A new characterization of finite groups in which every element has prime power order *

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Abstract

In this paper, we give a new characterization of finite groups in which every element has prime power order, and answered the problem 18.112 in "Unsolved Problem in Group Theory".

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A group is called a CP-group if every element of the group has prime power order. This definition is equivalent to the statement that the centralizer of every nontrivial p-element is a p-group for all p. This is a generalization of groups of prime power order. A group is called a C_{pp}-group if the centralizer of every nontrivial p-element is a p-group for a fixed p. Examples of CP-groups include p-groups, where p is a prime, and Tarski groups, which are simple groups whose proper subgroups have prime order.

In 1957, G. Higman first studied the finite CP-groups [4]. He showed that a finite solvable CP-group is a split extension of its Fitting subgroup, which must clearly be a p-group, by a complement acting fixed-point-freely. Moreover, the order of a finite solvable CP-group is divisible by at most two primes. In the same article, Higman studied the structure of finite non-solvable CP-groups and showed that such a group has a non-abelian simple section which largely determines its structure. M. Suzuki classified finite simple CP-groups in his celebrated work [9], finding that only eight finite simple CP-groups exist. R. Brandl [2] and W. Shi, W. Yang [7] continued this line of inquiry by classifying

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finite non-solvable CP-groups. For the special finite CP-groups, W. Shi, W. Yang [8] and M. Deaconescu etc.(see [1] and [6]) classified the finite groups with all elements of prime order except the identity.

In [7], the authors proved that a finite CP-group G satisfies the following property:

Property A: *For all subgroups H of G if $(|H|, d) = 1 (d > 1)$, then $|H|$ divides the number of elements of order d in G.*

Proof. Let H act on the set of elements of order d in G by conjugation $a \to h^{-1}ah$, where $o(a) = d$ and $h \in H$. We get the H-conjugate class $C = \{a^h | \forall a \in H\}$. Since G is a CP-group , $C_H(a) = 1$ and $|C| = |H|$, the conclusion holds.

Then the second author of this paper asked whether the inverse is true, that is, if a finite group G satisfies Property A, is G a CP-group? This question is collected in "Unsolved Problem in Group Theory (Edition 18)" [5], which is Problem 18.112. In this short paper, we give a positive answer to this problem.

Next we denote by π a set of primes and $\pi(n)$ the set of all primes of the integer n ($n > 1$). Denote by π' the complement of the set π, and n_π the π-part of the integer n, i.e., the largest divisor of n including all prime divisors of π. Note that the element x can write $x = x_\pi x_{\pi'}$ such that x_π and $x_{\pi'}$ are π- and π'-elements respectively. Moreover, such resolution is unique. All groups in the following are considered finite. Since Sylow subgroups exist always, we have Property A is equivalent to the following property:

Property B: *For all Sylow p-subgroups P of G if $(|P|, d) = 1 (d > 1)$, then $|P|$ divides the number of elements of order d in G.*

First, we cite a well known result due to Frobenius as follows.

Lemma 1[[3], Theorem 9.1.2]. *Let G be a finite group and n a divisor of $|G|$, and let $f_G(n) = |\{g \in G | g^n = 1\}|$. Then $f_G(n)$ is a multiple of n.*

Before starting our main result, we next give a lemma, which is the generalization of the result of Weisner [10].

Lemma 2. *Let G be a finite group and g an element of G. Suppose that the order of x is n and denote by g^G the conjugacy class including g.*

(1) if $g^G = \{g\}$ and m assumes all values prime to n, then the number of solutions of the equations $x^m = g$ is $\phi(n) f_G(|G|_{\pi(n)'})$, where ϕ is Euler function.

(2) if $g^G = \{g\}$ and m assumes all values not prime to n, suppose that the conjugacy classes of $\pi(n)$-elements satisfied $x^m = g$ are $g_1^G, g_2^G, \cdots, g_l^G$, then the number of solutions of the equations $x^m = g$ is $\sum_{i=1}^{l} |g_i^G| f_{C_G(g_i)}(|C_G(g_i)|_{\pi(n)'})$.

(3) let $|g^G| > 1$ and the number of solutions of the equation $x^m = g$ in the subgroup $C_G(g)$ be $\mu(C_G(g))$ for all m. Then the number of elements whose power are in the conjuagcy class g^G is $|g^G| \mu(C_G(g))$.

Proof. Let k be the greatest divisor of $|G|$ that is prime to n, that is $k = |G|_{\pi(n)'}$. First we assume that $g^G = \{g\}$, and so g is a center element of G.

(1) the case of $(m,n)=1$. So there exist integers m' and n' such that $mm'+nn'=1$. Since $x^m=g$, it leads to $x^m_{\pi(n)}x^m_{\pi(n)'}=g$ and then $x^m_{\pi(n)}=g$ and $x^m_{\pi(n)'}=1$. Furthermore $x_{\pi(n)}=x^{mm'+nn'}_{\pi(n)}=g^{m'}$ and $(m',n)=1$. It follows that the solution set of the equation $x^m=g$ is the set $\{g^{m'}z|(m',n)=1$ and z is a $\pi(n)'$-element$\}$. On the other hand, the number of $\pi(n)'$-elements of G is $f_G(k)$, and so the number of solutions of the equation of $x^m=g$ is $\phi(n)f_G(k)$.

(2) the case of $(m,n)\neq 1$. Similarly, we can resolve x into the $\pi(n)$-part and $\pi(n)'$-part, that is $x=x_{\pi(n)}x_{\pi(n)'}$, and then $x^m_{\pi(n)}=g$ and $x_{\pi(n)'}=1$. Hence the solution of equation $x^m=g$ can be only expressed by the form $x_{\pi(n)}z$, where $x^m_{\pi(n)}=g$ and z is a $\pi(n)'$-element of $C_G(x_{\pi(n)})$. Next suppose that the conjugacy classes of $\pi(n)$-elements satisfied $x^m=g$ are $g_1^G,g_2^G,\cdots,g_l^G$. Since the number of solutions of the equation $x^m=g$ generated by the conjugacy class g_i^G is $|g_i^G|f_{C_G(g_i)}(|C_G(g_i)|_{\pi(n)'})$, it follows that the number of all solutions is $\sum_{i=1}^{l}|g_i^G|f_{C_G(g_i)}(|C_G(g_i)|_{\pi(n)'})$.

(3) Let $g^G=\bigcup_{i=1}^{h}C(g_i)$ where $C(g_i)=\{g_i^{n_1},g_i^{n_2},\cdots,g_i^{n_r}\}$ for $1\leq i\leq h$ and $(n_i,n)=1$ such that any two elements in the same subset $C(g_i)$ are powers of each other, whereas no element is a power of an element in another subset $C(g_j)$ for $i\neq j$. Without loss of generalization, we set $n_1=1$. Note that if the power of the solution x of the equation $x^m=g$ is in the $C(g_i)$, then a power of x cannot be in the $C(g_j)$ for $i\neq j$. Otherwise, if $x^m=g_i^{n_t}$ and $x^{m'}=g_j^{n_l}$, then x^m and $x^{m'}$ are of same order n, so that each is a power of the other, whence $i=j$. Since each centralizer $C_G(g_i)$ is a conjugate subgroup of $C_G(g)$, the numbers of elements whose power includes in $C(g_i)$ are same. Let the number of elements of $C_G(g_i)$ whose power in $C(g_i)$ be $\mu(C_G(g_i))$. So the number of elements whose power are in the conjuagcy class g^G is $|g^G|\mu(C_G(g))$. □

From the above Lemma 1 and Lemma 2, we can get the following corollary which is first given by Weisner (Theorem 3, [10]).

Corollary. The number of elements whose orders are multiples of $n\,(n>1)$ is a multiple of $|G|_{\pi(n)'}$.

Now we give our main result as follows.

Theorem. *Let G be a finite group. Then the following conditions are equivalent:*

(1) *G is a CP-group;*

(2) *For every divisor d $(d>1)$ of $|G|$ and for every subgroup H of G of order coprime to d, the order $|H|$ divides the number of elements of G of order d;*

(3) *For every divisor d $(d>1)$ of $|G|$, the number of elements of order d is divided by the number $|G|_{\pi(d)'}$.*

Proof. It's obvious that the (2) and (3) are equivalent. Next we prove the item (1) is equivalent to the (3). From Property A, we have (1) $\Rightarrow$ (3). Next we prove (3) $\Rightarrow$ (1). Denote by s_n the number of elements whose orders are n. Suppose that for every divisor d $(d>1)$ of $|G|$, the number of elements of order d is divided by the number $|G|_{\pi(d)'}$. We will prove G is a CP-group. Now we assume that G has an element of order pq and $p \neq q$, that is $s_{pq} > 0$. By Corollary, since $p \neq q$, we have

$|G|_q \big| |G|_{p'} \big| \sum_{p|d} s_d$ (this is the number of all elements whose order is a multiple of p), and

$\sum_{p|d} s_d = (\sum_{pq|d} s_d + \sum_{p|d, q\nmid d} s_d)$. (we divide these elements into two parts, one is the set of all elements whose order can be divided by pq, the other cannot divided by pq). Then

$$|G|_q \Big| \sum_{pq|d} s_d + \sum_{p|d, q\nmid d} s_d. \qquad (*)$$

Next we consider $\sum_{p|d, q\nmid d} s_d$, i.e. the number of all elements whose order can not be divided by pq but can be divided by p. In view of our hypothesis(the condition (3)), if $q \nmid d$, then $|G|_q \big| |G|_{\pi(d)'} \big| s_d$, and so

$$|G|_q \big| s_d.$$

It follows that

$$|G|_q \Big| \sum_{p|d, q\nmid d} s_d \qquad (**)$$

By the $(*)$ and $(**)$ we have $|G|_q \big| \sum_{pq|d} s_d$. Similarly, we have $|G|_p \big| \sum_{pq|d} s_d$. It follows that

$$|G|_{\{p,q\}} \Big| \sum_{pq|d} s_d. \qquad (\dagger)$$

On the other hand, by Corollary, we have

$$|G|_{\{p,q\}'} \Big| \sum_{pq|d} s_d. \qquad (\dagger\dagger)$$

So by the $(\dagger)$ and $(\dagger\dagger)$ we can get

$$|G| \Big| \sum_{pq|d} s_d,$$

since $\sum_{pq|d} s_d < |G|$, and then $\sum_{pq|d} s_d = 0$, so $s_{pq} = 0$, which contradicts the hypothesis that there exists an element of order pq. Therefore G is a CP-group. □

Finally, we pose the following problem: the above theorem give a characterization of finite CP-groups using the number of elements and the orders of subgroups. Is there a similar characteristic property for finite C_{pp}-groups?

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