a shuffled deck of cards and being incorrect about every card is approximately $\frac{1}{e} (\approx 37\%)$. Thus formally the Principle of Inclusion-Exclusion is equivalent to the identity $(e^x)^{-1} = e^{-x}$.

In [2], Equivalence and uniformity of inclusion-exclusion principle in combinatorics, the total probability formula in probability theory are given. Let $S$ be an $n$-set. Let $V$ be the $2^n$-dimensional vector space (over some field $k$) of all functions $f : 2^n \to k$. Let $\phi : V \to V$ be a linear transformation defined by

$$
\phi f(T) = \sum_{Y \supseteq T} f(Y),
$$

for all $T \subseteq S$, then $\phi^{-1}$ exists and is given by

$$
\phi^{-1} f(T) = \sum_{Y \supseteq T} (-1)^{|Y-T|} f(Y),
$$

for all $T \subseteq S$.

We discuss equivalence and correspondence between max-min identity and GCD-LCM identity in this paper. Our goals is to give a connection between max-min function and GCD-LCM over partially ordered set. As are we known, complexity degree of algorithm between to prove max-min identity and GCD-LCM identity are very different in computation number theory.
References